UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2007/2008

Jun 2008

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

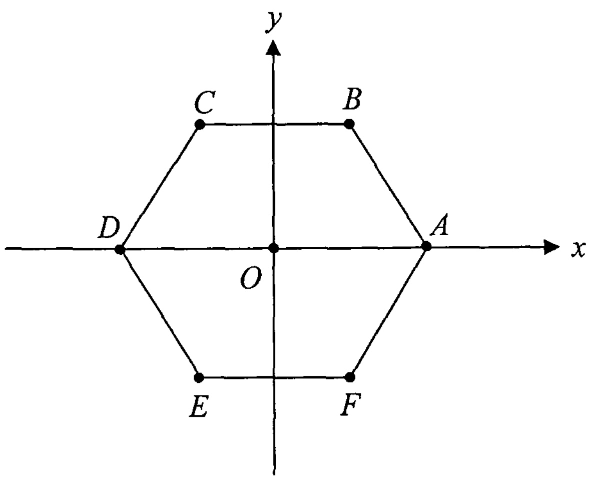
Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan**: Jawab **semua lima** [5] soalan.]*

1. (a) In the figure below, A, B, C, D, E, and F are vertices of a regular hexagon centered at the origin. Express $\overrightarrow{BC}+\overrightarrow{DE}+\overrightarrow{FA}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ where $\boldsymbol{a}=\overrightarrow{OA}$ and $\boldsymbol{b}=\overrightarrow{OB}$.



(b) (i) Given that vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal when $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{0}$. Prove that $|\boldsymbol{u}+\boldsymbol{v}|=|\boldsymbol{u}-\boldsymbol{v}|$ if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

(ii) Compute the angle between the vectors $\boldsymbol{u}=(2,1,-2)$ and $\boldsymbol{v}=(1,1,1)$.

(c) (i) Find the volume of the parallelepiped determined by the vectors $\boldsymbol{p}=(6,3,-1)$, $\boldsymbol{q}=(0,1,2)$ and $\boldsymbol{r}=(4,-2,5)$.

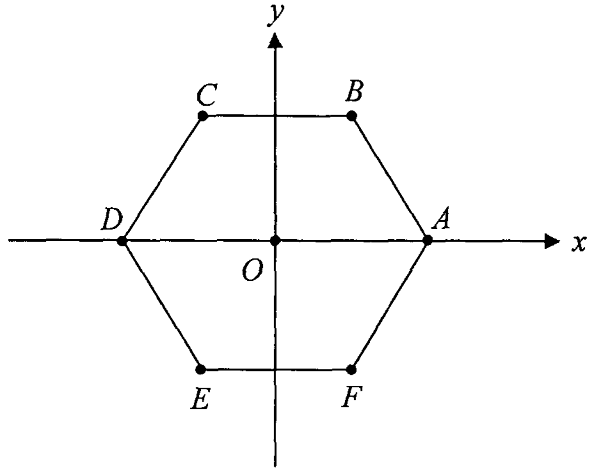
(ii) Use vectors to show that the area of a triangle determined by the points $A(2,2,0)$, $B(-1,0,2)$ and $C(0,4,3)$ is $\frac{15}{2}$ unit^2.

(d) (i) Determine the equation of the plane that passes through the point $P(1,-2,3)$ and the line with equation $(x,y,z)=(2,0,1)+t(2,-4,1)$.

(ii) Find the equation of the plane that contains the line $x=-1-2t$, $y=t$ and $z=5+3t$ and is perpendicular to the plane $3x-y+8z=17$.

[100 marks]

1. (a) *Dalam gambarajah di bawah, A, B, C, D, E, dan F adalah bucu-bucu sebuah heksagon lazim yang berpusat di asalan. Ungkapkan* $\overrightarrow{BC}+\overrightarrow{DE}+\overrightarrow{FA}$ *dalam sebutan* $\boldsymbol{a}$ *dan* $\boldsymbol{b}$ *di mana* $\boldsymbol{a}=\overrightarrow{OA}$ *dan* $\boldsymbol{b}=\overrightarrow{OB}$.



(b) (i) *Diberi bahawa vektor-vektor* $\boldsymbol{u}$ *dan* $\boldsymbol{v}$ *adalah berortogon bila* $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{0}$. *Buktikan bahawa* $|\boldsymbol{u}+\boldsymbol{v}|=|\boldsymbol{u}-\boldsymbol{v}|$ *jika dan hanya jika* $\boldsymbol{u}$ *dan* $\boldsymbol{v}$ *adalah berortogon.*

(ii) *Dapatkan sudut di antara vektor-vektor* $\boldsymbol{u}=(2,1,-2)$ *dan* $\boldsymbol{v}=(1,1,1)$.

(c) (i) *Cari isipadu paralelepiped yang ditentukan oleh vektor-vektor* $\boldsymbol{p}=(6,3,-1)$, $\boldsymbol{q}=(0,1,2)$ *dan* $\boldsymbol{r}=(4,-2,5)$.

(ii) *Gunakan vektor untuk menunjukkan bahawa luas sebuah segitiga yang ditentukan oleh titik-titik* $A(2,2,0)$, $B(-1,0,2)$ *dan* $C(0,4,3)$ *adalah* $\frac{15}{2}$ *unit*^{2}.

(d) (i) *Tentukan persamaan satah yang melalui titik* $P(1,-2,3)$ *dan garislurus dengan persamaan* $(x,y,z)=(2,0,1)+t(2,-4,1)$.

(ii) *Cari persamaan satah yang mengandungi garislurus* $x=-1-2t$, $y=t$ *dan* $z=5+3t$ *dan beseranjang dengan satah* $3x-y+8z=17$.

[100 markah]

2. (a) Prove that $\mathbb{R}^3$ is generated by the set $\{(0,1,-2),(1,1,1),(1,-3,2)\}$.

(b) Let $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a,b \in \mathbb{R} \right\}$. Show that V with the usual operations on $M_{2\times 2}$ is a vector space over $\mathbb{R}$.

(c) Let $W = \{(x+y, 2x, -y) \mid x, y \in \mathbb{R}\}$.

(i) Show that W is a subspace of $\mathbb{R}^3$.
(ii) Find the generating set S of W.
(iii) Explain why S cannot be a basis of $\mathbb{R}^3$.
(iv) Extend S so that we obtain a basis B of $\mathbb{R}^3$ that contains S. Verify that B is indeed a basis of $\mathbb{R}^3$.

(d) Show that if $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly independent and $\boldsymbol{w}$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ then $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent.

[100 marks]

3. (a) Let U and W be subspaces of a finite-dimensional vector space V. Prove that:

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

[Hint: Let B be a basis for $U \cap W$. Extend B to a basis C of U and a basis D of W. Show that $C \cup D$ is a basis for $U+W$.]

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V. Prove that $\{v_1, v_1+v_2, v_1+v_2+v_3, \ldots, v_1+v_2+\ldots+v_n\}$ is also a basis for V.

(c) (i) Show that $\mathscr{L}((-1,-1,1),(1,1,0)) = \mathscr{L}((-2,-2,1),(0,0,1))$.
(ii) Let S be a subspace of a vector space V. Show that $\mathscr{L}(S) = S$.

(d) Use the Gauss-Jordan method to find all solutions of the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & 4x_2 & - & x_3 & + & x_4 & = & 2 \\ & & 10x_2 & - & 4x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 5 \\ -2x_1 & - & 8x_2 & + & 2x_3 & - & 2x_4 & = & -4 \\ x_1 & - & 6x_2 & + & 3x_3 & & & = & 1 \end{array}$$

[100 marks]

2. (a) *Buktikan bahawa* $\mathbb{R}^3$ *dijana oleh set* $\{(0,1,-2),(1,1,1),(1,-3,2)\}$.

(b) *Biar* $V=\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a,b \in \mathbb{R}\right\}$. *Tunjukkan bahawa V dengan operasi-operasi biasa dalam* $M_{2\times 2}$ *adalah suatu ruang vektor ke atas* $\mathbb{R}$.

(c) *Biar* $W=\{(x+y,2x,-y) \mid x,y\in\mathbb{R}\}$.

(i) *Tunjukkan bahawa W adalah suatu subruang dari* $\mathbb{R}^3$.
(ii) *Cari set penjana S bagi W.*
(iii) *Terangkan mengapa S tidak boleh menjadi asas* $\mathbb{R}^3$.
(iv) *Lanjutkan S supaya kita memperoleh suatu asas B bagi* $\mathbb{R}^3$ *yang mengandungi S. Tentusahkan bahawa B memang suatu asas* $\mathbb{R}^3$.

(d) *Tunjukkan bahawa jika* $\{\mathbf{u},\mathbf{v}\}$ *adalah tak bersandar linear dan* $\mathbf{w}$ *bukan gabungan linear* $\mathbf{u}$ *dan* $\mathbf{v}$ *maka* $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ *adalah tak bersandar linear.*

[100 markah]

3. (a) *Biar U dan W subruang-subruang suatu ruang vektor terhingga V. Buktikan bahawa:*

$$dim(U+W)=dim\,U+dim\,W-dim(U\cap W)$$

[Hint: Biar B sebagai suatu asas bagi $U\cap W$. *Lanjutkan B kepada suatu asas C bagi U dan suatu asas D bagi W. Tunjukkan bahawa* $C\cup D$ *adalah asas bagi* $U+W$ *.]*

(b) *Biar* $\{v_1,v_2,\ldots,v_n\}$ *sebagai suatu asas bagi suatu ruang vektor V. Buktikan bahawa* $\{v_1,v_1+v_2,v_1+v_2+v_3,\ldots,v_1+v_2+\ldots+v_n\}$ *adalah suatu asas bagi V juga.*

(c) (i) *Tunjukkan bahawa* $\mathcal{L}((-1,-1,1),(1,1,0))=\mathcal{L}((-2,-2,1),(0,0,1))$.
(ii) *Biar S sebagai subruang dari suatu ruang vektor V. Tunjukkan bahawa* $\mathcal{L}(S)=S$.

(d) *Gunakan kaedah Gauss-Jordan untuk mencari semua penyelesaian bagi sistem persamaan:*

$$\begin{array}{rcrcrcrcr} x_1 & + & 4x_2 & - & x_3 & + & x_4 & = & 2 \\ & & 10x_2 & - & 4x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 5 \\ -2x_1 & - & 8x_2 & + & 2x_3 & - & 2x_4 & = & -4 \\ x_1 & - & 6x_2 & + & 3x_3 & & & = & 1 \end{array}$$

[100 markah]

4. (a) Suppose we have $T:V \rightarrow W$ and $S:V \rightarrow W$ where both are linear transformations. Show that $2T+3S$ is also a linear transformation from V to W.

(b) Find the standard matrix representing the linear transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^2$ if T is defined as $(x,y,z)T = (x-y+z, 2x+y-3z)$.

(c) Find the dimension of the row space of A where

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -3 & 2 \\ 2 & 4 & -2 & 0 \\ 3 & 8 & -5 & 4 \end{bmatrix}$$

(d) Let the matrix B represents a linear transformation $T:U \rightarrow V$ and the matrix C represents a linear transformation $S:V \rightarrow W$. If $\rho(B)$ denotes the rank of B, show that $\rho(BC) \leq \rho(C)$.

[100 marks]

4. (a) *Andai kita mempunyai* $T:V \to W$ *dan* $S:V \to W$ *di mana kedua-dua adalah transformasi linear. Tunjukkan bahawa* $2T+3S$ *adalah suatu transformasi linear dari V ke W juga.*

(b) *Cari matriks piawai yang mewakili* $T:\mathbb{R}^3 \to \mathbb{R}^2$ *jika T ditakrifkan sebagai* $(x,y,z)T=(x-y+z,2x+y-3z)$.

(c) *Cari dimensi ruang baris A di mana*

$$A=\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -3 & 2 \\ 2 & 4 & -2 & 0 \\ 3 & 8 & -5 & 4 \end{bmatrix}$$

(d) *Biar matriks B mewakili transformasi linear* $T:U \to V$ *dan matriks C mewakili transformasi linear* $S:V \to W$. *Jika* $\rho(B)$ *menandakan pangkat B, tunjukkan bahawa* $\rho(BC) \le \rho(C)$.

[100 markah]

5. (a) Given the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$(a,b,c)T = (2a-b, a+b-3c, -a+c)$$

(i) Find a basis for the kernel of T.
(ii) Find a basis for the image of T.
(iii) Verify the dimension theorem using your answers in (i) and (ii).
(iv) Use the Gram-Schmidt process on your basis from (ii) to find an orthonormal basis for the image of T.

(b) Find the parabola $y = ax^2 + bx + c$ that best fits the data

x	-2	-1	0	1	2
y	4	7	3	0	-1

(c) Given $U = \{(x,y,z) \mid x = t, y = -t, z = 2t, \text{ where } t \in \mathbb{R}\}$. Find $U^{\perp}$ (the orthogonal complement of U).

(d) Let $\alpha = \{b_1, b_2, b_3\}$ and $\beta = \{c_1, c_2, c_3\}$ two bases of $\mathbb{R}^3$ and T is a linear transformation where $T:\mathbb{R}^3 \to \mathbb{R}^3$. Suppose that

$$c_1T = b_1 - 2b_2 + b_3 \qquad c_2T = -b_2 + 3b_3 \qquad c_3T = -2b_1 + b_3$$

(i) Find the matrix $T_{\beta,\alpha}$.
(ii) Consider the vector $v = c_1 - 2c_2 + 2c_3$. Find $(vT)_\alpha$ using (i).

[100 marks]

5. (a) *Diberi transformasi linear* $T:\mathbb{R}^3 \to \mathbb{R}^3$ *yang ditakrifkan dengan*

$$(a,b,c)T = (2a-b, a+b-3c, -a+c)$$

(i) *Cari suatu asas bagi kernel dari T.*
(ii) *Cari suatu asas bagi imej dari T.*
(iii) *Tentusahkan teorem dimensi menggunakan jawapan anda dalam (i) dan (ii).*
(iv) *Guna proses Gram-Schmidt terhadap asas anda dalam (ii) untuk mencari asas ortonormal bagi imej T.*

(b) *Cari parabola* $y = ax^2 + bx + c$ *yang merupakan padanan terbaik dengan data*

x	−2	−1	0	1	2
y	4	7	3	0	−1

(c) *Diberi* $U = \{(x,y,z) \mid x = t, y = -t, z = 2t, \text{ where } t \in \mathbb{R}\}$. *Cari* $U^{\perp}$ *(pelengkap berortogon bagi U).*

(d) *Biar* $\alpha = \{b_1, b_2, b_3\}$ *dan* $\beta = \{c_1, c_2, c_3\}$ *adalah dua asas dari* $\mathbb{R}^3$ *dan T adalah suatu transformasi linear* $T:\mathbb{R}^3 \to \mathbb{R}^3$. *Andai*

$$c_1T = b_1 - 2b_2 + b_3 \qquad c_2T = -b_2 + 3b_3 \qquad c_3T = -2b_1 + b_3$$

(i) *Cari matriks* $T_{\beta,\alpha}$.
(ii) *Pertimbangkan vektor* $v = c_1 - 2c_2 + 2c_3$. *Cari* $(vT)_\alpha$ *menggunakan (i).*

[100 markah]

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