First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MAT100 - Mathematical Foundations
(Matematik Asas)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT (8) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN (8) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **ELEVEN (11)** questions.

*[**Arahan :** Jawab **SEBELAS (11)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

Question 1

Given $P(x): |x-1| \leq 1$, $Q(x): |x+2| > 2$ and $R(x): x^2 \leq 1$ be three open sentences with domain $\square$, find

(a) the truth value of $P(x)$, $Q(x)$ and $R(x)$, for each $x \in \square$.

[6 marks]

(b) $\{x \in \square : \square\,(P(x) \wedge Q(x))$ is false$\}$

[2 marks]

(c) $\{x \in \square : \square\, P(x) \vee (Q(x) \wedge R(x))$ is false$\}$

[2 marks]

(d) $\{x \in \square : (Q(x) \Rightarrow R(x)) \wedge P(x)$ is true$\}$

[2 marks]

(e) $\{x \in \square : P(x) \Leftrightarrow Q(x)$ is true$\}$

[2 marks]

Soalan 1

Diberi $P(x): |x-1| \leq 1$, $Q(x): |x+2| > 2$ *and* $R(x): x^2 \leq 1$ *adalah tiga ayat terbuka dengan domain* $\square$ *, dapatkan*

(a) *nilai kebenaran* $P(x)$, $Q(x)$ *dan* $R(x)$ *bagi setiap* $x \in \square$.

[6 markah]

(b) $\{x \in \square : \square\,(P(x) \wedge Q(x))$ adalah palsu$\}$

[2 markah]

(c) $\{x \in \square : \square\, P(x) \vee (Q(x) \wedge R(x))$ adalah palsu$\}$

[3 markah]

(d) $\{x \in \square : (Q(x) \Rightarrow R(x)) \wedge P(x)$ adalah benar$\}$

[4 markah]

(e) $\{x \in \square : P(x) \Leftrightarrow Q(x)$ adalah benar$\}$

[2 markah]

Question 2

For polynomial $q(x) = x^3 - 2x^2 - x + 2$, consider $A = \{a \in \mathbb{R} : q(a) = 0\}$

(a) Find all elements in A.

[3 marks]

(b) Give an open sentence $P(x)$ with domain $\mathbb{R}$ such that A exactly contains all elements in $\mathbb{R}$ for which $P(x)$ is true.

[1 mark]

Soalan 2

Untuk polinomial $q(x) = x^3 - 2x^2 - x + 2$, *pertimbangkan* $A = \{a \in \mathbb{R} : q(a) = 0\}$

(a) Dapatkan semua unsur dalam A.

[3 markah]

(b) Beri satu ayat terbuka $P(x)$ *dengan domain* $\mathbb{R}$ *sedemikian hingga A mengandungi semua unsur dalam* $\mathbb{R}$ *yang mana* $P(x)$ *adalah benar.*

[1 markah]

Question 3

Each of the following proposed proofs contains a mistake. Evaluate each proof by pointing out the mistake.

(a) **Result** Let $x, y \in \mathbb{Z}$ and let a and b be odd integers. If $ax + by$ is even, then x and y are of the same parity.

Proof Assume that x and y are of opposite parity. Then $x = 2p$ and $y = 2q + 1$ for some integers p and q. Since a and b are odd integers, $a = 2r + 1$ and $b = 2s + 1$ for some integers r and s. Hence

$$\begin{aligned} ax + by &= (2r+1)(2p) + (2s+1)(2q+1) \\ &= 4pr + 2p + 4qs + 2s + 2q + 1 \\ &= 2(2pr + p + 2qs + s + q) + 1. \end{aligned}$$

Since $2pr + p + 2qs + s + q$ is an integer, $ax + by$ is odd.

[2 marks]

(b) **Result** Let $x, y \in \mathbb{Z}$ such that $3 \mid x$. If $3 \mid (x + y)$, then $3 \mid y$.

Proof Since $3 \mid x$, it follows that $x = 3a$, where $a \in \mathbb{Z}$. Assume that $3 \mid (x+y)$. Then $x + y = 3b$ for some integer b. Hence

$$y = 3b - x = 3b - 3a = 3(b-a).$$

Since $b-a$ is an integer, $3 \mid y$.

For the converse, assume that $3 \mid y$. Therefore, $y = 3c$ where $c \in \mathbb{Z}$. Then

$$x + y = 3a + 3c = 3(a+c).$$

Since $a+c$ is an integer, $3 \mid (x + y)$.

[2 marks]

(c) **Result** If x is an irrational number and y is a rational number, then $z = x - y$ is irrational.

Proof Assume to the contrary that $z = x - y$ is rational. Then $z = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Recall that $\sqrt{2}$ is irrational and let $x = \sqrt{2}$. As y is rational, $y = \frac{c}{d}$, where $c, d \in \mathbb{Z}$ and $d \neq 0$. Therefore $\sqrt{2} = x = y + z = \frac{c}{d} + \frac{a}{b} = \frac{ad+bc}{bd}$. Since $ad + bc$ and bd are integers with $bd = 0$, it follows that $\sqrt{2}$ is rational, producing a contradiction.

[2 marks]

Soalan 3

Setiap satu daripada bukti-bukti yang dicadangkan berikut mengandungi satu kesilapan. Nilaikan setiap bukti ini dengan menjelaskan kesilapan tersebut.

(a) ***Keputusan*** *Biar* $x, y \in \mathbb{Z}$ *dan biar a dan b integer ganjil. Jika* $ax + by$ *adalah genap, maka x dan y mempunyai pariti yang sama.*

Bukti *Andaikan x dan y mempunyai pariti yang berlainan. Maka* $x = 2p$ *dan* $y = 2q + 1$ *bagi suatu integer p dan q. Memandangkan a dan b adalah integer ganjil,* $a = 2r + 1$ *dan* $b = 2s + 1$ *bagi suatu integer p dan q. Maka*

$$\begin{aligned} ax + by &= (2r+1)(2p) + (2s+1)(2q+1) \\ &= 4pr + 2p + 4qs + 2s + 2q + 1 \\ &= 2(2pr + p + 2qs + s + q) + 1. \end{aligned}$$

Memandangkan $2pr + p + 2qs + s + q$ *adalah integer,* $ax + by$ *adalah ganjil.*

[2 markah]

(b) ***Keputusan*** *Biar* $x, y \in \square$ *sedemikian hingga 3 | x. Jika 3 | (x + y), maka 3 | y.*

Bukti *Memandangkan 3|x, maka* $x = 3a$ *yang mana* $a \in \square$ *. Andaikan* $3|(x+y)$. *Maka* $x + y = 3b$ *bagi suatu integer b. Oleh yang demikian*

$$y = 3b - x = 3b - 3a = 3(b-a).$$

Memandangkan $b-a$ *adalah suatu integer, 3| y.*
Sebaliknya, andaikan 3 | y. Maka, $y = 3c$ *yang mana* $c \in \square$ *. Oleh itu,*

$$x + y = 3a + 3c = 3(a+c).$$

Memandangkan a+c adalah integer, 3 | (x + y).

[2 markah]

(c) ***Keputusan*** *Jika x adalah suatu nombor tak nisbah dan y adalah suatu nombor nisbah, maka* $z = x - y$ *adalah nombor tak nisbah.*

Bukti *Andaikan yang sebaliknya bahawa* $z = x - y$ *adalah nombor nisbah. Maka* $z = \frac{a}{b}$, *dengan* $a, b \in \square$ *dan* $b \neq 0$. *Ingat bahawa* $\sqrt{2}$ *adalah nombor tak nisbah dan biar* $x = \sqrt{2}$. *Memandangkan y adalah nombor nisbah,* $y = \frac{c}{d}$, *dengan* $c, d \in \square$ *dan* $d \neq 0$. *Maka* $\sqrt{2} = x = y + z = \frac{c}{d} + \frac{a}{b} = \frac{ad+bc}{bd}$. *Memandangkan* $ad + bc$ *dan bd adalah integer dengan* $bd \neq 0$, *ini diikuti dengan* $\sqrt{2}$ *adalah nombor nisbah, satu percanggahan berlaku.*

[2 markah]

Question 4

Write the statement of the result for each of the following proofs:

(a) **Proof** Let $x = a + bi \in \square$. Consider $\overline{x} = a - bi \in \square$. Then

$$x\overline{x} = (a+bi)(a-bi) = a^2 + b^2 \in \square.$$

[2 marks]

(b) **Proof** Assume that $B \not\subset A$. Then there must be some element $x \in B$ such that $x \notin A$. Since $x \in B$, it follows that $x \in A \cup B$. However, since $x \notin A$, we have $A \cup B \neq A$. Conversely assume that $B \subseteq A$. Then it is clear by definition that $A \cup B = A$.

[2 marks]

(c) **Proof** Consider the function $f : \square \to \square$ defined by $f(x) = 3x - 5$. For every $y \in \square$, let $x = \frac{y+5}{3} \in \square$. Thus $f(x) = f\left(\frac{y+5}{3}\right) = 3\left(\frac{y+5}{3}\right) - 5 = y$, that is, f is onto.

[2 marks]

Soalan 4

Berikan pernyataan bagi setiap pembuktian yang berikut:

(a) ***Bukti*** *Biar* $x = a + bi \in \square$. *Pertimbangkan* $\overline{x} = a - bi \in \square$. *Maka*

$$x\overline{x} = (a+bi)(a-bi) = a^2 + b^2 \in \square .$$

[2 markah]

(b) ***Bukti*** *Andaikan bahawa* $B \not\subset A$. *Maka wujud suatu unsur* $x \in B$ *sedemikian hingga* $x \notin A$. *Memandangkan* $x \in B$, *maka* $x \in A \cup B$. *Namun memandangkan* $x \notin A$, *diperoleh* $A \cup B \neq A$. *Sebaliknya andaikan bahawa* $B \subseteq A$. *Maka adalah jelas daripada takrif bahawa* $A \cup B = A$.

[2 markah]

(c) ***Bukti*** *Pertimbangkan fungsi* $f : \square \to \square$ *yang ditakrif sebagai* $f(x) = 3x - 5$.

Untuk setiap $y \in \square$, *biar* $x = \frac{y+5}{3} \in \square$. *Maka* $f(x) = f\left(\frac{y+5}{3}\right) = 3\left(\frac{y+5}{3}\right) - 5 = y$,

iaitu, f adalah keseluruhan.

[2 markah]

Question 5

Prove that for every odd integer *n*, $4n^2 + n + 1$ is even.

[3 marks]

Soalan 5

Buktikan bahawa untuk setiap integer ganjil n, $4n^2 + n + 1$ *adalah genap.*

[3 markah]

Question 6

Let *A* and *B* be sets. Prove that if $A \cap B \neq A$, then $A \not\subset B$.

[3 marks]

Soalan 6

Biar A dan B adalah set. Buktikan bahawa jika $A \cap B \neq A$, *maka* $A \not\subset B$.

[3 markah]

Question 7

Let n be an integer. Then the remainder of n^2 when divide by 4 is either 0 or 1.

[7 marks]

Soalan 7

Biar n ialah suatu integer. Maka baki bagi n^2 apabila dibahagi dengan 4 adalah sama ada 0 atau 1.

[7 markah]

Question 8

Show that for every real number ε greater than zero, there exist a real number δ such that $\delta > 3\varepsilon$.

[3 marks]

Soalan 8

Tunjukkan bahawa bagi setiap nombor nyata ε yang lebih besar daripada sifar, wujud suatu nombor nyata δ sedemikian hingga $\delta > 3\varepsilon$.

[3 markah]

Question 9

Let x be a nonzero real number. Prove by contradiction that if $x + \frac{4}{x} < 4$, then $x < 0$.

[5 marks]

Soalan 9

Biar x ialah suatu nombor nyata bukan sifar. Buktikan dengan percanggahan bahawa jika $x + \frac{4}{x} < 4$, maka $x < 0$.

[5 markah]

Question 10

Let $a, b \in \square$. Use mathematical induction to prove that, for every natural number n,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}.$$

[5 marks]

Soalan 10

Biar $a, b \in \square$. Guna aruhan matematik untuk membuktikan bahawa bagi setiap nombor tabii n,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}.$$

[5 markah]

Question 11

Let $x, y \in \mathbb{R}$. Prove that if $x < 0$, then $x^3 - x^2y \le x^2y - xy^2$. Include your backward scratch work as part of your answer.

[4 marks]

Soalan 11

Biar $x, y \in \mathbb{R}$. *Buktikan bahawa jika* $x < 0$, *maka* $x^3 - x^2y \le x^2y - xy^2$. *Masukkan kerja-kerja kebelakang anda sebagai sebahagian daripada jawapan.*

[4 markah]

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