First Semester Examination
Academic Session 2017/2018

January 2018

MST561 - Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **TWELVE (12)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **DUA BELAS (12)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions: Answer **all five (5)** questions.

*[**Arahan:** Jawab **semua lima (5)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

Question 1

(a) Given the joint probability density function $f_{X,Y}(x, y) = 2e^{-(x+y)}, 0 < x < y < \infty$. Find

(i) $P(Y < 1 | X < 1)$

(ii) $P(Y < 1 | X = 1)$

(iii) $f_{Y|x}(y)$

[30 marks]

(b) Let X be a positive continuous random variable with distribution function F and density function f. Find the formula for the density function g for the random variable $Y = \dfrac{1}{1 + X}$.

[20 marks]

(c) Let X and Y have joint density function $f(x, y) = 2e^{-(x+y)} I_{(0,y)}(x) I_{(x,\infty)}(y)$. Find the joint density function $S = X$ and $T = X + Y$, marginal density function S and marginal density function T.

[30 marks]

(d) Let X be a random variable having probability density function $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$, $-\infty < x < \infty$. If $Y = X^2$, find the expected value of Y.

[20 marks]

Soalan 1

(a) *Diberi fungsi ketumpatan kebarangkalian tercantum*
$f_{X,Y}(x,y) = 2e^{-(x+y)}, 0 < x < y < \infty.$
Cari

(i) $P(Y < 1 \mid X < 1)$

(ii) $P(Y < 1 \mid X = 1)$

(iii) $f_{Y|x}(y)$

[30 markah]

(b) *Biarkan X suatu pembolehubah rawak selanjar positif dengan fungsi taburan F dan fungsi ketumpatan f. Cari formula bagi fungsi ketumpatan g untuk pembolehubah rawak* $Y = \dfrac{1}{1+X}$.

[20 markah]

(c) *Biarkan X dan Y mempunyai fungsi ketumpatan kebarangkalian tercantum*
$f(x,y) = 2e^{-(x+y)} I_{(0,y)}(x)\, I_{(x,\infty)}(y).$
Cari fungsi ketumpatan tercantum bagi S = X dan T = X + Y, fungsi ketumpatan sut S dan fungsi ketumpatan sut T.

[30 markah]

(d) *Biarkan X suatu pembolehubah rawak dengan fungsi ketumpatan kebarangkalian* $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$, $-\infty < x < \infty$. *Jika* $Y = X^2$, *cari nilai jangkaan bagi Y.*

[20 markah]

Question 2

(a) Let $Y_1, Y_2, ..., Y_n$ be a random sample from the exponential probability density function $f_Y(y) = e^{-y}, y > 0$. What is the smallest n for which $P(Y_{\min} < 0.2) > 0.9$?

[30 marks]

(b) Differentiate the moment-generating function for the geometric random variable and verify the expressions given for $E(X) = (1-p)/p$ and $\mathrm{Var}(X) = (1-p)/p^2$.

[20 marks]

(c) (i) Let U have an F distribution with parameters r and s. Using the related theorem, prove that $\frac{1}{U}$ have an F distribution with parameters s and r.

(ii) Let $T = \frac{W}{\sqrt{V/r}}$, where the independent variables W and V follow the standard normal distribution and the chi-square distribution with r degrees of freedom, respectively. Using the related theorem, show that T^2 has an F distribution with parameters 1 and r.

[30 marks]

(d) Let the probability mass function of Y_n be $P(Y_n = n) = 1$. Show that Y_n does not have a limiting distribution.

[20 marks]

Soalan 2

(a) *Biarkan* $Y_1, Y_2, ..., Y_n$ *sebagai sampel rawak dari fungsi ketumpatan kebarangkalian eksponen* $f_Y(y) = e^{-y}, y > 0$. *Apakah nilai n yang terkecil bagi* $P(Y_{\min} < 0.2) > 0.9$?

[30 markah]

(b) *Bezakan fungsi penjana momen bagi pembolehubah rawak geometri dan tunjukkan ungkapan yang diberi bagi E(X) = (1-p)/p dan Var(X) = (1-p)/p*^{2}.

[20 markah]

(c) *(i)* *Biarkan U mempunyai taburan F dengan parameter r dan s. Dengan menggunakan teorem yang berkaitan, buktikan bahawa* $\frac{1}{U}$ *mempunyai taburan F dengan parameter s dan r.*

(ii) *Biarkan* $T = \frac{W}{\sqrt{V/r}}$, *yang mana pembolehubah tak bersandar W dan V, masing-masing mengikuti taburan normal piawai dan taburan khi kuasa dua dengan r darjah kebebasan. Dengan menggunakan teorem yang berkaitan, tunjukkan bahawa* T^2 *mempunyai taburan F dengan parameter 1 dan r.*

[30 markah]

(d) *Biarkan fungsi jisim kebarangkalian bagi* Y_n *sebagai* $P(Y_n = n)$ *= 1. Tunjukkan bahawa* Y_n *tidak mempunyai taburan penghad.*

[20 markah]

Question 3

(a) Let X_1, X_2, …, X_n be a random sample of size n from the Poisson distribution $p_x(k,\lambda) = \dfrac{\lambda^k e^{\lambda}}{k!}, k = 0,1,2,\ldots$. Show that $\hat{\lambda} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ is an efficient estimator for λ.

[30 marks]

(b) Show that $\hat{\sigma}^2 = \sum_{i=1}^{n} Y_i^2$ is sufficient for σ^2 if Y_1, Y_2, …, Y_n is a random sample from a normal pdf with $\mu = 0$.

[20 marks]

(c) Let Y_1, Y_2, …, Y_n be a random sample of size n from the probability density function $f_Y(y;\theta) = \theta y^{\theta-1},\ 0 \le y \le 1$. Show that $W = \prod_{i=1}^{n} Y_i$ is a sufficient estimator for θ. Is the maximum likelihood estimator of θ a function of W?

[30 marks]

(d) If $X_1, X_2, \ldots, X_n$ is a random sample from the Bernoulli distribution (p), $0 < p < 1$ and $\overline{X}_n$ is the sample mean, prove that $\overline{X}_n \xrightarrow{P} p$.

[20 marks]

Soalan 3

(a) *Biarkan X_1, X_2, …, X_n sebagai sampel rawak bersaiz n dari taburan Poisson* $p_x(k,\lambda)=\dfrac{\lambda^k e^{\lambda}}{k!}, k=0,1,2,\ldots$ *.Tunjukkan bahawa* $\hat{\lambda}=\dfrac{1}{n}\sum_{i=1}^{n}X_i$ *adalah penganggar cekap bagi* λ.

[30 markah]

(b) *Tunjukkan bahawa* $\hat{\sigma}^2=\sum_{i=1}^{n}Y_i^2$ *adalah cukup bagi* σ^2 *jika Y_1, Y_2, …, Y_n adalah sampel rawak daripada taburan normal yang mempunyai fungsi ketumpatan kebarangkalian dengan $\mu = 0$.*

[20 markah]

(c) *Biarkan Y_1, Y_2, …, Y_n adalah sampel rawak bersaiz n daripada fungsi ketumpatan kebarangkalian* $f_Y(y;\theta)=\theta y^{\theta-1},\ 0\le y\le 1$. *Tunjukkan bahawa* $W=\prod_{i=1}^{n}Y_i$ *adalah penganggar cukup bagi θ. Adakah penganggar kebolehjadian maksimum bagi θ suatu fungsi W?*

[30 markah]

(d) *Jika* $X_1, X_2, \ldots, X_n$ *sampel rawak daripada taburan Bernoulli (p), 0 < p < 1 dan* $\overline{X}_n$ *ialah min sampel, buktikan bahawa* $\overline{X}_n \xrightarrow{P} p$.

[20 markah]

Question 4

(a) Let X be a random variable from the Poisson distribution with parameter θ and assume that the prior distribution of the parameter Θ has a gamma distribution with parameters α and λ which are known. Find the posterior distribution of Θ given $X = x$ and hence write the mean of the posterior distribution of Θ.

[40 marks]

(b) Suppose the random variables Y_1, Y_2, … ,Y_n denote the number of successes or failures (1 or 0) in each of n independent trials, where p = P(success occurs at a given trial) is an unknown parameter such that $p_{Y_i}(k; p) = p^k(1- p)^{1-k}$, k=0,1; $0 < p < 1$.

If $Y = Y_1 + Y_2 + \ldots + Y_n$ = total number of successes and define $\hat{p} = \dfrac{Y}{n}$.

(i) Show that $\hat{p}$ is an unbiased estimator for p.

(ii) Calculate the variance for $\hat{p}$.

(iii) Compute the Cramer-Rao lower bound for $p_{Y_i}(k; p)$

(iv) Compare the results in (ii) and (iii) and discuss.

[40 marks]

(c) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a probability distribution with density function $f_X(x;\theta) = \theta x^{\theta-1}$, $0 \le x \le 1$. If a and b are two number such that $0 < a < b < 1$, construct a 100γ% confidence interval for θ using the pivotal quantity method.

[20 marks]

Soalan 4

(a) *Biarkan X sebagai suatu pembolehubah rawak dari taburan Poisson dengan parameter θ dan andaikan taburan prior bagi parameter Θ mempunyai taburan gama dengan parameter α dan λ yang diketahui. Cari taburan posterior bagi Θ diberi X = x dan seterusnya tulis min bagi taburan posterior untuk Θ.*

[40 markah]

(b) *Andaikan pembolehubah rawak* $Y_1, Y_2, \ldots, Y_n$ *adalah bilangan kejayaan atau kegagalan (1 atau 0) dalam setiap n percubaan yang tak bersandar, yang mana p = P(kejayaan berlaku pada suatu percubaan yang diberi) adalah suatu parameter yang tidak diketahui supaya* $p_{Y_i}(k; p) = p^k(1- p)^{1-k}, \quad k=0,1; \quad 0 < p < 1.$

Jika $Y = Y_1 + Y_2 + \ldots + Y_n$ *=jumlah bilangan kejayaan dan takrifkan* $\hat{p} = \dfrac{Y}{n}$ **.**

(i) *Tunjukkan bahawa* $\hat{p}$ *adalah suatu penganggar saksama bagi p.*

(ii) *Kira varians bagi* $\hat{p}$**.**

(iii) *Kira batas bawah Cramer-Rao bagi* $p_{Y_i}(k; p)$

(iv) *Bandingkan keputusan dalam (ii) dan (iii) dan bincang.*

[40 markah]

(c) *Biarkan* $X_1, X_2, \ldots, X_n$ *sebagai suatu sampel rawak bersaiz n dari taburan kebarangkalian dengan fungsi ketumpatan* $f_X(x;\theta) = \theta x^{\theta-1}, \; 0 \le x \le 1.$ *Jika a dan b adalah dua nombor supaya 0 < a < b < 1, bina suatu selang keyakinan 100γ% bagi θ dengan menggunakan kaedah kuantiti pangsian.*

[20 markah]

Question 5

(a) Assume that $\overline{X}$ denotes the sample mean for a random sample of size n from a $N(\mu, 9)$ distribution. Find the value of n so that $\left(\overline{X} - 1, \overline{X} + 1\right)$ is a 95% confidence interval for μ.

[30 marks]

(b) Assume that $X_1, X_2, \ldots, X_{10}$ is a random sample of size 10 from a $N(0, \theta)$ distribution, where $\theta > 0$. For testing $H_0 : \theta = 1$ vs. $H_1 : \theta > 1$, the following critical region is used:

$$C = \left\{ \left(x_1, x_2, \ldots, x_{10}\right) : \sum_{i=1}^{10} x_i^2 \geq c \right\}.$$

Find c if the size of the critical region C is 0.05.

[30 marks]

(c) Assume that X is a single observation from a distribution having probability density function

$$f(x;\theta) = \left(1 + \theta\right) x^{\theta} I_{(0,1)}(x); \quad \theta > -1.$$

(i) Find the most powerful test of size-α for testing

$$H_0 : \theta = 0 \text{ vs. } H_1 : \theta = 1.$$

(ii) For testing $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$, the following test is used:

Reject H_0 if and only if $X \geq \dfrac{3}{4}$.

Find the power function and the size of the test.

[40 marks]

Soalan 5

(a) *Andaikan bahawa* $\overline{X}$ *menandakan min sampel bagi suatu sampel rawak bersaiz n daripada taburan N(*μ*, 9). Cari nilai n supaya* $\left(\overline{X}-1, \overline{X}+1\right)$ *adalah suatu selang keyakinan 95% bagi* μ*.*

[30 markah]

(b) *Andaikan bahawa* $X_1, X_2, ..., X_{10}$ *adalah suatu sampel rawak bersaiz 10 daripada taburan N(0,* θ*), yang mana* $\theta > 0$*. Bagi menguji* $H_0 : \theta = 1$ *lawan.* $H_1 : \theta > 1$*, rantau genting berikut digunakan:*

$$C = \left\{ \left(x_1, x_2, ..., x_{10}\right) : \sum_{i=1}^{10} x_i^2 \geq c \right\}.$$

Cari c jika saiz rantau genting C adalah 0.05.

[30 markah]

(c) *Andaikan bahawa X adalah suatu cerapan tunggal daripada taburan yang mempunyai fungsi ketumpatan kebarangkalian*

$$f(x;\theta) = \left(1+\theta\right) x^{\theta} I_{(0,1)}(x); \quad \theta > -1.$$

(i) *Cari ujian paling berkuasa saiz* α *bagi menguji*

$$H_0 : \theta = 0 \text{ lawan } H_1 : \theta = 1.$$

(ii) *Bagi menguji* $H_0 : \theta \leq 0$ *lawan.* $H_1 : \theta > 0$*, ujian berikut digunakan :*

$$\text{Tolak } H_0 \text{ jika dan hanya jika } X \geq \frac{3}{4}.$$

Cari fungsi kuasa dan saiz ujian ini.

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x) = \frac{1}{N} I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q + pe^t\right)^n$
Geometri	$f(x) = pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}, \quad qe^t < 1$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}, \quad t \neq 0$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\{-(x-\mu)^2 / 2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t + (\sigma t)^2 / 2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}, \quad t < \lambda$
Gama	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha, \quad t < \lambda$
Khi Kuasa Dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}, \quad t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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