First Semester Examination
2017/2018 Academic Session

January 2018

MAT363 - Statistical Inference
(Pentaabiran Statistik)

Duration : 3 hours
(Masa : 3 jam)

Please check that this examination paper consists of **SEVEN (7)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **TUJUH (7)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions : Answer **all four (4)** questions.

*[**Arahan** : Jawab **semua empat (4)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

Question 1

(a) Find $P\left(A \cup \left(B^c \cup C^c\right)^c\right)$ if $P(A) = 2P(B \cap C) = 3P(A \cap B \cap C) = \frac{1}{2}$.

[30 marks]

(b) Proof that $E\left[X\,Y \middle| Y = y\right] = y\,E\left[X \middle| Y = y\right]$ for the discrete case.

[20 marks]

(c) Assume that $Y \sim U\,(0, 1)$. Find the function g so that $X = g(Y) \sim G(1, \lambda)$.

[30 marks]

(d) If X has a Poisson distribution such that $P(X = 0) = \frac{1}{2}$, find the probability mass function of X.

[20 marks]

Soalan 1

(a) Cari $P\left(A \cup \left(B^c \cup C^c\right)^c\right)$ *jika* $P(A) = 2P(B \cap C) = 3P(A \cap B \cap C) = \frac{1}{2}$.

[30 markah]

(b) Buktikan bahawa $E\left[X\,Y \middle| Y = y\right] = y\,E\left[X \middle| Y = y\right]$ *untuk kes diskret.*

[20 markah]

(c) Andaikan bahawa $Y \sim U\,(0, 1)$. *Cari fungsi g supaya* $X = g(Y) \sim G(1, \lambda)$.

[30 markah]

(d) Jika X mempunyai taburan Poisson sedemikian sehingga $P(X = 0) = \frac{1}{2}$, *cari fungsi jisim kebarangkalian untuk X.*

[20 markah]

Question 2

(a) Assume that X and Y are independent random variables having a common exponential distribution with parameter $\lambda = 1$. Find the probability density function of $Z = X + Y$.

[20 marks]

(b) Assume that $X_1, X_2, ..., X_n$ is a random sample from the exponential distribution with parameter λ.

(i) Find the exact distribution of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(ii) Find the approximate distribution of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ when n is large.

[40 marks]

(c) Let $X_1, X_2, ..., X_n, ...$ represent a sequence of random variables, where F_n and f_n are the cumulative distribution function and probability density function of X_n, respectively. Find the limiting distribution of X_n, where $X_n \sim N\left(\frac{1}{n}, \frac{1}{n^2}\right)$.

[40 marks]

Soalan 2

(a) Andaikan bahawa X dan Y adalah pembolehubah rawak tak bersandar yang mempunyai taburan eksponen sepunya dengan parameter $\lambda = 1$. Cari fungsi ketumpatan kebarangkalian untuk $Z = X + Y$.

[20 markah]

(b) Andaikan bahawa $X_1, X_2, ..., X_n$ ialah suatu sampel rawak daripada taburan eksponen dengan parameter λ.

(i) Cari taburan tepat untuk $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(ii) Cari taburan hampiran untuk $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ apabila n adalah besar.

[40 markah]

(c) Biarkan $X_1, X_2, ..., X_n, ...$ mewakili suatu jujukan pembolehubah rawak, yang mana F_n dan f_n masing-masing adalah fungsi taburan longgokan dan fungsi ketumpatan kebarangkalian untuk X_n. Cari taburan penghad untuk X_n, yang mana $X_n \sim N\left(\frac{1}{n}, \frac{1}{n^2}\right)$.

[40 markah]

Question 3

(a) Let $X_1, X_2, ..., X_n$ represent a random sample from a distribution with probability density function $f_\lambda(x) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha}$, $x > 0$ and α is known. Find the maximum likelihood estimator of λ.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ represent a random sample from an exponential distribution with probability density function $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x > 0$ and $\theta > 0$.

(i) Find *m* if *m* is the distribution median.

(ii) By using the result in (i), find the uniformly minimum variance of unbiased estimators (UMVUE) of $\frac{m}{\theta^2}$.

[40 marks]

(c) Suppose that a random sample $X_1, X_2, ..., X_n$ is taken from a normal $N(\mu, \sigma^2)$ distribution, where σ^2 is known.

(i) Find the Cramer-Rao's lower bound for the variance of an unbiased estimator of μ.

(ii) Is $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ an efficient estimator of μ?

[30 marks]

Soalan 3

(a) Biarkan $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan dengan fungsi ketumpatan kebarangkalian* $f_\lambda(x) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha}$, $x > 0$ *dan* α *adalah diketahui. Cari penganggar kebolehjadian maksimum untuk* λ.

[30 markah]

(b) Biarkan $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan eksponen dengan fungsi ketumpatan kebarangkalian* $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x > 0$ *dan* $\theta > 0$.

(i) Cari m jika m ialah median taburan.

(ii) Dengan menggunakan keputusan dalam (i), cari penganggar saksama bervarians minimum secara seragam (PSVMS) untuk $\frac{m}{\theta^2}$.

[40 markah]

(c) *Andaikan bahawa sampel rawak* $X_1, X_2, ..., X_n$ *diambil daripada taburan normal* $N\left(\mu, \sigma^2\right)$, *yang mana* σ^2 *adalah diketahui.*

(i) *Cari batas bawah Cramer-Rao untuk varians penganggar saksama* μ.

(ii) *Adakah* $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ *penganggar cekap untuk* μ?

[30 markah]

Question 4

(a) Let $X_1, X_2, ..., X_n$ represent a random sample from the normal, $N\left(\theta, \theta^2\right)$ distribution.

(i) Find a $100\gamma\%$ ($0 < \gamma < 1$) confidence interval for θ and the length of this confidence interval.

(ii) Find a $100\gamma\%$ ($0 < \gamma < 1$) confidence interval for θ^2 and the length of this confidence interval.

[40 marks]

(b) Let $X_1, X_2, ..., X_n$ denote a random sample with the probability density function $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, $\Theta = \{\theta : \theta > 0\}$. Find the generalized likelihood ratio test of size α to test $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$.

[30 marks]

(c) Let X has the probability mass function $f\left(x;\theta\right) = \theta^x \left(1-\theta\right)^{1-x}$, $x = 0,\ 1$; and zero elsewhere. We test $H_0 : \theta = \dfrac{1}{2}$ versus $H_1 : \theta < \dfrac{1}{2}$ by taking a random sample X_1, X_2, X_3, X_4 of size 4. Find a uniformly most powerful test to test these hypotheses.

[30 marks]

Soalan 4

(a) *Biarkan* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan normal,* $N\left(\theta, \theta^2\right)$.

(i) *Cari suatu selang keyakinan* 100γ % ($0 < \gamma < 1$) *bagi* θ *dan panjang selang keyakinan ini.*

(ii) *Cari suatu selang keyakinan* 100γ % ($0 < \gamma < 1$) *bagi* θ^2 *dan panjang selang keyakinan ini.*

[40 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *menandai suatu sampel rawak dengan fungsi ketumpatan kebarangkalian* $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, $\Theta = \{\theta : \theta > 0\}$. *Cari ujian nisbah kebolehjadian teritlak saiz* α *untuk menguji* $H_0 : \theta \le \theta_0$ *lawan* $H_1 : \theta > \theta_0$.

[30 markah]

(c) *Biarkan X mempunyai fungsi jisim kebarangkalian* $f\left(x;\theta\right) = \theta^x \left(1-\theta\right)^{1-x}$, x = 0, 1; *dan sifar di tempat lain. Kita menguji* $H_0 : \theta = \frac{1}{2}$ *lawan* $H_1 : \theta < \frac{1}{2}$ *dengan mengambil suatu sampel rawak* X_1, X_2, X_3, X_4 *saiz 4. Cari ujian paling berkuasa secara seragam untuk menguji hipotesis-hipotesis ini.*

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\dots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\dots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\dots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\dots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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