UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MSS 301 – Complex Analysis
[Analisis Kompleks]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all twelve** [12] questions.

*[**Arahan**: Jawab **semua dua belas** [12] soalan.]*

1. A complex number is an ordered pair $z = (a, b),\ a, b \in \mathbb{R}$. Define the operations addition and multiplication in the complex plane $\mathbb{C}$. With $i = (0,1)$, show that every complex number z can be expressed in its Cartesian form $z = a + ib$.

   Show that the complex plane $\mathbb{C}$ is not an ordered field, that is, there does not exist a non-empty subset $\mathbb{C}_p$ satisfying :

   (i) $z + w \in \mathbb{C}_p$ for $z, w \in \mathbb{C}_p$,
   (ii) $zw \in \mathbb{C}_p$ for $z, w \in \mathbb{C}_p$, and
   (iii) for each $z \in \mathbb{C}$, only one of the following statements holds : either $z = 0$; $z \in \mathbb{C}_p$; or $-z \in \mathbb{C}_p$.

   [20/250]

2. Solve each equation, and express the solution in its Cartesian form :
   (a) $z^2 = 2 - 2\sqrt{3}i$
   (b) $\sin z = 3$

   [25/250]

3. Find in Cartesian form all roots to $z^3 = -27i$. Show that the roots form the vertices of an equilateral triangle inscribed in a circle centered at the origin of radius 3. Provide a sketch of the triangle and circle.

   [15/250]

4. If $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, show that the Cauchy –Riemann equations hold at $z_0 = x_0 + iy_0$:

   $$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

   Deduce that $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$.

   [20/250]

5. Determine where $f(z) = f(x + iy) = 2 - x^3 - xy^2 + i(x^2y + y^3 - 16y)$ is differentiable, and find its derivative. Where is f analytic?

   [15/250]

6. If $u(x, y) = x^2 - y^2 + 4$, find its harmonic conjugate. Find the maximum and minimum values of u on $|z| \leq 1$.

   [20/250]

7. Show that $|\cos z|^2 + |\sin z|^2 = 1 + 2\sinh^2 y,\ z = x + iy$. Hence deduce that $|\cos z|^2 + |\sin z|^2 \geq 1$ with equality if and only if z is real.

   [15/250]

1. *Nombor kompleks ialah pasangan bertertib* $z = (a, b),\ a, b \in \mathbb{R}$. *Takrifkan operasi hasil tambah dan hasil darab pada satah kompleks* $\mathbb{C}$. *Dengan* $i = (0,1)$, *tunjukkan bahawa setiap nombor kompleks* z *dapat diungkapkan dalam bentuk Cartesan* $z = a + ib$.

   Tunjukkan satah kompleks $\mathbb{C}$ *bukan medan bertertib, iaitu tidak wujud subset tak kosong* $\mathbb{C}_p$ *yang memenuhi :*

   (i) $z + w \in \mathbb{C}_p$ *untuk* $z, w \in \mathbb{C}_p$,
   (ii) $zw \in \mathbb{C}_p$ *untuk* $z, w \in \mathbb{C}_p$, *dan*
   (iii) *bagi setiap* $z \in \mathbb{C}$, *hanya satu pernyataan berikut adalah benar: sama ada* $z = 0$; $z \in \mathbb{C}_p$; *atau* $-z \in \mathbb{C}_p$.

   [20/250]

2. *Selesaikan setiap persamaan berikut, dan ungkapkan jawapan dalam bentuk Cartesan :*

   (a) $z^2 = 2 - 2\sqrt{3}i$
   (b) $\sin z = 3$

   [25/250]

3. *Cari dalam bentuk Cartesan semua punca* $z^3 = -27i$. *Tunjukkan punca-punca tersebut membentuk bucu-bucu segi tiga sama sisi yang terterap dalam bulatan berpusat di asalan dengan jejari 3. Lakarkan segi tiga dan bulatan tersebut.*

   [15/250]

4. *Jika* $f(z) = u(x,y) + iv(x,y)$ *adalah terbezakan di titik* z_0, *tunjukkan bahawa persamaan Cauchy –Riemann dipenuhi pada titik* $z_0 = x_0 + iy_0$:

   $$u_x(x_0, y_0) = v_y(x_0, y_0) \quad and \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

   Deduksikan bahawa $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$.

   [20/250]

5. *Tentukan di mana fungsi* $f(z) = f(x + iy) = 2 - x^3 - xy^2 + i(x^2y + y^3 - 16y)$ *adalah terbezakan, dan dapatkan terbitan f. Dimanakah fungsi f analisis?*

   [15/250]

6. *Jika* $u(x,y) = x^2 - y^2 + 4$, *dapatkan konjugat harmonik u. Dapatkan nilai maksimum dan nilai minimum u pada* $|z| \leq 1$.

   [20/250]

7. *Tunjukkan* $|\cos z|^2 + |\sin z|^2 = 1 + 2\sinh^2 y,\ \ z = x + i$. *Justeru deduksikan* $|\cos z|^2 + |\sin z|^2 \geq 1$ *dengan kesamaan jika dan hanya jika z adalah nyata.*

   [15/250]

8. Evaluate the following integrals over the given positively oriented simple closed contour :

(a) $$\oint_{|z|=2} \frac{ze^z}{2z-3}\,dz$$

(b) $$\oint_{|z|=1} \frac{\sin z}{z^4}\,dz$$

(c) $$\oint_{|z|=4} \frac{e^z}{z(z-1)}\,dz$$

[30/250]

9. Evaluate

(a) $\int_{\gamma} (y - x - i\,3x^2)\,dz$ where γ is the line segment from i to $2-i$

(b) $\int_{0}^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$, $0 < \alpha < 1$

[25/250]

10. Find three Laurent series expansion for the function

$$f(z) = \frac{2z-3}{z^2-3z-4}$$

in powers of z.

[25/250]

11. Let f be an entire function satisfying $|f(z)| \le a + b|z|$, where a and b are positive real numbers. Show that

$$|f(z_0)| \le a + br + b|z_0|$$

on every circle $|z - z_0| = r$. Hence show that $f''(z_0) = 0$ for all z_0, and deduce that $f(z) = \alpha z + \beta$, where α and β are complex constants. In general, if f is entire and $|f(z)| \le a + b|z|^k$, k a positive integer, show that f is a polynomial of the form

$$f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k.$$

[25/250]

12. If $\beta_1, \beta_2, \cdots, \beta_n$ are the roots of $z^n = a$, $a \in \mathbb{C}$, show that

$$\sum_{k=1}^{n} \beta_k = 0$$

[15/250]

8. *Nilaikan setiap kamiran berikut pada kontur tertutup ringkas berarah positif:*

(a) $$\oint_{|z|=2} \frac{ze^z}{2z-3} dz$$

(b) $$\oint_{|z|=1} \frac{\sin z}{z^4} dz$$

(c) $$\oint_{|z|=4} \frac{e^z}{z(z-1)} dz$$

[30/250]

9. *Nilaikan*

(a) $\int_{\gamma} (y - -i\, 3x^2)\, dz$ *dengan* γ *sebagai tembereng garis dari* i *ke* $2 - i$

(b) $$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}, \qquad 0 < \alpha < 1$$

[25/250]

10. *Cari tiga perwakilan siri Laurent bagi fungsi*

$$f(z) = \frac{2z-3}{z^2-3z-4}$$

dalam kuasa menaik z.

[25/250]

11. *Andaikan* f *fungsi seluruh yang memenuhi* $|f(z)| \leq a + b|z|$, *dengan* a *dan* b *sebagai nombor nyata positif. Tunjukkan*

$$|f(z_0)| \leq a + br + b|z_0|$$

pada setiap bulatan $|z - z_0| = r$. *Justeru tunjukkan bahawa* $f''(z_0) = 0$ *untuk setiap* z_0, *dan deduksikan* $f(z) = \alpha z + \beta$, *dengan* α *dan* β *sebagai pemalar kompleks. Umumnya jika* f *adalah seluruh dan* $|f(z)| \leq a + b|z|^k$, k *suatu integer positif, tunjukkan* f *ialah polinomial berbentuk*

$$f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k.$$

[25/250]

12. *Jika* $\beta_1, \beta_2, \cdots, \beta_n$ *adalah punca-punca persamaan* $z^n = a$, $a \in \mathbb{C}$, *tunjukkan*

$$\sum_{k=1}^{n} \beta_k = 0$$

[15/250]

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