UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2007/2008

Jun 2008

MAT 102 – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Let

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

(i) Find f_x and f_y at $(x,y)\neq(0,0)$.

(ii) Show that f_x and f_y exist at $(0,0)$.

(iii) Is f continuous at $(0,0)$? Give your reason.

(b) Let $z=f(x,y)$ be a differentiable function and x and y be related to r and θ by $x=r\cos\theta$ and $y=r\sin\theta$.

(i) Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

(ii) Show that $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\left(\dfrac{\partial z}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

(c) Find (if any) the local maximum and minimum values and saddle point(s) of the function $f(x,y)=x^2+y^2+x^2y+4$.

[100 marks]

2. (a) (i) Sketch the region of integration and then change the order of integration for $\int_0^1\int_{4x}^4 f(x,y)\,dydx$.

(ii) Evaluate the integral $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dxdy$ by converting to polar coordinates.

(b) Let D be the region on the xy-plane bounded by the parabola $y=4-x^2$ and the line $y=3x$. Set up, but DO NOT evaluate, an integral to find the volume of the solid bounded above by the surface $z=x^2+4$ and below by the region D.

(c) Show that $\int_1^\infty \dfrac{1}{x^p}dx$ is convergent if $p>1$ and divergent if $p\leq 1$.

(*Hint*: Prove by considering three cases: $p=1$, $p<1$ and $p>1$)

[100 marks]

3. (a) Find the following limits:

(i) $\lim\limits_{x\to\infty}(xe^{\frac{1}{x}}-x)$ (ii) $\lim\limits_{x\to\frac{1}{2}^-}\dfrac{\ln(1-2x)}{\tan(\pi x)}$ (iii) $\lim\limits_{x\to 0}x^2\cos\left(\dfrac{1}{x}\right)$

1. (a) Andaikan

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

(i) Cari f_x dan f_y pada $(x,y)\neq(0,0)$.

(ii) Tunjukkan bahawa f_x dan f_y wujud pada $(0,0)$.

(iii) Adakah f selanjar pada $(0,0)$? Berikan alasan anda.

(b) Andaikan $z=f(x,y)$ fungsi terbezakan dan x dan y dihubungkan kepada r dan θ melalui $x=r\cos\theta$ dan $y=r\sin\theta$.

(i) Cari $\dfrac{\partial z}{\partial r}$ dan $\dfrac{\partial z}{\partial \theta}$.

(ii) Tunjukkan bahawa $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\left(\dfrac{\partial z}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

(c) Cari (jika ada) nilai-nilai maksimum dan minimum setempat dan titik pelana untuk fungsi $f(x,y)=x^2+y^2+x^2y+4$.

[100 markah]

2. (a) (i) Lakarkan rantau kamiran dan kemudiannya tukar tertib pengamiran untuk $\int_0^1\int_{4x}^4 f(x,y)\,dydx$.

(ii) Nilaikan kamiran $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dxdy$ dengan menukar ke koordinat kutub.

(b) Andaikan D suatu rantau pada satah-xy yang dibatasi oleh $y=4-x^2$ dan garis $y=3x$. Bentukkan, TANPA menilai, satu kamiran untuk mencari isipadu pepejal yang dibatasi atasnya oleh permukaan $z=x^2+4$ dan bawah oleh rantau D.

(c) Tunjukkan bahawa $\int_1^\infty \dfrac{1}{x^p}dx$ adalah menumpu jika $p>1$ dan mencapah jika $p\leq 1$.
(Petunjuk: Buktikan dengan mempertimbangkan tiga kes: $p=1$, $p<1$ dan $p>1$)

[100 markah]

3. (a) Cari had berikut:

(i) $\lim\limits_{x\to\infty}(xe^{\frac{1}{x}}-x)$ (ii) $\lim\limits_{x\to\frac{1}{2}^-}\dfrac{\ln(1-2x)}{\tan(\pi x)}$ (iii) $\lim\limits_{x\to 0} x^2\cos\left(\dfrac{1}{x}\right)$

(b) Determine whether each of the following series converges:

(i) $\sum_{k=1}^{\infty} \ln\left(\frac{k}{2k+5}\right)$ (ii) $\sum_{k=2}^{\infty} \frac{k^2+1}{k^3-1}$

(c) If f' is continuous, use the l'Hospital Rule to show that

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x).$$

[100 marks]

4. (a) The sequence $\{a_n\}$ is defined as $a_n = \frac{n}{2^n}$, $\forall n \geq 1$.

(i) Show that $\frac{a_{n+1}}{a_n} \leq 1$.

Hence, determine whether the sequence $\{a_n\}$ is increasing or decreasing.

(ii) Is $\{a_n\}$ bounded?

(iii) Does $\lim_{n\to\infty} a_n$ exist? Give your reason.

(b) (i) Find the first three nonzero terms in the Maclaurin series for $\sin x$.

(ii) By using part (i), evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

(c) Evaluate the following integrals:

(i) $\int_0^3 \frac{1}{x-1}\,dx$ (ii) $\int_0^2 \frac{1}{\sqrt{3-x}}\,dx$

[100 marks]

(b) Tentukan sama ada siri berikut menumpu:

(i) $\sum_{k=1}^{\infty} \ln\left(\frac{k}{2k+5}\right)$ (ii) $\sum_{k=2}^{\infty} \frac{k^2+1}{k^3-1}$

(c) Jika f' adalah selanjar, tunjuk dengan menggunakan Petua l'Hospital bahawa

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x).$$

[100 markah]

4. (a) Jujukan $\{a_n\}$ ditakrif sebagai $a_n = \frac{n}{2^n}$, $\forall n \geq 1$.

(i) Tunjukkan bahawa $\frac{a_{n+1}}{a_n} \leq 1$.

Seterusnya tentukan sama ada jujukan $\{a_n\}$ adalah menokok atau menyusut.

(ii) Adakah $\{a_n\}$ terbatas?

(iii) Adakah $\lim_{n\to\infty} a_n$ wujud? Beri alasan anda.

(b) (i) Cari tiga sebutan tak sifar pertama dalam siri Maclaurin untuk $\sin x$.

(ii) Dengan menggunakan bahagian (i), nilaikan $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

(c) Nilaikan kamiran berikut:

(i) $\int_0^3 \frac{1}{x-1}\,dx$ (ii) $\int_0^2 \frac{1}{\sqrt{3-x}}\,dx$

[100 markah]

- ooo O ooo -