ON THE THEORY OF DIOPHANTINE APPROXIMATIONS

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The question on how the irrational numbers can be approximated by the rational numbers has long been raised by many Mathematicians. In 1963, Niven in his monograph had shown a sketch proof of several results related to this problem. Now, in this paper we attempt to write a more comprehensive proofs and also filling in the gaps left out by Niven [1963]. Given a real number θ, how closely can it be approximated by rational numbers?. To make this more precise, for any given positive ε is there a rational number $\frac{a}{b}$ within ε of θ, so that the inequality $\left|\theta - \frac{a}{b}\right| < \varepsilon$ is satisfied?. The answer is yes because the rational numbers are dense on real line. In fact, we proved that given any irrational number θ, there are infinitely many rational numbers $\frac{a}{b}$, where a and $b > 0$ are integers, such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{b^2}$. Although the exponent cannot be improved, this result can be strengthened by a constant factor. Specifically $\frac{1}{b^2}$ can be replaced by $\frac{1}{\sqrt{5}b^2}$ and no larger constant than $\sqrt{5}$ can be used. In addition to this, an attempt also has been made to improve this constant but not in generalized form. We then have put several restrictions to help improved the constant.

Key words: Diophantine approximation, continued fraction, irrational number, and rational number.

1. Introduction

To 15 decimal places, π is given by 3.141592653589793... For simple calculations, it is widely known that 22/7 = 3.142857... is a good approximation of π, valid to 2 decimal places. It is also true that 355/113 = 3.14159292... is accurate to 6 decimal places. For a relatively small denominator 113, we obtain accuracy up to a large number of decimal places. This kind of consideration is an example of the problem of Diophantine approximation: How close can irrational numbers are approximated by rational numbers. For instance, given any irrational numbers θ, how close can it be approximated by rational numbers $\frac{p}{q}$? Mathematically we can conclude this statement as follows, for any $\varepsilon > 0$, is there any rational number $\frac{p}{q}$ approximates the irrational number θ, so that the inequality $\left|\theta - \frac{p}{q}\right| < \varepsilon$ is satisfied and the distance between these two numbers is less than ε ?. This paper explores this question. We first introduce the most useful theorem in Diophantine approximation which is the Hurwitz Theorem. We give a detailed proof of the Hurwitz Theorem, which has filled the gaps to minimize complexity. We then give a precise approximation, which considers the extension results.

2. The Approximation Of Irrationals By The Rationals

In 1918, Hurwitz proved a useful result in approximating irrational numbers by rational numbers. Before we go further, some basic results are needed.

2.1. *Farey sequence.*

Given any positive number n, the Farey sequence F_n is a sequence ordered in size, of all rational fractions $\frac{a}{b}$ in lowest terms with $0 < b \le n$. For instance

$$F_8 = \cdots, -\frac{1}{7}, -\frac{1}{8}, 0, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{8}, \frac{2}{7}, \frac{1}{3}, \cdots$$

The following theorem mentions two properties of Farey sequence which is required in our discussion.

Theorem 2.1. *If* $\frac{a}{b}$ *and* $\frac{c}{d}$ *are two consecutive terms in* F_n, *then presuming* $\frac{a}{b}$ *to be a small and* $bc - ad = 1$. *If* θ *is any given irrational number and r is any positive integer, then for all n sufficiently large, the two fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$ *adjacent to* θ *in* F_n *have denominators larger than r, that is* $b > r$ *and* $d > r$.

Lemma 2.1. *There is no positive integers x and y which satisfy simultaneously the inequalities*

$$\frac{1}{xy} \ge \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \text{ and } \frac{1}{x(x+y)} \ge \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{(x+y)^2}\right). \tag{1}$$

Proof of Theorem 2.1 and Lemma 2.1 refer to Niven[1963].

2.2. *The Theorem of Hurwitz*

Theorem 2.2 (Hurwitz). *Given any irrational number* θ *there exist infinitely many rational numbers* $\frac{p}{q}$ *in lowest terms such that*

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}. \tag{2}$$

The value $\sqrt{5}$ *is the best constant. This inequality become false if* $\sqrt{5}$ *is replaced by any larger constant.*

***Proof*:**

Two parts will be proved here which are:

i) There exist infinitely many rational numbers $\frac{p}{q}$ in lowest terms such that the inequality $\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$ is satisfied.

ii) This inequality become false if $\sqrt{5}$ is replaced by any larger constant.

We start our proof with the first part. Let say $\frac{a}{b}$ and $\frac{c}{d}$ are two adjacent fractions in Farey sequence, F_n and θ is between of these two fractions with $b > 0$ and $d > 0$. So,

$$\frac{a}{b} < \theta < \frac{c}{d}.$$

Consider two cases which are either $\theta > \frac{(a+c)}{(b+d)}$ or $\theta < \frac{(a+c)}{(b+d)}$. In case one, prove that not all of the following inequalities

$\theta - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}$, $\theta - \frac{a+c}{b+d} \geq \frac{1}{\sqrt{5}(b+d)^2}$ and $\frac{c}{d} - \theta \geq \frac{1}{\sqrt{5}d^2}$ are satisfied. Add the first and the third inequality, then add the second and third inequality, we will get (1) with $x = b$ and $y = d$ (from Lemma 2.1). In case two, prove that not all of the three inequalities

$\theta - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}$, $\frac{a+c}{b+d} - \theta \geq \frac{1}{\sqrt{5}(b+d)^2}$, and $\frac{c}{d} - \theta \geq \frac{1}{\sqrt{5}d^2}$ hold. If we add the first and third, then add the second and third inequality we get (2.1) with $x = b$ and $y = d$. Hence, the inequality (2) holds if we replace $\frac{p}{q}$ by at least one of $\frac{a}{b}, \frac{c}{d}$ and $\frac{(a+c)}{(b+d)}$. Then we will prove there are infinitely many solutions $\frac{p}{q}$ which satisfy the inequality (2). We argue by contradiction. Suppose there are finitely many solutions to inequality (2), and let r denote the maximum denominator among these solutions. For sufficiently large n, Theorem 2.1 guarantees the consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ adjacent to θ in F_n have denominators greater than r. The solution of $\frac{p}{q}$ to (2) is one of the three forms $\frac{a}{b}, \frac{c}{d}$ or $\frac{(a+c)}{(b+d)}$. By definition of Farey sequences $\frac{a}{b}$ and $\frac{c}{d}$ is in the lowest terms. Also $\frac{(a+c)}{(b+d)}$ is in the lowest terms because $c(b+d) - d(a+c) = bc - ad = 1$. What will happen if $\sqrt{5}$ is replaced by any larger constant? This is impossible, because if $\sqrt{5}$ is replaced by any larger constant the result become false. There exist finitely many rational numbers $\frac{p}{q}$ in lowest terms that satisfy the inequality (2). This can be seen in the following argument. Now, define $\theta_0 = \frac{1+\sqrt{5}}{2}$ and $\theta_1 = \frac{1-\sqrt{5}}{2}$, such that $(x-\theta_0)(x-\theta_1) = x^2 - x - 1$. Hence, for any integers p and q, with $q > 0$, we find that

$$\left|\frac{p}{q} - \theta_0\right| \cdot \left|\frac{p}{q} - \theta_1\right| = \left|\left(\frac{p}{q}\right)^2 - \frac{p}{q} - 1\right| \neq 0, \text{ and}$$

$$\theta_1 = \theta_0 - \sqrt{5} \Rightarrow \left|\frac{p}{q} - \theta_0\right| \cdot \left|\frac{p}{q} - \theta_0 + \sqrt{5}\right| = \frac{\left|p^2 - pq - q^2\right|}{q^2} \geq \frac{1}{q^2}.$$

Using applications of the triangle inequality gives

$$\frac{1}{q^2} \le \left|\frac{p}{q} - \theta_0\right| \cdot \left\{\left|\frac{p}{q} - \theta_0\right| + \sqrt{5}\right\} \qquad (3)$$

For some $\beta > 0$, there exist infinitely many solutions $\frac{p_j}{q_j}$ where $j = 1,2,3,\dots$ such that $\left|\frac{p_j}{q_j} - \theta_0\right| < \frac{1}{\beta q_j^2}$. As $j \to \infty \Rightarrow q_j \to \infty$, from inequality (3), we found that $\frac{1}{q_j^2} < \frac{1}{\beta q_j^2}\left(\frac{1}{\beta q_j^2} + \sqrt{5}\right) \Rightarrow \beta < \frac{1}{\beta q_j^2} + \sqrt{5}$. When $j \to \infty \Rightarrow q_j \to \infty, \frac{1}{\beta q_j^2} \to 0$. Hence, the largest constant we can use is $\sqrt{5}$. If $\sqrt{5}$ is replaced by any larger constant, we can see that j actually is finite. So that there exist finite solutions $\frac{p_j}{q_j}$ and this contradict the hypothesis in the Theorem. Note that the exponent two on the q^2 in inequality (3) is the best value. If any real number which is larger than 2 is replaced, then the result become false. That means there exist finitely many solutions to inequality (3). These complete the proof of the theorem of Hurwitz. ∎

3. An Extension Result

Can the value $\sqrt{5}$ be replaced by any larger constant? The answer is yes. The value $\sqrt{5}$ can be replaced by any larger constant if we use certain constraint to the irrational numbers [Eggan(1961)].

3.1. *More Precise Approximation.*

The constant $\sqrt{5}$ can be replaced by any larger constant if the irrational number $\theta = \frac{1-\sqrt{5}}{2}$ is omitted from consideration. In [Nat] small changes in Cohn's proof was made a much stronger result was obtained. Consider $k \ge 1$ and let $F(k)$ be the set of all real numbers x such that $0 \le x \le 1$ and the continued fractions for x has no partial quotient greater than k and $F(0) = \phi$.

Theorem 3.1. *Let $k \ge 1$ and x be a real irrational number and not equivalent to the element in $F(k-1)$. Then there exists infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{\left(k^2 + 4\right)^{1/2} q^2} \quad . \qquad (4)$$

The constant $\dfrac{1}{\left(k^2+4\right)^{1/2}}$ *is the best possible. Given* $k \geq 1$ *and x be a real irrational number and not equivalent to the element in* $F(k-1)$. *We need to show that, there exists infinitely many rational numbers* $\dfrac{p}{q}$ *such that* $\left|x-\dfrac{p}{q}\right| < \dfrac{1}{\left(k^2+4\right)^{1/2} q^2}$ *and the constant* $\dfrac{1}{\left(k^2+4\right)^{1/2}}$ *is the best possible.*

Proof:

Let $\dfrac{p_n}{q_n}$ denote the nth convergent of the continued fractions $[a_0, a_1, a_2, \ldots]$ of x and $\theta_n = \left|q_n^2 x - p_n q_n\right|$, $\phi_n = \min(\theta_{n-1}, \theta_n, \theta_{n+1})$. Hence

$$\frac{\theta_n}{q_n^2} + \frac{\theta_{n+1}}{q_{n+1}^2} = \left|\frac{q_n^2 x - p_n q_n}{q_n^2}\right| + \left|\frac{q_{n+1}^2 x - p_{n+1} q_{n+1}}{q_{n+1}^2}\right| = \left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}}$$

$$\Rightarrow \quad \frac{\theta_n}{q_n^2} + \frac{\theta_{n+1}}{q_{n+1}^2} = \frac{1}{q_n q_{n+1}}$$

$$\Rightarrow \quad \frac{\theta_n q_{n+1}^2 + \theta_{n+1} q_n^2}{q_n^2 q_{n+1}^2} = \frac{1}{q_n q_{n+1}}$$

$$\Rightarrow \quad \frac{\theta_n q_{n+1}^2 + \theta_{n+1} q_n^2}{q_n q_{n+1}^2} = \frac{1}{q_n}$$

$$\Rightarrow \quad \frac{\theta_n q_{n+1}^2 + \theta_{n+1} q_n^2}{q_n^2} = \frac{q_{n+1}}{q_n}$$

$$\Rightarrow \quad \theta_n \left(\frac{q_{n+1}}{q_n}\right)^2 + \theta_{n+1} - \frac{q_{n+1}}{q_n} = 0 \qquad (5)$$

The equation (5) is a quadratic equation. So, we can get the solutions of this equation by using the formula

$$\frac{q_{n+1}}{q_n} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{1 - 4\theta_n \theta_{n+1}}}{2\theta_n},$$

and the solutions are

$$\frac{q_{n+1}}{q_n} = \frac{1 \pm \sqrt{1 - 4\theta_n \theta_{n+1}}}{2\theta_n}.$$

From our solution we can say that,

$$\frac{q_{n+1}}{q_n} \le \frac{1+\sqrt{1-4\theta_n\theta_{n+1}}}{2\theta_n}, \tag{6}$$

due to $\frac{1-\sqrt{1-4\theta_n\theta_{n+1}}}{2\theta_n} < \frac{1+\sqrt{1-4\theta_n\theta_{n+1}}}{2\theta_n}$ and $\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n}$.

Also $\frac{q_n}{q_{n-1}} = \frac{1 \pm \sqrt{1-4\theta_n\theta_{n-1}}}{2\theta_{n-1}} \Rightarrow \frac{q_{n-1}}{q_n} = \frac{2\theta_{n-1}}{1 \pm \sqrt{1-4\theta_n\theta_{n-1}}}$.

It is obvious that

$$\frac{2\theta_{n-1}}{1+\sqrt{1-4\theta_n\theta_{n-1}}} \le \frac{2\theta_{n-1}}{1-\sqrt{1-4\theta_n\theta_{n-1}}},$$

$$\Rightarrow a_{n+1} + \frac{q_{n-1}}{q_n} \ge a_{n+1} + \frac{2\theta_{n-1}}{1-\sqrt{1-4\theta_n\theta_{n-1}}}$$

$$\Rightarrow \frac{1+\sqrt{1-4\theta_n\theta_{n+1}}}{2\theta_n} \ge \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} \ge a_{n+1} + \frac{2\theta_{n-1}}{1-\sqrt{1-4\theta_n\theta_{n-1}}}.$$

In Theorem 3.1, given that $\phi_n = \min(\theta_{n-1}, \theta_n, \theta_{n+1})$, we have $2\phi_n a_{n+1} \le 2\theta_n a_{n+1}$, and it has already been shown above that $\frac{1+\sqrt{1-4\theta_n\theta_{n+1}}}{2\theta_n} \ge \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n}$, hence

$$1 + \left(1-4\theta_n\theta_{n+1}\right)^{1/2} \ge 2\theta_n a_{n+1} + 2\theta_n \frac{q_{n-1}}{q_n},$$

$$\Rightarrow 1 + \sqrt{1-4\theta_n\theta_{n+1}} - 2\theta_n \frac{q_{n-1}}{q_n} \ge 2\theta_n a_{n+1}. \Rightarrow \sqrt{1-4\theta_n\theta_{n+1}} + \sqrt{1-4\theta_n\theta_{n-1}} \ge 2\theta_n a_{n+1}.$$

Given $\phi_n = \min(\theta_{n-1}, \theta_n, \theta_{n+1})$, we have

$$\sqrt{1-4\theta_n\theta_{n+1}} + \sqrt{1-4\theta_n\theta_{n-1}} \ge 2\left(1-4\phi_n^2\right)^{1/2}$$

$$\Rightarrow 2\phi_n a_{n+1} \le 2\theta_n a_{n+1} \le \left(1-4\theta_n\theta_{n+1}\right)^{1/2} + \left(1-4\theta_n\theta_{n-1}\right)^{1/2} \le 2\left(1-4\phi_n^2\right)^{1/2}$$

$$\Rightarrow \phi_n a_{n+1} \le \left(1-4\phi_n^2\right)^{1/2}$$

$$\Rightarrow \left(\phi_n a_{n+1}\right)^2 \le \left(1-4\phi_n^2\right)$$

$$\Rightarrow \left(\phi_n a_{n+1}\right)^2 + 4\phi_n^2 \le 1$$

$$\Rightarrow \phi_n^2 (a_{n+1}^2 + 4) \le 1, \quad \Rightarrow \phi_n^2 \le \frac{1}{(a_{n+1}^2 + 4)}, \quad \Rightarrow \phi_n \le \frac{1}{(a_{n+1}^2 + 4)^{1/2}}.$$

There are two possible conditions which are $\phi_n = \frac{1}{(a_{n+1}^2 + 4)^{1/2}}$ and $\phi_n < \frac{1}{(a_{n+1}^2 + 4)^{1/2}}$. But, if $\phi_n = \frac{1}{(a_{n+1}^2 + 4)^{1/2}}$, implies $\theta_n = \theta_{n+1} = \phi_n$. From (6) we have $\frac{q_{n+1}}{q_n} \le \frac{1 \pm \sqrt{1 - 4\theta_n \theta_{n+1}}}{2\theta_n}$, (undefined). So, it is impracticable to have $\theta_n = \theta_{n+1} = \phi_n$.

Therefore for all values of n, $\phi_n < \frac{1}{(a_{n+1}^2 + 4)^{1/2}}$. If x is not equivalent to the element in $F(k-1)$, thus for infinitely many n, $a_{n+1} \ge k$ and the continued fractions for $x = \frac{\left(k^2 + 4\right)^{1/2} - k}{2}$ is $[0, k, k, k, \ldots]$. The smallest value of a_{n+1} is k, so that the distance between both two numbers is less than or equal to $\frac{1}{\left(k^2 + 4\right)^{1/2} q^2}$. Note that $\phi_n < \frac{1}{(a_{n+1}^2 + 4)^{1/2}} \Rightarrow \phi_n < \frac{1}{(k^2 + 4)^{1/2}}$. Let say $\phi_n = \theta_n$, hence

$$\phi_n = \left| q_n^2 x - p_n q_n \right| < \frac{1}{\left(k^2 + 4\right)^{1/2}}$$

$$= \left| x - \frac{p_n}{q_n} \right| < \frac{1}{\left(k^2 + 4\right)^{1/2} q^2}$$

for $n = 1, 2, 3, \ldots$ an d these implies $\phi = \left| x - \frac{p}{q} \right| < \frac{1}{\left(k^2 + 4\right)^{1/2} q^2}$.

For infinitely many solutions $\frac{p}{q}$, the largest distance between rational number $\frac{p}{q}$ and irrational number x is at most $\frac{1}{\left(k^2 + 4\right)^{1/2} q^2}$. For $k = 1$, this theorem gives Hurwitz's Theorem. For $k = 2$, the constant is $\frac{1}{\sqrt{8}}$, $k = 3$, the constant is $\frac{1}{\sqrt{13q}}$, for $k = 4$, the largest constant is $\frac{1}{\sqrt{20}}$ and so on.

4. Conclusion

The largest and the best constant can be chosen depends on the form of the irrational number. From our discussion above, the form of the irrational numbers are $\theta = \frac{\left(k^2 + 4\right)^{1/2} - k}{2} = [0, k, k, k, \ldots]$, for integer $k \ge 1$ and the largest constant for such irrational numbers are of form $\frac{1}{\left(k^2 + 4\right)^{1/2}}$, where $k \geq 1$. But what happen to other constants or other irrational numbers that is not in the mentioned form?. That is part of the future works. We expect the constant would change if we modify the form of the irrational numbers. That means it requires another form of irrational numbers to obtain different constants.

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