UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2007/2008

Jun 2008

MAT 202 – Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all three** [3] questions.

*[**Arahan** : Jawab **semua tiga** [3] soalan.]*

1. (a) State the completeness axiom. Show that if the supremum and infimum of a non-empty set exist, they must be unique.

   (b) State the Archimedean principle. Then show that for each positive real number x, there exists a positive integer n such that $nx > 1$.

   (c) For every pair of real numbers x and y with $x < y$, show that there is a rational number q such that $x < q < b$.

   (d) Let A, B and C be non-empty sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (e) Give the definition of a countable set. If A and B are countable sets, then show that the set $A \times B$ is also countable.
   [$A \times B = \{(a,b) : a \in A, b \in B\}$]

[100 marks]

2. (a) (i) Let $\{a_n\} = \left\{\dfrac{1+2n}{3+4n} : n \in \mathbb{N}\right\}$. Use the definition of limit to verify that $\lim\limits_{n\to\infty} \dfrac{1+2n}{3+4n} = \dfrac{1}{2}$.

   (ii) Let $\{a_n\}$ be a sequence of real numbers. Show that if the limit of $\{a_n\}$ exists, then it is unique.

   (b) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be a closed interval on $\mathbb{R}$. Given that $I_n \supset I_{n+1}$, prove that $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

   (c) Suppose the real sequences $\{a_n\}$ and $\{b_n\}$ converge to ℓ and m, respectively. Show that the sequence $\{a_n + b_n\}$ converges to $\ell + m$.

   (d) By using the definition of a Cauchy sequence, determine whether the sequence $\{(-1)^n\}$ is Cauchy.

[100 marks]

1. (a) *Nyatakan aksiom kelengkapan. Tunjukkan bahawa jika supremum dan infimum bagi set tak kosong wujud, ianya adalah unik.*

(b) *Nyatakan Prinsip Archimedes. Tunjukkan bahawa untuk setiap nombor nyata positif* x, *wujudnya integer posotif* n *supaya* $nx > 1$.

(c) *Untuk setiap pasangan nombor nyata* x *dan* y *dimana* $x < y$, *tunjukkan bahawa wujudnya suatu nombor nisbah* q *supaya* $x < q < b$.

(d) *Biarkan* A, B *dan* C *adalah set tak kosong. Tunjukkan bahawa* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(e) *Berikan takrifan bagi set terbilangkan. Jika* A *dan* B *adalah set terbilangkan, buktikan bahawa set* $A \times B = \{(a,b) : a \in A, b \in B\}$ *adalah terbilangkan.*

[100 markah]

2. (a) (i) *Biarkan* $\{a_n\} = \left\{\dfrac{1+2n}{3+4n} : n \in \mathbb{N}\right\}$. *Gunakan takrifan bagi jujukan menumpu untuk menentusahkan bahawa* $\displaystyle had_{n\to\infty} \frac{1+2n}{3+4n} = \frac{1}{2}$.

(ii) *Biarkan* $\{a_n\}$ *adalah jujukan nombor nyata. Buktikan bahawa jika had* $\{a_n\}$ *wujud, maka ia adalah unik.*

(b) *Untuk setiap* $n \in \mathbb{N}$, *biarkan* $I_n = [a_n, b_n]$ *adalah selang tertutup pada* $\mathbb{R}$. *Diberi* $I_n \supset I_{n+1}$, *buktikan bahawa* $\displaystyle\bigcap_{n=1}^{\infty} I_n \neq \phi$.

(c) *Andaikan jujukan nombor nyata* $\{a_n\}$ *dan* $\{b_n\}$ *masing-masing menumpu ke* ℓ *dan* m. *Buktikan bahawa jujukan* $\{a_n + b_n\}$ *menumpu ke* $\ell + m$.

(d) *Dengan menggunakan takrifan jujukan Cauchy, tentukan samada jujukan* $\{(-1)^n\}$ *adalah Cauchy.*

[100 markah]

3. (a) Consider the set $A=(-1,10)-Q$,
[Q is the set of all rational numbers].

Find the interior points, accumulation/limit points and isolated points of A. Furthermore, determine whether A is open or closed or neither.

(b) Consider the set $A=(0,1)$, and the collection

$$\tau=\left\{\left(-\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},2\right)\right\}.$$

(i) Determine whether τ is an open covering for A. Give your reason.
(ii) Use the definition of compactness to show that A is not compact.

(c) Let G_k be open sets on $\mathbb{R}$, $k=1,2,3,\ldots$ Show that $\bigcap_{k=1}^{n} G_k$ is open.

(d) Given a function $f(x)=\frac{1}{x^2}$. Discuss the uniform continuity of f when

(i) $D_f=[c,\infty)$ for any fixed $c>0$,

(ii) $D_f=[0,\infty)$.

[D_f = domain of $f(x)$]

(e) Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on $\mathbb{R}$ by $f_n(x)=x^n$. Determine for what values of x on the given set, the sequence converges point wise. Next, find the point wise limit of the sequence.

[100 marks]

3. (a) *Pertimbangkan set* $A=(-1,10)-Q$,
[Q *adalah set semua nombor nisbah].*

Cari titik pedalaman, titik had dan titik terpencil bagi A. *Kemudian, tentukan samada* A *terbuka atau tertutup atau bukan keduanya.*

(b) *Pertimbangkan set* $A=(0,1)$, *dan pungutan*

$$\tau=\left\{\left(-\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},2\right)\right\}.$$

(i) *Tentukan samada* τ *adalah tudung terbuka bagi* A. *Berikan alasan.*
(ii) *Gunakan takrifan kepadatan (dalam sebutan tudung terbuka) untuk menunjukkan* A *adalah bukan padat.*

(c) *Andaikan* G_k *adalah set terbuka pada* $\mathbb{R}$, $k=1,2,3,\ldots$ *Tunjukkan bahawa* $\bigcap_{k=1}^{n} G_k$ *adalah terbuka.*

(d) *Diberikan fungsi* $f(x)=\dfrac{1}{x^2}$. *Bincangkan keselanjaran secara seragam fungsi bila*
(i) $D_f=[c,\infty)$ *untuk sebarang* $c>0$,
(ii) $D_f=[o,\infty)$.

[D_f = *domain bagi* $f(x)$]

(e) *Suatu jujukan fungsi* $\{f_n\}_{n=1}^{\infty}$ *ditakrfikan pada* $\mathbb{R}$ *oleh* $f_n(x)=x^n$. *Tentukan nilai-nilai* x *pada set dimana jujukan menumpu secara titik demi titik. Kemudian, cari titik penumpuan bagi jujukan tersebut.*

[100 markah]

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