UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2007/2008

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MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Let $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, be the probability density function of X. Find the cumulative distribution and the probability density function of $Y = X^2$.
(Hint: Consider $P(X^2 \leq y)$ for two cases $0 \leq y < 1$ and $1 \leq y < 4$)

[20 marks]

(b) Let $f(x, y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, zero elsewhere, be the joint probability density function of X and Y. Then if $Z = X + Y$, find $P(Z \leq z)$, for $0 < z < \infty$. What is the probability density function of Z?

[30 marks]

(c) Find the variance of Y if the moment generating function is $M_Y(t) = e^{2t}/(1-t^2)$.

[20 marks]

(d) Let (X, Y) be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{2}, & 0 < x < y < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the conditional probability density of X given $Y = y$.

[30 marks]

2. (a) Cast a die a number of independent times until a six appear on the up side of the die.

(i) Find the probability mass function $p(x)$ of X, the number of casts needed to obtain the first six.

(ii) Show that $\sum_{x=1}^{\infty} p(x) = 1$.

(iii) Determine $P(X = 1, 3, 5, 7, \dots)$

[30 marks]

(b) If continuous random variables (X, Y) have joint density function $f(x,y) = 2$, $0 < x < y < 1$, and zero elsewhere, find the correlation coefficient for X and Y.

[20 marks]

(c) If $X_1, X_2, \dots, X_n$ is a random sample from the $N(\mu, 4\sigma^2)$ distribution, and $\overline{X}_m$ is defined as

$$\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i, \quad m \leq n,$$

find the distribution for each of the following statistics:

1. (a) *Biarkan* $f(x) = \frac{1}{3}$, $-1 < x < 2$, *sifar di tempat lain, adalah fungsi ketumpatan kebarangkalian bagi X. Cari taburan longgokan dan fungsi ketumpatan kebarangkalian bagi* $Y = X^2$.
(Petunjuk: Pertimbangkan $P(X^2 \leq y)$ *bagi dua kes* $0 \leq y < 1$ *dan* $1 \leq y < 4$*)*

[20 markah]

(b) *Biarkan* $f(x, y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, *sifar di tempat lain, adalah fungsi ketumpatan kebarangkalian tercantum bagi X dan Y. Kemudian jika* $Z = X + Y$, *cari* $P(Z \leq z)$, *bagi* $0 < z < \infty$. *Apakah fungsi ketumpatan kebarangkalian bagi Z?*

[30 markah]

(c) *Cari varians bagi Y jika fungsi penjana momen adalah* $M_Y(t) = e^{2t}/(1-t^2)$.

[20 markah]

(d) *Biarkan (X, Y) sebagai pembolehubah rawak selanjar dengan fungsi ketumpatan kebarangkalian tercantum*

$$f(x,y) = \begin{cases} \frac{1}{2}, & 0 < x < y < 2 \\ 0, & \text{di tempat lain} \end{cases}$$

Cari ketumpatan kebarangkalian bersyarat bagi X diberi $Y = y$.

[30 markah]

2. (a) *Lambung sebiji dadu sebilangan kali yang tak bersandar sehingga satu nombor enam muncul pada sebelah atas dadu tersebut.*

(i) *Cari fungsi jisim kebarangkalian, p(x) bagi X, bilangan kali yang diperlukan untuk memperoleh nombor enam yang pertama.*

(ii) *Tunjukkan bahawa* $\sum_{x=1}^{\infty} p(x) = 1$.

(iii) *Tentukan* $P(X = 1, 3, 5, 7, \dots)$

[30 markah]

(b) *Jika pembolehubah rawak selanjar (X, Y) mempunyai fungsi kebarangkalian tercantum* $f(x,y) = 2$, $0 < x < y < 1$, *dan sifar di tempat lain, cari pekali korelasi bagi X dan Y.*

[20 markah]

(c) *Jika* $X_1, X_2, \dots, X_n$ *merupakan sampel rawak daripada taburan* $N(\mu, 4\sigma^2)$, *dan* $\overline{X}_m$ *ditakrifkan sebagai,*

$$\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i, \quad m \leq n,$$

cari taburan bagi setiap statistik berikut:

(i) $m\overline{X}_m - n\overline{X}_n$

(ii) $(m-1)\overline{X}_m + (n-1)\overline{X}_n$

(iii) $\dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\sum_{i=1}^{m} \dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

[30 marks]

(d) Let $X_1, X_2, \ldots, X_n$ represent a random sample from a Be(θ) distribution, $0<\theta<1$, and $\overline{X}_n$ represents the corresponding sample mean. Show that $\overline{Y}_n = 1 - \overline{X}_n \xrightarrow{P} 1 - \theta$.

[20 marks]

3. (a) Let $X_1, X_2, \ldots, X_n$ represent a random sample from a Bernoulli distribution with parameter λ, $0 \le \lambda \le 1$. Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\lambda(1-\lambda)$.

[20 marks]

(b) Assume that $X_1, X_2, \ldots, X_n$ is a random sample of size n that has a distribution with density function

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta}, & 0 < x < 1 \\ 0 & , \text{ elsewhere} \end{cases}$$

(i) Is this distribution a member of the exponential family? Explain.

(ii) Find a uniformly minimum variance of unbiased estimator (UMVUE) for $\theta + 1$.

[40 marks]

(c) If $X_1, X_2, \ldots, X_n$ is a random sample from the gamma, G(3, α) distribution and the sample mean, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, find

(i) an approximate 100γ% confidence interval for α when n is large, where $0 < \gamma < 1$.

(ii) a 100γ% confidence interval for α when n is small.

[40 marks]

(i) $m\overline{X}_m - n\overline{X}_n$

(ii) $(m-1)\overline{X}_m + (n-1)\overline{X}_n$

(iii) $\dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\displaystyle\sum_{i=1}^{m} \frac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

[30 markah]

(d) *Biarkan* $X_1, X_2, \ldots, X_n$ *mewakili sampel rawak daripada taburan Be(θ), 0<θ<1, dan* $\overline{X}_n$ *mewakili min sampel yang sepadan. Tunjukkan bahawa* $\overline{Y}_n = 1 - \overline{X}_n \xrightarrow{P} 1 - \theta$.

[20 markah]

3. (a) *Biarkan* $X_1, X_2, \ldots, X_n$ *menandakan suatu sampel rawak daripada taburan Bernoulli dengan parameter* λ, $0 \le \lambda \le 1$. *Cari batas bawah Cramer-Rao bagi varians penganggar-penganggar saksama* $\lambda(1-\lambda)$.

[20 markah]

(b) *Andaikan* $X_1, X_2, \ldots, X_n$ *ialah suatu sampel rawak bersaiz n yang mempunyai taburan dengan fungsi ketumpatan*

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta}, & 0 < x < 1 \\ 0 & , \textit{di tempat lain} \end{cases}$$

(i) *Adakah taburan ini ahli bagi famili eksponen? Jelaskan.*

(ii) *Cari penganggar saksama bervarians minimum secara seragam (PSVMS) bagi* $\theta + 1$.

[40 markah]

(c) *Jika* $X_1, X_2, \ldots, X_n$ *suatu sampel rawak daripada taburan gama, G(3, α) dan min sampel,* $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, *cari*

(i) *suatu selang keyakinan hampiran 100γ% bagi α apabila n besar, yang mana* $0 < \gamma < 1$.

(ii) *suatu selang keyakinan 100γ% bagi α apabila n kecil.*

[40 markah]

4. (a) Assume that X is a single observation from a distribution having density function

$$f(x;\theta) = \theta x^{\theta-1} I_{(0,1)}(x);\ \ \theta > 0.$$

For testing $H_0 : \theta \le 1$ vs. $H_1 : \theta > 1$, the test that is used is reject H_0 if and only if $X \ge \frac{1}{2}$. Find the power function and the size of the test.

[30 marks]

(b) Let $X_1, X_2, \ldots, X_n$ represents a random sample of size n from a distribution with density function $f(x;\alpha) = \alpha^2 x e^{-\alpha x} I_{(0,\infty)}(x)$.

(i) Find the uniformly most powerful critical region for testing $H_0 : \alpha = 1$ vs. $H_1 : \alpha > 1$.

(ii) For testing $H_0 : \alpha = 1$ vs. $H_1 : \alpha \ne 1$, the following test is used: Reject H_0 if and only if $|\overline{X} - 2| \ge k$. Find an approximate value of k so that the size of the Type-I error is 0.05. Assume that n is sufficiently large so that the central limit theorem can be used to find an approximate value of k.

[40 marks]

(c) Assume that the single observation X follows a binomial, $b(n, \theta)$ distribution with probability mass function

$$f(x;\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x},\ x = 0, 1, 2, \ldots, n.$$

Find the generalized likelihood ratio test for testing $H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$.

[30 marks]

4. (a) *Andaikan X cerapan tunggal daripada taburan yang mempunyai fungsi ketumpatan*

$$f(x;\theta)=\theta x^{\theta-1}I_{(0,1)}(x);\ \ \theta>0.$$

Untuk menguji $H_0:\theta\leq 1$ *lawan* $H_1:\theta>1$, *ujian yang digunakan ialah tolak* H_0 *jika dan hanya jika* $X\geq\frac{1}{2}$. *Cari fungsi kuasa dan saiz ujian tersebut.*

[30 markah]

(b) *Biarkan* $X_1,X_2,\ldots,X_n$ *mewakili suatu sampel rawak saiz n daripada taburan dengan fungsi ketumpatan* $f(x;\alpha)=\alpha^2xe^{-\alpha x}I_{(0,\infty)}(x)$.

(i) *Cari rantau genting paling berkuasa secara seragam bagi menguji* $H_0:\alpha=1$ *lawan* $H_1:\alpha>1$.

(ii) *Untuk menguji* $H_0:\alpha=1$ *lawan* $H_1:\alpha\neq 1$, *ujian berikut digunakan: Tolak* H_0 *jika dan hanya jika* $|\overline{X}-2|\geq k$. *Cari nilai hampiran k supaya saiz ralat Jenis-I ialah 0.05. Andaikan bahawa n adalah cukup besar supaya teorem had memusat boleh digunakan untuk mencari nilai hampiran k.*

[40 markah]

(c) *Andaikan cerapan tunggal, X bertaburan binomial, b(n, θ) dengan fungsi jisim kebarangkalian*

$$f(x;\theta)=\binom{n}{x}\theta^x(1-\theta)^{n-x},\ \ x=0,1,2,\ldots,n.$$

Cari ujian nisbah kebolehjadian teritlak bagi menguji $H_0:\theta\leq\theta_0$ *lawan* $H_1:\theta>\theta_0$.

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N} I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x} I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!} I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}} exp\{-(x-\mu)^2/2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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