UNIVERSITI SAINS MALAYSIA

First Semester Examination
2017/2018 Academic Year

January 2018

MAT 514 - Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer all **FOUR** (4) questions.

*[**Arahan**: Jawab semua **EMPAT** (4) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]

<u>Question 1</u>

(a) Discuss briefly the problem-solving technique in mathematical modelling of engineering problems.

(b) Define

(i) shear and normal stresses,

(ii) no-temperature-jump condition,

(iii) convective heat and mass transfer process.

[20 marks]

<u>Soalan 1</u>

(a) Bincangkan secara ringkas teknik yang digunakan untuk menyelesaikan masalah kejuruteraan dalam pemodelan matematik.

(b) Takrifkan

(i) tegasan-tegasan ricih dan normal,

(ii) syarat suhu tak terlompat,

(iii) proses pemindahan haba dan jisim berolak.

[20 markah]

<u>Question 2</u>

Consider a steady flow along a semi-infinite two dimensional surface of small curvature with a free-stream velocity u_∞, with x measured along the surface and y normal to the surface. Cut out an infinitesimal stationary control volume of unit depth within the boundary layer and consider the external forces acting on this control volume in the x direction and the momentum fluxes crossing the control surface as shown in Figure 1.

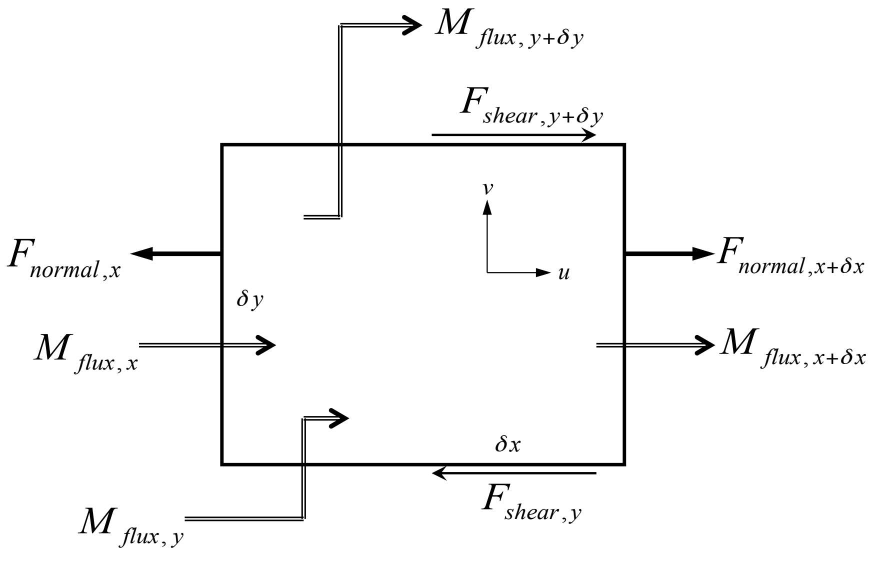


Figure 1: Control volume for development of the momentum equation

Table 1: Basic rates of momentum transfer

$$F_{normal,x} = \sigma_x \delta y$$

$$F_{shear,y} = \tau_{yx} \delta x$$

$$M_{flux,x+\delta x} = \left[G_x u + \frac{\partial}{\partial x}(G_x u)\delta x \right] \delta y$$

(a) State the momentum theorem.

(b) Explain the concept of momentum boundary layer. Then, state the boundary layer approximations for this problem.

(c) Based on Figure 1, Table 1 and applying 2(a), the two-dimensional boundary layer continuity equation ($\nabla \cdot \mathbf{G} = 0$) and the boundary layer approximation from 2(b), show that the momentum equation for the two-dimensional boundary layer flow is

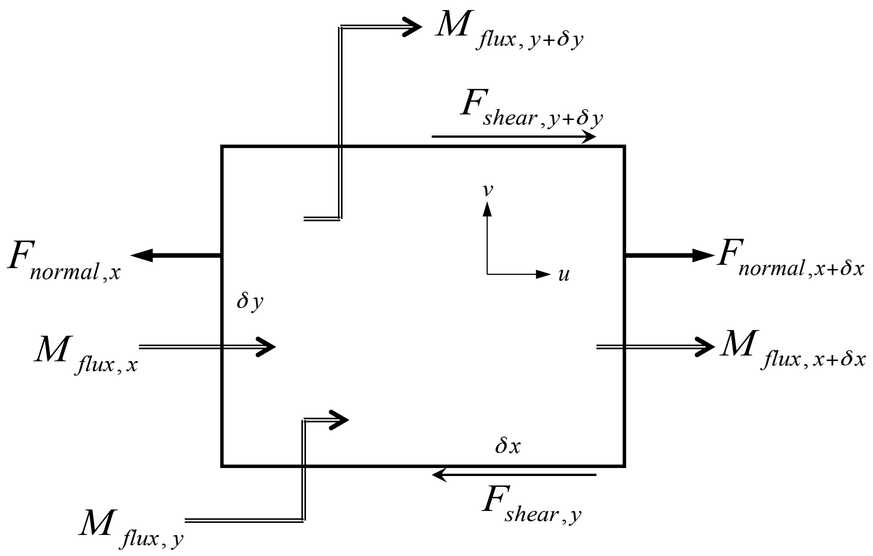
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right),$$

where u and v are the velocity components along x- and y-axes, respectively, $\mathbf{G}$ is the mass flux where $(G_x, G_y) = (\rho u, \rho v)$, ρ is the density of the fluid and μ is the dynamic viscosity of the fluid.

[25 marks]

<u>Soalan 2</u>

Pertimbangkan suatu aliran mantap terhadap permukaan berkelengkungan kecil separuh tak terhingga dengan halaju strim bebas u_∞, dengan x diukur di sepanjang permukaan dan y berserenjang terhadap permukaan. Satu unit kedalaman unsur isipadu kawalan pegun dalam lapisan sempadan dikeluarkan dan daya-daya luar yang bertindak ke atas unsur isipadu kawalan ini dalam arah x serta fluks-fluks momentum merentasi permukaan kawalan seperti dalam Rajah 1 dipertimbangkan.



Rajah 1: Isipadu kawalan untuk menerbitkan persamaan momentum

Jadual 1: Kadar-kadar asas pemindahan momentum

$$F_{normal,x} = \sigma_x \delta y$$

$$F_{shear,y} = \tau_{yx} \delta x$$

$$M_{flux,x+\delta x} = \left[G_x u + \frac{\partial}{\partial x}(G_x u)\delta x \right] \delta y$$

(a) *Nyatakan teorem momentum.*

(b) *Terangkan konsep lapisan sempadan momentum. Kemudian, nyatakan penghampiran-penghampiran lapisan sempadan bagi masalah ini.*

(c) *Berdasarkan Rajah 1, Jadual 1 dan menggunakan 2(a), persamaan keselanjaran lapisan sempadan dua dimensi* $(\nabla \cdot \mathbf{G} = 0)$ *dan penghampiran lapisan sempadan daripada 2(b), tunjukkan bahawa persamaan momentum bagi aliran lapisan sempadan dua dimensi adalah*

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right),$$

dengan u dan v masing-masing ialah komponen-komponen halaju sepanjang paksi x dan y, $\mathbf{G}$ *ialah fluks jisim dengan* $(G_x, G_y) = (\rho u, \rho v)$, ρ *ialah ketumpatan bendalir dan* μ *ialah kelikatan dinamik bendalir.*

[25 markah]

Question 3

Suppose that the momentum and energy equations with constant surface heat rate for axisymmetric flow in a circular tube of diameter d_s in x-direction is given by the following equations, respectively:

$$\begin{aligned} \frac{\partial}{\partial r}\left(\sqrt{r}\,\mu\frac{\partial u}{\partial r}\right) - \sqrt{r}\frac{dP}{dx} &= 0, \\ \frac{1}{u}\frac{\partial}{\partial r}\left(\sqrt{r}\,\alpha\frac{\partial T}{\partial r}\right) - \sqrt{r}\frac{dT_m}{dx} &= 0, \end{aligned}$$

where x and r are the axial and radial coordinates of circular tube, respectively, and u and T are the fully developed velocity and temperature profiles, respectively. The pressure P and the mass-averaged fluid temperature T_m are independent of r, and the dynamic viscosity μ and the thermal diffusivity α of the fluid are constants.

(a) Derive a fully developed temperature profile T by using the momentum and energy equations subject to the following boundary conditions:

$$\frac{\partial u}{\partial r} = \frac{\partial T}{\partial r} = \sqrt{rP} \text{ at the centerline of tube, and}$$

$$u = 0, \quad T = T_s \text{ at the surface of tube,}$$

where T_s is the fluid temperature at the surface of circular tube.

(b) What are the velocity and temperature gradients near the surface of the tube?

[25 marks]

Soalan 3

Andaikan bahawa persamaan-persamaan momentum dan tenaga dengan kadar haba permukaan malar bagi aliran simetri sepaksi dalam tiub bulat mendatar bergaris pusat d_s pada arah x masing-masing diberikan oleh persamaan-persamaan berikut:

$$\begin{aligned} \frac{\partial}{\partial r}\left(\sqrt{r}\,\mu\frac{\partial u}{\partial r}\right) - \sqrt{r}\frac{dP}{dx} &= 0, \\ \frac{1}{u}\frac{\partial}{\partial r}\left(\sqrt{r}\,\alpha\frac{\partial T}{\partial r}\right) - \sqrt{r}\frac{dT_m}{dx} &= 0, \end{aligned}$$

dengan x dan r masing-masing ialah koordinat paksian dan jejarian tiub bulat, dan u dan T masing-masing ialah profil-profil halaju dan suhu terbangun penuh. Tekanan P dan suhu jisim-purata bendalir T_m tidak bergantung terhadap r, dan kelikatan dinamik μ dan resapan terma α malar.

(a) *Terbitkan profil suhu terbangun penuh T dengan menggunakan persamaan-persamaan momentum dan tenaga tertakluk kepada syarat-syarat sempadan berikut:*

$$\frac{\partial u}{\partial r} = \frac{\partial T}{\partial r} = \sqrt{rP} \text{ pada garis tengah tiub, dan}$$

$$u = 0, \quad T = T_s \text{ pada permukaan tiub,}$$

dengan T_s ialah suhu bendalir pada permukaan tiub bulat.

(b) *Apakah kecerunan-kecerunan halaju dan suhu berhampiran permukaan tiub?*

[25 markah]

Question 4

Consider a steady mixed convection boundary layer flow and heat transfer near a semi-infinite vertical flat plate embedded in a saturated porous medium with a Newtonian fluid. x is the coordinate measured from the origin O along the plate and y is the coordinate normal to it. Under these assumptions, the governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{\zeta \rho_\infty g \beta}{\mu} \frac{\partial T}{\partial y}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$T = T_m, \quad k\frac{\partial T}{\partial y} = \rho\left[L + C_{ps}\left(T_m - T_0\right)\right] v \quad \text{at} \quad y = 0, \tag{4}$$
$$T = T_\infty, \quad u = U_\infty \quad \text{as} \quad y \to \infty,$$

where u and v are velocity components along x and y axes, respectively, T is the temperature of the fluid, T_m is the constant melting temperature of the material, T_∞ is the constant temperature of the liquid phase far from the plate, T_0 is the constant temperature of the solid phase far from the interface, ζ is the constant permeability of the porous medium, μ is the constant dynamic viscosity of the fluid, ρ is the density of the fluid, g is the acceleration due to gravity, P is the pressure, $\alpha = k/(\rho C_{pf})$ is the thermal diffusivity of the fluid, k is the thermal conductivity of the fluid, β is the coefficient of thermal expansion of the fluid, ρ_∞ is the constant density of the free stream, C_{ps} and C_{pf} are the specific heats of solid region and fluid, respectively, L is the plate length and U_∞ is the constant free stream velocity.

$$\left[\psi \text{ is a stream function which is defined as } u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}.\right]$$

(a) By using similarity variables of the following form

$$\eta = \sqrt{\frac{U_\infty}{2\alpha x}}\, y, \quad \psi = \sqrt{2\alpha U_\infty x}\, f(\eta), \quad T - T_m = [T_\infty - T_m]\ \theta(\eta),$$

$$Ra = \frac{\rho_\infty g \beta \zeta}{\alpha \mu}[T_\infty - T_m]x, \quad Pe = \frac{U_\infty}{\alpha}x,$$

show that the boundary layer equations (1) – (3) can be reduced to the following system of ordinary differential equations

$$f'' - \frac{Ra}{Pe}\ \theta' = 0, \tag{5}$$

$$\theta'' + f\theta' = 0, \tag{6}$$

with the boundary conditions (4) becoming

$$\theta(0) = 0, \quad f(0) + M\theta'(0) = 0, \quad \theta(\infty) = 1, \quad f'(\infty) = 1, \tag{7}$$

where primes ($'$) denotes differentiation with respect to η, and M is the dimensionless melting parameter defined by

$$M = \frac{C_{pf}(T_\infty - T_m)}{L + C_{ps}(T_m - T_0)}.$$

(b) Table 2 shows the values of skin friction coefficient $f''(0)$ obtained from the system of equations (5) and (6) subject to boundary conditions (7) for $M = 0$ with various values of parameter Ra/Pe. Find the values of surface heat transfer rate $[-\theta'(0)]$ for all considered parameter Ra/Pe in Table 2. Then, interpret the obtained results based on your calculation and this table.

Table 2: The values of skin friction coefficient $f''(0)$ for $M = 0$ with various values of parameter Ra/Pe.

M	Ra/Pe	$f''(0)$
0	-1	0.4696
	-1.1	0.4611
	-1.2	0.4302
	-1.3	0.3566

[30 marks]

Soalan 4

Pertimbangkan suatu aliran lapisan sempadan dan pemindahan haba olakan campuran yang mantap menghampiri sebuah plat rata menegak separuh tak terhingga dalam bahantara beron -gga tepu dengan bendalir Newtonan. x ialah koordinat yang diukur di sepanjang plat bermula dari asalan O dan y ialah koordinat berserenjang dengan plat. Berdasarkan andaian-andaian tersebut, persamaan-persamaan menakluk adalah

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial y}=\frac{\zeta\rho_\infty g\beta}{\mu}\frac{\partial T}{\partial y}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

tertakluk kepada syarat-syarat sempadan

$$T=T_m,\quad k\frac{\partial T}{\partial y}=\rho\left[L+C_{ps}\left(T_m-T_0\right)\right]v\quad at\quad y=0, \tag{4}$$
$$T=T_\infty,\quad u=U_\infty\quad as\quad y\to\infty,$$

dengan u dan v masing-masing ialah komponen-komponen halaju pada paksi x dan y, T ialah suhu bendalir, T_m ialah suhu bahan melebur malar, T_∞ ialah suhu fasa cecair berjauhan daripada plat malar, T_0 ialah suhu fasa pepejal berjauhan daripada plat malar, ζ ialah kebolehtelapan bahantara berongga malar, μ ialah kelikatan dinamik bendalir berolak, ρ ialah ketumpatan bendalir, g ialah pecutan graviti, P ialah tekanan, $\alpha=k/(\rho C_{pf})$ ialah resapan terma bendalir, k ialah kekonduksian terma bendalir, β ialah pekali pengembangan terma bendalir, ρ_∞ ialah ketumpatan strim bebas malar, C_{ps} dan C_{pf} masing-masing ialah haba tentu bagi kawasan pejal dan bendalir berolak, L ialah panjang plat dan U_∞ ialah halaju strim bebas malar.

$$\left[\psi \text{ adalah fungsi strim yang ditakrifkan sebagai } u=\frac{\partial\psi}{\partial y} \text{ dan } v=-\frac{\partial\psi}{\partial x}.\right]$$

(a) Dengan menggunakan pemboleh-pemboleh ubah keserupaan berbentuk berikut

$$\eta=\sqrt{\frac{U_\infty}{2\alpha x}}\,y,\quad \psi=\sqrt{2\alpha U_\infty x}\,f(\eta),\quad T-T_m=\left[T_\infty-T_m\right]\,\theta(\eta),$$
$$Ra=\frac{\rho_\infty g\beta\zeta}{\alpha\mu}\left[T_\infty-T_m\right]x,\quad Pe=\frac{U_\infty}{\alpha}x,$$

tunjukkan bahawa persamaan-persamaan lapisan sempadan (1) – (3) boleh diturunkan kepada sistem persamaan pembezaan biasa berikut

$$f''-\frac{Ra}{Pe}\,\theta'=0, \tag{5}$$

$$\theta''+f\theta'=0, \tag{6}$$

dengan syarat-syarat sempadan (4) menjadi

$$\theta(0) = 0, \quad f(0) + M\theta'(0) = 0, \quad \theta(\infty) = 1, \quad f'(\infty) = 1, \tag{7}$$

yang tandaan $(')$ *merujuk kepada pembezaan terhadap* η *dan* M *ialah parameter peleburan tak berdimensi yang ditakrifkan sebagai*

$$M = \frac{C_{pf}(T_\infty - T_m)}{L + C_{ps}(T_m - T_0)}.$$

(b) *Jadual 2 menunjukkan nilai-nilai pekali geseran kulit* $f''(0)$ *diperoleh daripada sistem persamaan (5) dan (6) tertakluk kepada syarat-syarat sempadan (7) untuk* $M = 0$ *dengan beberapa nilai parameter* Ra/Pe*. Dapatkan nilai kadar pemindahan haba permukaan* $[-\theta'(0)]$ *bagi semua parameter* Ra/Pe *yang dipertimbangkan dalam Jadual 2. Seterusnya, tafsirkan keputusan yang diperoleh berdasarkan pengiraan tersebut dan jadual ini.*

Jadual 2: Nilai-nilai pekali geseran kulit $f''(0)$ *untuk* $M = 0$ *dengan beberapa nilai parameter* Ra/Pe*.*

M	Ra/Pe	$f''(0)$
0	-1	0.4696
	-1.1	0.4611
	-1.2	0.4302
	-1.3	0.3566

[30 markah]

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