First Semester Examination
2017/2018 Academic Session

January 2018

MAT517 - Computational Linear Algebra
[Aljabar Linear Pengkomputeran]

Duration : 3 hours
(Masa : 3 jam)

Please check that this examination paper consists of **TEN (10)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEPULUH (10)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **all ten (10)** questions.

*[**Arahan :** Jawab **semua sepuluh (10)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

Question 1

The sequence of matrices below is resulted from the Gaussian elimination procedure with partial pivoting, performed on matrix $\mathbf{A}$. Write down explicitly the permutation and elementary matrices involved in each step.

$$\underset{\mathbf{A}}{\begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & -2 \\ 2 & 5 & 4 \end{pmatrix}} \to \begin{pmatrix} 2 & 5 & 4 \\ 1 & 2 & -2 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & -\frac{1}{2} & -4 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 1 \\ 0 & -\frac{1}{2} & -4 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{23}{6} \end{pmatrix}.$$

[15 marks]

Soalan 1

Jujukan matriks di bawah dihasilkan daripada prosedur penghapusan Gauss dengan pemangsian separa, dilaksanakan terhadap matriks $\mathbf{A}$. *Tulis secara jelas matriks-matriks putaran dan asas yang terlibat dalam setiap langkah.*

$$\underset{\mathbf{A}}{\begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & -2 \\ 2 & 5 & 4 \end{pmatrix}} \to \begin{pmatrix} 2 & 5 & 4 \\ 1 & 2 & -2 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & -\frac{1}{2} & -4 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 1 \\ 0 & -\frac{1}{2} & -4 \end{pmatrix} \to \begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{23}{6} \end{pmatrix}.$$

[15 markah]

Question 2

Consider

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix}.$$

Write down the Householder matrix needed to create zeroes in positions $k+1, k+2, \ldots, n$ of vector $\mathbf{x}$.

[15 marks]

Soalan 2

Pertimbangkan

$$\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix}.$$

Tulis matriks Householder yang diperlukan untuk menghasilkan sifar pada kedudukan $k+1, k+2, \ldots, n$ *vektor* $\mathbf{x}$.

[15 markah]

Question 3

The pseudocode below describes an algorithm for LU factorization without pivoting.

For steps $i=1,2,\ldots,n-1$
 For $k=i+1,\ldots,n$ (Row operations)
  Set $l_{ki}=a_{ki}^{(i)} / a_{ii}^{(i)}$ (provided that $a_{ii}^{(i)} \neq 0$)
  For $j=i+1,\ldots,n$ (New row k)
   Set $a_{kj}^{(i+1)}=a_{kj}^{(i)}-l_{ki}a_{ij}^{(i)}$
  End for loop
 End for loop
End for loop

Determine the cost of the algorithm to leading order in n .

[15 marks]

Soalan 3

Pseudokod di bawah menghuraikan satu algoritma pemfaktoran LU tanpa pemangsian.

For steps $i = 1, 2, \ldots, n-1$
    For $k = i+1, \ldots, n$ *(Operasi baris)*
        Set $l_{ki} = a_{ki}^{(i)} \big/ a_{ii}^{(i)}$ *(jika diberi* $a_{ii}^{(i)} \neq 0$ *)*
        For $j = i+1, \ldots, n$ *(New row k)*
            Set $a_{kj}^{(i+1)} = a_{kj}^{(i)} - l_{ki} a_{ij}^{(i)}$
        End for loop
    End for loop
End for loop

Tentusahkan kos algoritma ini berdasarkan darjah tertinggi n *.*

[15 markah]

Question 4

Let $\mathbf{A}$ be an $m \times n$ matrix of rank n with singular value decomposition $\mathbf{A} = \mathbf{U\Sigma V}^T$ with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. Let $\mathbf{\Sigma}^\dagger$ denote the $n \times n$ matrix

$$\mathbf{\Sigma}^\dagger = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n} \end{pmatrix},$$

and define $\mathbf{A}^\dagger = \mathbf{V\Sigma}^\dagger \mathbf{U}^T$. Show that $\hat{\mathbf{x}} = \mathbf{A}^\dagger \mathbf{b}$ satisfies $\mathbf{A}^T \mathbf{A x} = \mathbf{A}^T \mathbf{b}$.

[15 marks]

Soalan 4

Biar $\mathbf{A}$ *menjadi suatu matriks* $m \times n$ *berpangkat* n *dengan penghuraian nilai singular* $\mathbf{A} = \mathbf{U\Sigma V}^T$ *dengan nilai-nilai singular* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *. Biar* $\mathbf{\Sigma}^\dagger$ *melambangkan matriks* $n \times n$

$$\mathbf{\Sigma}^\dagger = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n} \end{pmatrix},$$

dan definisikan $\mathbf{A}^\dagger = \mathbf{V\Sigma}^\dagger\mathbf{U}^T$. *Tunjukkan bahawa* $\hat{\mathbf{x}} = \mathbf{A}^\dagger\mathbf{b}$ *memuaskan persamaan* $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

[15 markah]

Question 5

Let

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 4 & 5 \\ 0 & 2 \end{pmatrix}.$$

Write down the Givens matrices needed to transform $\mathbf{A}$ into an upper triangular matrix

[20 marks]

Soalan 5

Biar

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 4 & 5 \\ 0 & 2 \end{pmatrix}.$$

Tulis matriks-matriks Givens yang diperlukan untuk mentransformasikan $\mathbf{A}$ *kepada matriks segitiga atas.*

[20 markah]

Question 6

Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -3 \\ 10 \\ 3 \\ 6 \end{pmatrix}.$$

(a) Use Householder reflections to reduce $\mathbf{A}$ to the form

$$\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \times & \times \\ 0 & \times \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and apply the same transformations to $\mathbf{b}$.

(b) Use the results from part (a) to find the least squares solution to $\mathbf{Ax} = \mathbf{b}$.

[20 marks]

Soalan 6

Diberi

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -3 \\ 10 \\ 3 \\ 6 \end{pmatrix}.$$

(a) Guna pantulan-pantulan Householder untuk menurunkan $\mathbf{A}$ *kepada bentuk*

$$\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \times & \times \\ 0 & \times \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

dan aplikasikan transformasi yang sama terhadap $\mathbf{b}$.

(b) Guna keputusan daripada bahagian (a) untuk mencari penyelesaian kuasa dua terkecil kepada $\mathbf{Ax} = \mathbf{b}$.

[20 markah]

Question 7

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ is the singular value decomposition of $\mathbf{A}$. Justify the following

(a) the diagonal entries of Σ are the square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$;

(b) the columns of $\mathbf{V}$ are orthonormal eigenvectors of $\mathbf{A}^T\mathbf{A}$

[20 marks]

Soalan 7

Biar $\mathbf{A}$ *suatu matriks* $m \times n$ *dan* $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ *ialah penghuraian nilai singular* $\mathbf{A}$.
Justifikasikan yang berikut:

(a) pemasukan pepenjuru Σ *adalah punca kuasa dua nilai-nilai eigen* $\mathbf{A}^T\mathbf{A}$ *;*

(b) lajur-lajur $\mathbf{V}$ *adalah vektor eigen ortonormal* $\mathbf{A}^T\mathbf{A}$

[20 markah]

Question 8

Matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix},$$

has singular value decomposition (SVD)

$$\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

(a) What is the dimension of the column space of $\mathbf{A}$?

(b) Use the SVD to calculate

(i) $cond_2(\mathbf{A})$;

(ii) $\|\mathbf{A}\|_2$.

(iii) $\|\mathbf{A}\|_F$.

[20 marks]

Soalan 8

Matriks

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix},$$

mempunyai penghuraian nilai singular (SVD)

$$\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

(a) *Apakah dimensi ruang lajur* $\mathbf{A}$ *?*

(b) *Guna SVD untuk mengira*

(i) $cond_2(\mathbf{A})$ *;*

(ii) $\|\mathbf{A}\|_2$.

(iii) $\|\mathbf{A}\|_F$.

[20 markah]

Question 9

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_2 & 0 & 0 & 0 \\ b_2 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & b_4 & 0 \\ 0 & 0 & b_4 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix},$$

where a_1, a_2, a_3, a_4, a_5, b_2 and b_4 are nonzero.

(a) Name ONE eigenvalue of $\mathbf{A}$.

(b) Describe a simplified QR method for computing the rest of the eigenvalues of $\mathbf{A}$ in which the method takes into account the special structure of $\mathbf{A}$.

(c) Perform two complete iterations of QR algorithm without shift on the matrix

$$\mathbf{B} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Write down the approximation of the eigenvalues of $\mathbf{B}$.

[30 marks]

Soalan 9

Pertimbangkan matriks

$$\mathbf{A} = \begin{pmatrix} a_1 & b_2 & 0 & 0 & 0 \\ b_2 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & b_4 & 0 \\ 0 & 0 & b_4 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix},$$

di mana a_1, a_2, a_3, a_4, a_5, b_2 *dan* b_4 *adalah bukan sifar.*

(a) *Namakan SATU nilai eigen* $\mathbf{A}$.

(b) *Huraikan kaedah QR mudah untuk mengira nilai-nilai eigen* $\mathbf{A}$ *yang lain di mana kaedah itu mengambil kira struktur istimewa* $\mathbf{A}$.

(c) *Laksanakan dua lelaran penuh algoritma QR tanpa anjakan ke atas matriks*

$$\mathbf{B} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Tulis penghampiran nilai-nilai eigen $\mathbf{B}$.

[30 markah]

Question 10

Let $\mathbf{A} \in R^{m \times n}$ with $m \geq n$ and $\mathbf{A}$ is full rank. Consider the factorization of $\mathbf{A}$ of the form

$$\mathbf{A} = \mathbf{QR} \ ,$$

where $\mathbf{Q} \in R^{m \times n}$ has orthonormal columns and $\mathbf{R} \in R^{n \times n}$ is upper triangular.

(a) Show that the columns of $\mathbf{Q}$ form an orthonormal basis for the column space of $\mathbf{A}$.

(b) Denote

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}^T , \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}^T ,$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the consecutive columns of $\mathbf{A}$ and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are the consecutive columns of $\mathbf{Q}$. Suppose a method is constructed to produce $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ from $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ such that

$$\mathbf{q}_{k+1} = \mathbf{v}_{k+1} \big/ \|\mathbf{v}_{k+1}\|_2 , \qquad k = 0, 1, \ldots, n-1,$$

where

$$\mathbf{v}_{k+1} = \mathbf{a}_{k+1} - \left(\alpha_{k+1}^{(1)} \mathbf{q}_1 + \alpha_{k+1}^{(2)} \mathbf{q}_2 + \cdots + \alpha_{k+1}^{(k)} \mathbf{q}_k \right),$$

with $\alpha_{k+1}^{(j)} = \mathbf{a}_{k+1}^T \mathbf{q}_j$, $\qquad j = 1, 2, \ldots, k.$

Discuss the numerical stability of this method when performed using floating point arithmetic. You may use specific examples or simplified diagrams to illustrate your arguments.

[30 marks]

Soalan 10

Biar $\mathbf{A} \in R^{m \times n}$ *dengan* $m \geq n$ *dan* $\mathbf{A}$ *berpangkat penuh. Pertimbangkan pemfaktoran* $\mathbf{A}$ *dalam bentuk*

$$\mathbf{A} = \mathbf{QR} \ ,$$

di mana $\mathbf{Q} \in R^{m \times n}$ *mempunyai lajur ortonormal dan* $\mathbf{R} \in R^{n \times n}$ *berbentuk segi tiga atas.*

(a) *Tunjukkan bahawa lajur-lajur* $\mathbf{Q}$ *membentuk asas ortonormal kepada ruang lajur* $\mathbf{A}$.

(b) *Lambangkan*

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}^T , \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}^T ,$$

di mana $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *adalah lajur* $\mathbf{A}$ *yang berturutan dan* $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ *adalah lajur* $\mathbf{Q}$ *yang berturutan. Seandainya suatu kaedah dibina untuk menghasilkan* $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ *daripada* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *sedemikian rupa sehinggakan*

$$\mathbf{q}_{k+1} = \mathbf{v}_{k+1} / \|\mathbf{v}_{k+1}\|_2 , \qquad k = 0, 1, \ldots, n-1 \ ,$$

di mana

$$\mathbf{v}_{k+1} = \mathbf{a}_{k+1} - \left(\alpha_{k+1}^{(1)} \mathbf{q}_1 + \alpha_{k+1}^{(2)} \mathbf{q}_2 + \cdots + \alpha_{k+1}^{(k)} \mathbf{q}_k \right),$$

dengan $\alpha_{k+1}^{(j)} = \mathbf{a}_{k+1}^T \mathbf{q}_j , \qquad j = 1, 2, \ldots, k.$

Bincangkan kestabilan berangka kaedah ini apabila dilaksanakan menggunakan aritmetik titik apungan. Anda boleh menggunakan contoh-contoh khusus dan gambarajah yang dipermudahkan untuk mengilustrasikan hujah anda.

[30 Markah]

-ooo00ooo-