First Semester Examination
2017/2018 Academic Session

January 2018

MST562 - Stochastic Processes
[Proses Stokastik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **NINE (9)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN (9)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions: Answer **all nine (9)** questions.

*[**Arahan:** Jawab **semua sembilan (9)** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]

Question 1

To cater to the growing customer demand, Pak Ali's Cendol Stall has started a drive-thru service. The number of customers who arrived at the drive-thru counter last week is in accordance with a Poisson process having rate 50 per hour. The amount of money the customers spent for each order is exponentially distributed with mean RM10, independent of the total number of customers who arrived at the counter.

(i) What is the expected time until the tenth customer arrives?

(ii) Determine the mean and variance of the amount of money spent by the customers from the counter in 4 hours.

[20 marks]

Soalan 1

Bagi memenuhi permintaan pelanggan yang semakin meningkat, Gerai Cendol Pak Ali telah memulakan perkhidmatan jualan secara pandu-lalu. Bilangan pelanggan yang tiba di kaunter pandu-lalu minggu lepas adalah mengikut suatu proses Poisson dengan kadar 50 orang per jam. Amaun wang yang dibelanjakan oleh pelanggan bagi setiap pesanan yang dibuat tertabur secara eksponen dengan min RM10, tidak bersandar dengan jumlah bilangan pelanggan yang tiba di kaunter.

(i) Apakah masa jangkaan sehingga pelanggan kesepuluh tiba?

(ii) Tentukan min dan varians bagi amaun wang yang dibelanjakan oleh pelanggan-pelanggan daripada kaunter tersebut dalam 4 jam.

[20 markah]

Question 2

A small bank in Bayan Lepas has two customer service counters, one for handling savings account and another for unit trust business. The savings account counter is served by Ahmad while the unit trust counter is served by Bakar. Suppose that the service time distributions of Ahmad and Bakar are independent and exponential with mean $1/\lambda_1$ and $1/\lambda_2$, respectively. Chandra and Daud arrive at the bank at the same time, Chandra for saving account business and Daud for unit trust business. When they arrive, the two counters have no customers and ready to serve.

(i) Show that the probability Chandra gets done first is $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

(ii) Now, suppose that Emran enters the bank for unit trust business and find that Chandra and Daud are being served. Emran will be served as soon as Daud is done. What is the probability that, of the three customers, Chandra is the last to leave the bank?

[30 marks]

Soalan 2

Sebuah bank kecil di Bayan Lepas mempunyai dua kaunter perkhidmatan pelanggan, satu bagi urusan akaun simpanan dan satu lagi bagi urusan unit amanah. Kaunter akaun simpanan diuruskan oleh Ahmad manakala kaunter unit amanah diurus oleh Bakar. Andaikan taburan-taburan masa layan Ahmad dan Bakar adalah tidak bersandar dan eksponen masing-masing dengan min $1/\lambda_1$ *dan* $1/\lambda_2$. *Chandra dan Daud tiba di bank pada masa yang sama, Chandra bagi urusan akaun simpanan dan Daud bagi urusan unit amanah. Apabila mereka tiba, kedua-dua kaunter tiada pelanggan dan bersedia untuk melayan.*

(i) *Tunjukkan bahawa kebarangkalian Chandra selesai urusan dahulu adalah* $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

(ii) *Sekarang, andaikan Emran memasuki bank tersebut untuk urusan unit amanah dan mendapati Chandra dan Daud sedang dilayan. Emran akan dilayan sebaik urusan Daud selesai. Apakah kebarangkalian bahawa antara ketiga-tiga pelanggan, Chandra adalah yang terakhir meninggalkan bank tersebut?*

[30 markah]

Question 3

Suppose $\{N(t), t > 0\}$ is a Poisson process of rate λ, where $N(t)$ represents the total number of arrivals that have occurred up to time *t*. Denote $N_1(s,t)$ as the number of arrivals in (s, t].

(i) Determine the joint distribution of $N(t)$ and $N(t + s)$ for $s > 0$.

(ii) Find $E\left[N(t).N(t+s)\right]$ for $s > 0$.

(iii) Find $E\left[N_1(t_1,t_3).N_1(t_2,t_4)\right]$ where $t_1 < t_2 < t_3 < t_4$.

[20 marks]

Soalan 3

Andaikan $\{N(t), t > 0\}$ *adalah suatu proses Poisson dengan kadar* λ, *yang mana N(t) mewakili bilangan ketibaan yang berlaku sehingga masa t. Tandakan* $N_1(s,t)$ *sebagai bilangan ketibaan dalam (s, t].*

(i) *Tentukan taburan tercantum bagi* $N(t)$ *dan* $N(t + s)$ *untuk* $s > 0$.

(ii) *Cari* $E\left[N(t).N(t+s)\right]$ *untuk* $s > 0$.

(iii) *Cari* $E\left[N_1(t_1,t_3).N_1(t_2,t_4)\right]$ *yang mana* $t_1 < t_2 < t_3 < t_4$.

[20 markah]

Question 4

Anthony runs a café that opens at 3:00 pm until 9:00 pm. From 3:00 to 6:00 pm, customers arrive on the average rate that seems to be constant at 20 customers per hour. The average rate then steadily increases from the initial average of 20 customers per hour at 6:00 pm and reaches a maximum of 40 customers per hour at 8:00 pm. However, the average arrival rate then drops steadily from 8:00 to 9:00 pm at which it has the value of 5 customers per hour at 9:00 pm.

(i) Anthony is concerned about the number of customers arriving at his café. Could you suggest a good probability model for him? State any assumptions that you made.

(ii) What is the expected number of arrivals between 4:00 to 7:00 pm?

(iii) What is the probability that there is no customer between 8:30 to 9:00pm?

[30 marks]

Soalan 4

Anthony menjalankan sebuah kafetaria yang dibuka pada pukul 3:00 petang hingga 9:00 malam. Dari 3:00 petang hingga 6:00 petang, pelanggan tiba pada kadar purata yang kelihatan tetap pada 20 pelanggan sejam. Kadar purata terus meningkat dari purata awal sebanyak 20 pelanggan sejam pada jam 6:00 petang dan mencapai maksimum 40 pelanggan sejam pada jam 8:00 malam. Walau bagaimanapun, kadar purata ketibaan kemudian jatuh secara tetap dari 8:00 hingga 9:00 malam di mana ia hanya mempunyai kadar purata 5 pelanggan sejam pada pukul 9:00 malam.

(i) Anthony mengambil berat tentang bilangan pelanggan yang tiba di kafetarianya. Bolehkah anda mencadangkan suatu model kebarangkalian yang baik untuk beliau? Nyatakan sebarang andaian yang anda buat.

(ii) Apakah jangkaan bilangan ketibaan antara pukul 4:00 petang ke 7:00 petang?

(iii) Apakah kebarangkalian tiada pelanggan antara 8:30 malam hingga 9:00 malam?

[30 markah]

Question 5

Let X_t, $t = 0, 1, 2, \ldots$ be a Markov chain on a finite state space $\boldsymbol{S} = \{1, 2, \ldots, n\}$.

(i) State the Markov property clearly.

(ii) By using the Markov property, show that

$$P\left(X_4 = j \middle| X_0 = i\right) = \sum_{k=1}^{n} p_{ik}^{(2)} p_{kj}^{(2)} ,$$

where $p_{ij}^{(2)} = P\left(X_{t+2} = j \middle| X_t = i\right),\ i, j \in S$ are the 2-step transition probabilities.

[15 marks]

Soalan 5

Andaikan X_t, $t = 0, 1, 2, \ldots$ ialah suata rantai Markov pada ruang keadaan terhingga $\boldsymbol{S} = \{1, 2, \ldots, n\}$.

(i) Nyatakan ciri Markov dengan jelas.

(ii) Dengan menggunakan ciri Markov, tunjukkan bahawa

$$P\left(X_4 = j \mid X_0 = i\right) = \sum_{k=1}^{n} p_{ik}^{(2)} p_{kj}^{(2)} ,$$

dengan $p_{ij}^{(2)} = P\left(X_{t+2} = j \mid X_t = i\right)$, $i, j \in S$ sebagai kebarangkalian peralihan 2-langkah.

[15 markah]

Question 6

A Markov chain X_t, $t = 0, 1, 2, \ldots$ on state space $\boldsymbol{S} = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \end{bmatrix}$$

with the 2-step transition probability matrix

$$\boldsymbol{P}^2 = \begin{bmatrix} 0.04 & 0.10 & 0.16 & 0.34 & 0.36 \\ 0.00 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0.00 & 0.00 & 0.09 & 0.45 & 0.46 \\ 0.00 & 0.00 & 0.00 & 0.52 & 0.48 \\ 0.00 & 0.00 & 0.00 & 0.48 & 0.52 \end{bmatrix}$$

(i) Find all closed classes. Which classes are irreducible? Determine which states are recurrent and which are transient.

(ii) If the initial probability distribution is $p_1 = p_2 = p_3 = 0.2$, $p_4 = 0.3$, $p_5 = 0.1$, compute $P\{ X_2 = 4 \text{ or } 5 \}$.

[25 marks]

Soalan 6

*Suatu rantai Markov X_t, t = 0, 1, 2, … pada ruang keadaan **S** = {1, 2, 3, 4, 5} mempunyai matriks kebarangkalian peralihan*

$$\boldsymbol{P} = \begin{Vmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \end{Vmatrix}$$

dengan matriks kebarangkalian peralihan 2-langkah

$$\boldsymbol{P}^2 = \begin{Vmatrix} 0.04 & 0.10 & 0.16 & 0.34 & 0.36 \\ 0.00 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0.00 & 0.00 & 0.09 & 0.45 & 0.46 \\ 0.00 & 0.00 & 0.00 & 0.52 & 0.48 \\ 0.00 & 0.00 & 0.00 & 0.48 & 0.52 \end{Vmatrix}$$

(i) *Dapatkan semua kelas tertutup. Kelas yang manakah tidak terturunkan? Tentukan keadaan-keadaan yang berulang dan keadaan-keadaan yang fana.*

(ii) *Jika taburan kebarangkalian awal ialah* $p_1 = p_2 = p_3 = 0.2,\ p_4 = 0.3,\ p_5 = 0.1$, *hitung* $P\{X_2 = 4 \text{ } atau \text{ } 5\}$.

[25 markah]

Question 7

Consider a branching process where $\{X_n,\ n \geq 0\}$ denotes the population size of the *n*th generation. The number of offspring produced by an individual is a random variable *Z* with mean, μ.

(i) Show that $E[X_n] = \mu^n E[X_0]$.

(ii) Suppose Z has a binomial distribution with parameters *N* = 3 and *p* = 1/2. Starting with $X_0 = 1$, find p_0, the probability that the population will eventually die out.

[20 marks]

Soalan 7

Pertimbangkan suatu proses bercabang dengan $\{X_n,\ n \geq 0\}$ mewakili saiz populasi generasi ke-n. Bilangan anak yang dihasilkan oleh seorang individu ialah suatu pembolehubah rawak Z dengan min, μ.

(i) *Tunjukkan bahawa* $E[X_n] = \mu^n E[X_0]$.

(ii) *Andaikan Z mempunyai taburan binomial dengan parameter N = 3 dan p = 1/2. Bermula dengan $X_0 = 1$, dapatkan p_0, kebarangkalian populasi tersebut akan pupus.*

[20 markah]

Question 8

Customers arrive at a one window drive-through bank according to a Poisson process with mean arrival rate of 10 per hour. The average service time per customer is exponential at 5 minutes. There are 3 spaces in front of the window, including the space occupied by the car in service. Other cars can wait outside these 3 spaces.

(i) What is the probability that an arriving customer can enter one of the 3 spaces in front of the window?

(ii) How long is an arriving customer expected to wait before starting service?

[25 marks]

Soalan 8

Pelanggan-pelanggan tiba di sebuah bank pandu-lalu bertingkap satu menurut suatu proses Poisson dengan kadar ketibaan purata 10 setiap jam. Masa layanan purata bagi setiap pelanggan tertabur secara eksponen pada 5 minit. Terdapat 3 ruang di depan tingkap termasuk ruang yang diisi oleh kereta dalam layanan. Kereta-kereta lain boleh menunggu di luar 3 ruang tersebut.

(i) *Apakah kebarangkalian bahawa seorang pelanggan yang tiba dapat memasuki salah satu daripada 3 ruang di depan tingkap?*

(ii) *Berapa lamakah seorang pelanggan yang tiba dijangka menunggu sebelum dimulakan layanan?*

[25 markah]

Question 9

Potential customers arrive at a single-server bank in accordance with a Poisson process having rate $\lambda = 2$. A customer will only enter the bank if the server is free when he arrives. The amount of time spent in the bank by an entering customer is a random variable having a distribution *G* with mean 2. Suppose that the amounts deposited in the bank by successive customers are independent random variables uniformly distributed over [5, 10].

(i) What proportion of potential customers actually enter the bank?

(ii) What is the rate at which deposits accumulate?

[15 marks]

Soalan 9

Pelanggan-pelanggan berpotensi tiba di sebuah bank satu-pelayan mengikut suatu proses Poisson dengan kadar $\lambda = 2$. Seorang pelanggan hanya akan memasuki bank jika pelayan bank tidak sibuk apabila ia tiba. Amaun masa yang dihabiskan oleh seorang pelanggan yang memasuki bank ialah suatu pembolehubah rawak yang mempunyai taburan G dengan min 2. Andaikan amaun wang yang disimpan di dalam bank oleh pelanggan-pelanggan yang berturutan ialah suatu pembolehubah rawak yang tertabur secara seragam pada [5, 10].

(i) *Apakah kadaran pelanggan-pelanggan berpotensi yang benar-benar memasuki bank?*

(ii) *Apakah kadar pengumpulan wang simpanan?*

[15 markah]

APPENDIX

1. If X is distributed as Poisson with parameter $\lambda > 0$, then

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0,1,2,\ldots$$

2. If X is distributed as geometric with parameter p, $0 < p < 1$, then

$$P(X = x) = p(1-p)^{x-1} \quad ; \quad x = 1,2,\ldots$$

3. If X is distributed as Binomial with parameter p, $0 < p < 1$, then

$$P(X = x) = \binom{n}{x} p^{x} q^{n-x} \quad ; \quad x = 0,1,2,\ldots,n$$

4. If X is distributed as exponential with parameter $\lambda > 0$, then

$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

5. If X is distributed as Gamma with parameter $\alpha > 0$ and $\beta > 0$ then

$$f(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1} e^{-\beta x} \quad ; \quad x > 0$$

6. If X is distributed as normal with parameter μ and $\alpha^2 > 0$ then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad ; \quad -\infty < x < \infty$$

7. Formula of geometric series

$$\sum_{k=0}^{\infty} ar^{k} = \frac{a}{1-r} \quad ; \quad |r| < 1$$

8. For an arbitrary event A and for any random variable Y,

$$\Pr\{E\} = E\left[\Pr\{A|Y\}\right]$$

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