INCLUSION PROPERTIES OF LINEAR OPERATORS AND ANALYTIC FUNCTIONS

by

MAHNAZ MORADI NARGESI

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SYMBOLS

Symbol	Description	page
$A(f)$	Alexander operator	28
$\mathcal{A}_b$	Class of analytic functions f with fixed second coefficient of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad \|a_2\| = 2b \quad (z \in \mathcal{U})$	171
$\mathcal{A}_m$	Class of m-valent analytic functions f of the form $f(z) = z^m + \sum_{k=1+m}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	2
$\mathcal{A}$	Class of analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	2
$\mathcal{B}(\alpha, \rho, \lambda)$	α-Bazilevič function of order ρ and type λ in $\mathcal{A}$	66
$\mathcal{B}(\alpha)$	α-Bazilevič function in $\mathcal{A}$	67
$(a)_n$	Pochhammer symbol	29
$\mathbb{C}$	Complex plane	1
$C^*(m, a, h)$	$\left\{ f \in \Sigma : -\frac{z(k_a * f)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(k_a * g_j)(z)} \prec h(z), \; z \in \mathcal{U}^*, \; g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m, a, h) \right\}$	50
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	5
$\mathcal{C}_a(h)$	$\left\{ f \in \mathcal{A} : \frac{z(k_a * f)'(z)}{(k_a * \psi)(z)} \prec h(z), \; \psi \in \mathcal{S}_a(h), \; z \in \mathcal{U} \right\}$	37
$\mathcal{CCV}(g, h)$	$\left\{ f \in \mathcal{A} : \frac{z(g * f)'(z)}{(g * \psi)(z)} \prec h(z), \; \psi \in \mathcal{ST}(g, h), \; z \in \mathcal{U} \right\}$	37
$\mathcal{CCV}_m(h)$	$\left\{ \hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : \frac{mzf_k'(z)}{\sum_{j=1}^{m} \psi_j(z)} \prec h(z), \; z \in \mathcal{U}, \; \hat{\psi} \in \mathcal{ST}_m(h), \; f_k \in \mathcal{A}, \; k = 1, \cdots, m \right\}$	39
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	4

$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}_\alpha$	$\left\{f \in \mathcal{A} : \left\|\frac{zf''(z)}{f'(z)}\right\| < 1-\alpha, z \in \mathcal{U}\right\}$	37
$\mathcal{CV}(g,h)$	$\left\{f \in \mathcal{A} : 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathcal{CV}_m(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle : \frac{m(zf_k')'(z)}{\sum_{j=1}^m f_j'(z)} \prec h(z),\ f_k \in \mathcal{A},\ z \in \mathcal{U},\ k = 1, \cdots, m\right\}$	38
$\mathcal{CV}(h)$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	36
$\overline{\text{co}}(D)$	The closed convex hull of a set D	4
D_λ^n	Generalized Sălăgean operator for meromorphic functions	49
$f * g$	Convolution or Hadamard product of functions f and g	17
$\mathcal{H}(\mathcal{U})$	Class of analytic functions in $\mathcal{U}$	2
$H_{a,b,c}(f)$	Hohlove operator	30
$\prec$	Subordinate to	24
$\mathcal{I}_{\lambda,\mu}^\beta$	$\mathcal{I}_{\lambda,\mu}^\beta f(z) = (f * p_\mu * q_{\beta,\lambda})(z)$	48
k	Koebe function $k(z) = z/(1-z)^2$	2
k_a	$k_a(z) := \frac{z}{(1-z)^a}, \quad a > 0$	37
$\mathcal{K}_a(h)$	$\left\{f \in \mathcal{A} : \frac{z(f*k_a)''(z)}{(f*k_a)'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathbb{N}$	$\mathbb{N} := \{1, 2, \cdots\}$	2
$k - \mathcal{UCV}$	Class of k-uniformly convex functions in $\mathcal{A}$	119
$L(f)$	Libera operator	28
$L_\gamma(f)$	Bernardi-Libera-Livingston operator	29
$\mathcal{L}(\alpha,\beta)$	$\left\{f \in \mathcal{A} : \alpha\frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},\right.$	

	$z \in \mathcal{U},\ \beta \in \mathbb{R} \setminus \{1\},\ \alpha \geq 0 \Big\}$	172
M	Möbius transformations	6
p_μ	$p_\mu(z) := \frac{1}{z(1-z)^\mu}$	48
$\mathcal{P}_\gamma$	$\left\{ f \in \mathcal{A} : \left\| \arg\left((1-\gamma)\dfrac{f(z)}{z} + \gamma f'(z) \right) \right\| < \dfrac{\pi}{2},\ z \in \mathcal{U} \right\}$	74
$\mathcal{P}(\beta)$	$\Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi} \big(f'(z) - \beta \big) > 0,\ z \in \mathcal{U} \Big\}$	22
$\mathcal{P}_\alpha(\beta)$	$\Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi} \Big((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta \Big) > 0,\ z \in \mathcal{U} \Big\}$	77
$\mathcal{PST}$	Class of parabolic starlike functions in $\mathcal{A}$	9
$\mathcal{PST}(\alpha)$	Class of parabolic starlike functions of order α in $\mathcal{A}$	8
Q	$\Big\{ q$: analytic and injective on $\bar{\mathcal{U}} \backslash E(q)$, $q'(\zeta) \neq 0,\ \zeta \in \partial\mathcal{U} \backslash E(q) \Big\}$	24
$q_{\beta,\lambda}$	$q_{\beta,\lambda}(z) := \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\lambda}{k+1+\lambda} \right)^\beta z^k$	48
$Q(m,n,\lambda,h)$	$\Big\{ f \in \Sigma : -\frac{z(D_\lambda^n f(z))'}{\frac{1}{m}\sum_{j=1}^m D_\lambda^n g_j(z)} \prec h(z),$ $z \in \mathcal{U}^*, g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m,n,\lambda,h) \Big\}$	49
$\mathcal{QCV}$	Class of quasi-convex functions in $\mathcal{A}$	5
$\mathcal{QCV}(g,h)$	$\Big\{ f \in \mathcal{A} : \frac{z(g*f)'(z)}{(g*\phi)(z)} \prec h(z),\ \phi \in \mathcal{CV}(g,h),\ z \in \mathcal{U} \Big\}$	37
$\mathcal{QCV}_m(h)$	$\Big\{ \hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : \frac{m(zf_k')'(z)}{\sum_{j=1}^m \phi_j'(z)} \prec h(z),$ $z \in \mathcal{U},\ \hat{\varphi} \in \mathcal{CV}_m(h),\ f_k \in \mathcal{A},\ k = 1, \cdots, m \Big\}$	39
$\mathbb{R}$	Set of all real numbers	21
Re	Real part of a complex number	4

$\mathcal{R}_\alpha$	Class of prestarlike functions of order α in $\mathcal{A}$	19
$\mathcal{R}^\alpha$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + \alpha z f''(z)\right) > 0,\ z \in \mathcal{U}\right\}$	68
$\mathcal{R}(\beta)$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + z f''(z)\right) > \beta,\ z \in \mathcal{U}\right\}$	114
$\mathcal{R}_\alpha(\beta)$	$\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left(f'(z) + \alpha z f''(z) - \beta\right) > 0,\ z \in \mathcal{U}\Big\}$	77
$\mathcal{S}$	Class of all normalized univalent functions f of the form $f(z) = z + a_2 z^2 + \cdots,\ z \in \mathcal{U}$	2
$S(f, z)$	Schwarzian derivative of analytic function f	6
$\mathcal{S}_a(h)$	$\left\{f \in \mathcal{A} : \frac{z(f * k_a)'(z)}{(f * k_a)(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
Σ	Class of all normalized meromorphic functions f of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathcal{U}^*)$	10
$\Sigma^{\beta}_{\lambda,\mu}(m, h)$	$\left\{g := \{g_1, g_2, \ldots, g_m\} : -\frac{z(\mathcal{I}^{\beta}_{\lambda,\mu} g_k(z))'}{\frac{1}{m}\sum_{j=1}^{m} \mathcal{I}^{\beta}_{\lambda,\mu} g_j(z)} \prec h(z),\ z \in \mathcal{U}^*,\ g_k \in \Sigma,\ k = 1, \cdots, m\right\}$	48
$\Sigma(m, n, \lambda, h)$	$\left\{f = \{f_1, f_2, \cdots, f_m\} : -\frac{z(D^n_\lambda f_i(z))'}{\frac{1}{m}\sum_{j=1}^{m} D^n_\lambda f_j(z)} \prec h(z),\ z \in \mathcal{U}^*,\ f_i \in \Sigma,\ i = 1, \cdots, m\right\}$	49
$\Sigma(m, a, h)$	$\left\{g = \{g_1, g_2, \cdots, g_m\} : -\frac{z(k_a * g_i)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(k_a * g_j)(z)} \prec h(z),\ z \in \mathcal{U}^*,\ g_i \in \Sigma,\ i = 1, \cdots, m\right\}$	49
$\Sigma^{ccv}_m(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle : -\frac{m z f_k'(z)}{\sum_{j=1}^{m} \psi_j(z)} \prec h(z),\ \hat{\psi} \in \Sigma^{st}_m(h),\ z \in \mathcal{U}^*,\ f_k \in \Sigma,\ k = 1, \cdots, m\right\}$	51
$\Sigma^{cv}(\alpha)$	$\left\{f \in \Sigma :\ -\operatorname{Re}\frac{(z f'(z))'}{f'(z)} > \alpha,\ z \in \mathcal{U}\right\}$	11

$\Sigma_m^{cv}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : -\frac{m(zf_k')'(z)}{\sum_{j=1}^{m} f_j'(z)} \prec h(z),\right.$ $\left. f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	50
$\Sigma_m^{qcv}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : -\frac{m(zf_k')'(z)}{\sum_{j=1}^{m} \varphi_j'(z)} \prec h(z),\right.$ $\left. \hat{\varphi} \in \Sigma_m^{cv}(h),\ f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	51
$\Sigma^{st}(\alpha)$	$\left\{f \in \Sigma :\ -\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathcal{U}\right\}$	11
$\Sigma_m^{st}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m \rangle :\right.$ $\left. -\frac{mzf_k'(z)}{\sum_{j=1}^{m} f_j(z)} \prec h(z), f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	50
$\mathcal{SST}(\alpha)$	Class of strongly starlike functions of order α in $\mathcal{A}$	59
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	4
$\mathcal{ST}[A, B]$	$\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U},\ -1 \leq B < A \leq 1\}$	15
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}_\alpha$	$\left\{f \in \mathcal{A} : \left\lvert\frac{zf'(z)}{f(z)} - 1\right\rvert < 1 - \alpha,\ z \in \mathcal{U}\right\}$	114
$\mathcal{ST}(h)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	36
$\mathcal{ST}(g, h)$	$\left\{f \in \mathcal{A} : \frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathcal{ST}_m(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m \rangle :\ \frac{mzf_k'(z)}{\sum_{j=1}^{m} f_j(z)} \prec h(z),\right.$ $\left. f_k \in \mathcal{A},\ z \in \mathcal{U},\ k = 1, \cdots, m\right\}$	38
$\mathcal{T}$	Class of analytic functions with negative coefficients	12
$\mathcal{TCV}(\alpha)$	Class of convex functions of order α in $\mathcal{T}$	12
$\mathcal{TL}(\alpha, \beta)$	$\left\{f \in \mathcal{T} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right) > \beta,\ z \in \mathcal{U}\right\}$	127
$\mathcal{TST}(\alpha)$	Class of starlike functions of order α in $\mathcal{T}$	12

SIFAT-SIFAT RANGKUMAN PENGOPERASIAN LINEAR DAN FUNGSI ANALISIS

ABSTRAK

Tesis ini mengkaji kelas $\mathcal{A}$ terdiri daripada fungsi analisis ternormalkan di dalam cakera unit terbuka $\mathcal{U}$ pada satah kompleks. Kelas fungsi meromorfi di dalam cakera unit berliang tidak termasuk titik asalan turut dikaji. Secara keseluruhannya, tesis ini merangkumi enam permasalahan kajian. Pertama, subkelas fungsi-fungsi bak-bintang, cembung, hampir cembung dan kuasi cembung diitlakkan dengan memperkenalkan subkelas baru fungsi-fungsi analisis dan meromorfi. Sifat tutupan kelas-kelas baru ini akan dikaji dan akan dibuktikan bahawa konvolusi kelas-kelas ini dengan fungsi pra bak-bintang dan pengoperasi kamiran Bernardi-Libera-Livingston adalah bersifat tertutup.

Keunivalenan fungsi $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ dikaji dengan menyarankan terbitan Schwarzian $S(f, z)$ dan pekali kedua a_2 fungsi f memenuhi ketaksamaan tertentu. Kriteria baru untuk fungsi analisis menjadi α-Bazilevič kuat tertib tak negatif dibangunkan dalam sebutan terbitan Schwarzian dan pekali kedua. Juga syarat-syarat serupa untuk pekali kedua dan terbitan Schwarzian $S(f, z)$ bagi f diperoleh yang menjamin fungsi f tersebut terkandung di dalam subkelas tertentu untuk $\mathcal{S}$. Untuk suatu fungsi analisis $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ yang memenuhi ketaksamaan $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$, batas tajam β diperoleh supaya f sama ada bak-bintang atau cembung tertib α. Batas tajam untuk η juga diperoleh agar fungsi f yang memenuhi $\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| \leq 1 - \beta$ adalah bak bintang atau cembung tertib α. Beberapa ketaksamaan pekali lain berkaitan dengan subkelas-subkelas tertentu juga dikaji. Andaikan $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analisis pada $\mathcal{U}$ dengan pekali kedua a_2 memenuhi $|a_2| = 2b$, $0 \leq b \leq 1$, dan katakan f memenuhi sama ada $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ atau $|a_n| \leq c/n$ $(c > 0)$

untuk $n \geq 3$. Jejari tajam bak-bintang Janowski dan beberapa jejari berkaitan untuk fungsi sedemikian juga diperoleh.

Sifat kecembungan pengoperasi kamiran umum $V_\lambda(f)(z) := \int_0^1 \lambda(t) f(tz)/t dt$ pada suatu subkelas fungsi analisis yang mengandung beberapa subkelas tersohor akan dikaji. Beberapa aplikasi menarik dengan pilihan λ berbeza akan dibincang. Sifat-sifat geometrik untuk pengoperasi kamiran teritlak berbentuk $\mathcal{V}_\lambda(f) = \rho z + (1-\rho) V_\lambda(f)$, $\rho < 1$ akan juga diterangkan. Akhir sekali, sifat subordinasi dan superordinasi untuk pengoperasi linear teritlak yang memenuhi suatu hubungan jadi semula pembeza peringkat pertama telah dikaji. Suatu kelas fungsi teraku yang sesuai telah dipertimbangkan untuk mendapatkan syarat cukup bagi domainan dan subordinan terbaik. Keputusan yang diperoleh menyatukann hasil kajian terdahulu.

INCLUSION PROPERTIES OF LINEAR OPERATORS AND ANALYTIC FUNCTIONS

ABSTRACT

This thesis studies the class $\mathcal{A}$ of normalized analytic functions in the open unit disk $\mathcal{U}$ of the complex plane. The class of meromorphic functions in the punctured unit disk which does not include the origin is also studied. This thesis investigates six research problems. First, the classical subclasses of starlike, convex, close-to-convex and quasi-convex functions are extended by introducing new subclasses of analytic and meromorphic functions. The closure properties of these newly defined classes are investigated and it is shown that these classes are closed under convolution with prestarlike functions and the Bernardi-Libera-Livingston integral operator.

The univalence of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ is investigated by requiring the Schwarzian derivative $S(f, z)$ and the second coefficient a_2 of f to satisfy certain inequalities. New criterion for analytic functions to be strongly α-Bazilevič of nonnegative order is established in terms of the Schwarzian derivatives and the second coefficients. Also, similar conditions on the second coefficient of f and its Schwarzian derivative $S(f, z)$ are obtained that would ensure the function f belongs to particular subclasses of $\mathcal{S}$. For an analytic function $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$, a sharp bound on β is determined so that f is either starlike or convex of order α. A sharp bound on η is obtained that ensures functions f satisfying $\sum_{n=2}^{\infty} \left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n| \leq 1 - \beta$ is either starlike or convex of order η. Several other coefficient inequalities related to certain subclasses are also investigated. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disk $\mathcal{U}$ with the second coefficient a_2 satisfying $|a_2| = 2b$, $0 \leq b \leq 1$, and let f satisfy either $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$

for $n \geq 3$. Sharp radius of Janowski starlikeness for such functions is obtained. Several related radii are also obtained.

The convexity property of a general integral operator $V_\lambda(f)(z) := \int_0^1 \lambda(t)f(tz)/tdt$ on a new class of analytic functions which includes several well-known classes is investigated. Several interesting applications for different choices of λ are discussed. The geometric properties of the generalized integral operator of the form $\mathcal{V}_\lambda(f) = \rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$ are also inquired. Finally, subordination and superordination properties of general linear operators satisfying a certain first-order differential recurrence relation are investigated. An appropriate class of admissible functions is considered to determine sufficient conditions for best dominant and best subordinant. The results obtained unify earlier works.

CHAPTER 1

INTRODUCTION

Geometric function theory is a remarkable area in complex analysis. This field is more often associated with geometric properties of analytic functions. Geometric function theory has raised the interest of many researchers since the beginning of the 20th century. The purpose of this chapter is to review and assemble for references, relevant definitions and known results in geometric function theory which underlie the theory of univalent functions.

1.1 Univalent Functions

A function f is *analytic* at z_0 in a domain D if it is differentiable in some neighborhood of z_0, and it is *analytic on a domain* D if it is analytic at all points in D. An analytic function f is said to be *univalent* in a domain D of the complex plane $\mathbb{C}$ if it is one-to-one in D. It is *locally univalent* in D if f is univalent in some neighborhood of each point $z_0 \in D$. It is known that a function f is locally univalent in D provided $f'(z) \neq 0$ for any $z \in D$ [48, p. 5]. In 1851, Riemann proved that any simply connected domain which is not the entire plane and the unit disk $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ are conformally equivalent.

Theorem 1.1 (Riemann Mapping Theorem) [48, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathcal{U}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Therefore, the study of conformal mappings on simply connected domains may be confined to the study of functions that are univalent on the unit disk $\mathcal{U}$. The Riemann Mapping Theorem shows that there is a one-to-one correspondence between proper simply connected domains (geometric objects) and suitably normalized univalent functions (analytic objects).

Let $\mathcal{H}(\mathcal{U})$ denote the set of all analytic functions defined in the unit disk $\mathcal{U}$.

Let $\mathcal{A}$ be the class of normalized analytic functions f defined in $\mathcal{U}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

More generally, let $\mathcal{A}_m$ denote the subclass of $\mathcal{A}$ consisting of normalized analytic functions f of the form

$$f(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad (m \in \mathbb{N} := \{1, 2, \cdots\}).$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. The class $\mathcal{S}$ are treated extensively in the books [48,61,151]. Bernardi [33] provided a comprehensive list of papers on univalent functions theory published before 1981.

The *Koebe function* defined by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n,$$

and its rotations $e^{-i\beta} k(e^{i\beta} z)$, play an important role in the class $\mathcal{S}$. The Koebe function maps $\mathcal{U}$ in a one-to-one manner onto a domain D consisting of the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -1/4$. A significant problem in the theory of univalent functions is the Bieberbach's conjecture which asserts that the Koebe function has the largest coefficients in $\mathcal{S}$.

Theorem 1.2 (Bieberbach's Conjecture) [48, p. 37] *If* $f = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{S}$, *then*

$$|a_n| \leq n \quad (n \geq 2).$$

Equality occurs only for the Koebe function and its rotations.

In 1916, Bieberbach [36] proved the inequality for $n = 2$, and conjectured that it is true for any n. In 1985, de Branges [37] proved this conjecture for all coefficients $n \geq 2$. Before de Branges's proof, the Bieberbach's conjecture was known for $n \leq 6$. Löwner [101] developed parametric representation of slit mapping and used it to prove the Bieberbach's conjecture for $n = 3$. The cases $n = 4, 5, 6$ were proved by Garabedian and Schiffer [57], Pederson and Schiffer [147], and Pederson [146]. In 1925, Littlewood [95] showed that the coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq en$ $(n \geq 2)$. Duren [48], Goodman [61] and Pommerenke [151] provided the history of this problem.

As an application, a famous covering theorem due to Koebe can be proved by Bieberbach's conjecture for the second coefficient. This theorem states that if $f \in \mathcal{S}$, then the image of $\mathcal{U}$ under f must cover an open disk centered at the origin with radius 1/4.

Theorem 1.3 (Koebe One-Quarter Theorem) [61, p. 62] *The range of every function $f \in \mathcal{S}$ contains the disk $\{w : \; |w| < 1/4\}$.*

The Koebe function and its rotations are the only functions in $\mathcal{S}$ which omit a value of modulus 1/4. The sharp upper and lower bounds for $|f(z)|$ and $|f'(z)|$ where $f \in \mathcal{S}$ are a consequence of the Bieberbach's conjecture for the second coefficient.

Theorem 1.4 (Distortion and Growth Theorem) [61, p. 68] *Let $f \in \mathcal{S}$. Then for each $z = re^{i\theta} \in \mathcal{U}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3},$$

and

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

The above inequalities are sharp with equality occurring for the Koebe function and its rotations.

1.2 Subclasses of Univalent Functions

The long gap between the formulation of the Bieberbach's conjecture (1916) and its proof by de Branges (1985) motivated researchers to investigate its validity on several subclasses of $\mathcal{S}$. These classes are defined by geometric conditions, and include the class of starlike functions, convex functions, close-to-convex functions, and quasi-convex functions. A set D in the plane is said to be *starlike* with respect to an interior point w_0 in D if the line segment joining w_0 to every other point w in D lies entirely in D. A set D in the plane is *convex* if it is starlike with respect to each of its points; that is, if the line segment joining any two points of D lies entirely in D. The *closed convex hull* of a set D in $\mathbb{C}$ is the closure of intersection of all convex sets containing D. It is the smallest closed convex set containing D and is denoted by $\overline{\text{co}}(D)$.

A function $f \in \mathcal{A}$ is *starlike* if $f(\mathcal{U})$ is a starlike domain with respect to the origin, and f is *convex* if $f(\mathcal{U})$ is a convex domain. Analytically, these are respectively equivalent to the conditions

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathcal{U}).$$

In 1915, Alexander [4] showed that there is a close connection between convex and starlike functions.

Theorem 1.5 (Alexander Theorem) [4] *Suppose that $f'(z) \neq 0$ in $\mathcal{U}$. Then f is convex in $\mathcal{U}$ if and only if zf' is starlike in $\mathcal{U}$.*

Denote the classes of starlike and convex functions by $\mathcal{ST}$ and $\mathcal{CV}$ respectively.

More generally, for $\alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ be subclasses of $\mathcal{A}$ consisting respectively of starlike functions of order α and convex functions of order α. For $0 \leq \alpha < 1$, these functions are known to be univalent [48, p. 40], and are defined analytically by

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}, \tag{1.2}$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}. \tag{1.3}$$

Clearly, $\mathcal{ST} = \mathcal{ST}(0)$ and $\mathcal{CV} = \mathcal{CV}(0)$.

In 1952, Kaplan [77] introduced the class of close-to-convex functions. A function $f \in \mathcal{A}$ is *close-to-convex* in $\mathcal{U}$ if there is a starlike function ψ and a real number α such that

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{\psi(z)} > 0 \quad (z \in \mathcal{U}). \tag{1.4}$$

The class of all such functions is denoted by $\mathcal{CCV}$. Geometrically, f is close-to-convex if and only if the image of $|z| = r$ has no large hairpin turns; that is, there is no sections of the curve $f(C_r)$ in which the tangent vector turns backward through an angle greater than π. Starlike functions are evidently close-to-convex.

Another subclass of $\mathcal{S}$ is the class of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be *quasi-convex* in $\mathcal{U}$ if there is a function ϕ in $\mathcal{CV}$ such that

$$\operatorname{Re} \frac{(zf'(z))'}{\phi'(z)} > 0 \quad (z \in \mathcal{U}).$$

This set of functions denoted by $\mathcal{QCV}$ was introduced by Noor and Thomas [129]. Note that $\mathcal{CV} \subset \mathcal{QCV}$ where $\phi(z) \equiv f(z)$. Every close-to-convex function is univalent. This can be inferred from the following simple but important criterion for univalence proved by Noshiro [130] and Warschawski [207].

Theorem 1.6 (Noshiro-Warschawski Theorem) [61, p. 47] *If f is analytic* *in a convex domain D and* $\operatorname{Re} f'(z) > 0$ *there, then f is univalent in D.*

The subclasses of $\mathcal{S}$ consisting of starlike, convex and close-to-convex functions satisfy the following chain:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV} \subset \mathcal{S}.$$

There are many criteria for functions to be univalent. In 1949, Nehari [123] obtained univalence criterion which involves the Schwarzian derivative. Let $S(f, z)$ denote the *Schwarzian derivative* of a locally univalent analytic function f defined by

$$S(f, z) := \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2. \tag{1.5}$$

A *Möbius transformation* M is defined by

$$M(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0). \tag{1.6}$$

The function M is univalent on the closed complex plane containing the point at ∞. A function of the form (1.6) always maps "circles" onto "circles" where a "circle" means a straight line or a circle [61, p. 10]. It can be shown that the Schwarzian derivative is invariant under Möbius transformations, that is, $S(M \circ f, z) = S(f, z)$. Also, the Schwarzian derivative of an analytic function f is identically zero if and only if it is a Möbius transformation [48, p. 259].

The following univalence criterion was given by Nehari.

Theorem 1.7 [123] *If $f \in \mathcal{S}$, then*

$$|S(f, z)| \leq \frac{6}{(1 - |z|^2)^2}. \tag{1.7}$$

Conversely, if an analytic function f *in* $\mathcal{U}$ *satisfies*

$$|S(f,z)| \leq \frac{2}{(1-|z|^2)^2}, \tag{1.8}$$

then f *is univalent in* $\mathcal{U}$*. The results are sharp.*

The preceding result was first proved by Kraus [85] but had been forgotten for a long time. Nehari re-discovered and proved Theorem 1.7. The Koebe function satisfies (1.7) and shows that the constant 6 is sharp. Also, the function

$$L(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \tag{1.9}$$

which maps $\mathcal{U}$ univalently onto the parallel strip $|\operatorname{Im} w| < \pi/2$ satisfies (1.8) and shows that the constant 2 is sharp. Nehari [125] also showed that inequality (1.8) holds if f is convex and this result is sharp for the function L defined by (1.9).

By considering two particular positive functions, Nehari [123] obtained a bound on the Schwarzian derivative that ensures univalence of an analytic function in $\mathcal{A}$. In fact, the following theorem was proved.

Theorem 1.8 [123, Theorem II, p. 549] *If* $f \in \mathcal{A}$ *satisfies*

$$|S(f,z)| \leq \frac{\pi^2}{2} \quad (z \in \mathcal{U}),$$

then $f \in \mathcal{S}$*. The result is sharp for the function* f *given by* $f(z) = (\exp(i\pi z) - 1)/i\pi$*.*

The problem of finding similar bounds on the Schwarzian derivatives that would imply univalence, starlikeness or convexity of functions was investigated by a number of authors including Gabriel [55], Friedland and Nehari [54], and

Ozaki and Nunokawa [139]. Chiang [41] investigated convexity of functions f by requiring the Schwarzian derivative $S(f,z)$ and the second coefficient a_2 of f to satisfy certain inequalities. In Chapter 3, it is assumed that the second coefficient of an analytic function f is small enough and that the Schwarzian derivative $S(f,z)$ satisfies a certain inequality. Under these assumptions, it is shown that f is univalent. Also, similar conditions on the second coefficient of f and its Schwarzian derivative $S(f,z)$ are obtained that would ensure the function f belongs to particular subclasses of $\mathcal{S}$.

Various subclasses of $\mathcal{ST}$ and $\mathcal{CV}$ were later introduced that possess certain geometric features. Goodman [62] introduced the class of uniformly convex functions $\mathcal{UCV}$. Geometrically, a function $f \in \mathcal{S}$ is *uniformly convex* if it maps every circular arc γ contained in $\mathcal{U}$ with center $\zeta \in \mathcal{U}$ onto a convex arc. Goodman [62] gave a two-variable analytic characterization for the class $\mathcal{UCV}$, that is,

$$\mathcal{UCV} := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \ \zeta, z \in \mathcal{U} \right\},$$

while Rønning [167], and Ma and Minda [103] independently gave a one-variable characterization for $f \in \mathcal{UCV}$ by using the minimum principle for harmonic functions:

$$f \in \mathcal{UCV} \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (z \in \mathcal{U}). \tag{1.10}$$

For $0 \le \alpha < 1$, let Ω_α be the parabolic region in the right-half plane defined by

$$\Omega_\alpha = \{w = u + iv : \ v^2 < 4(1-\alpha)(u-\alpha)\} = \{w : \ |w-1| < 1 - 2\alpha + \operatorname{Re} w\}.$$

The class $\mathcal{PST}(\alpha)$ of *parabolic starlike functions of order* α is the subclass of $\mathcal{A}$ consisting of functions f such that $zf'(z)/f(z) \in \Omega_\alpha$, $z \in \mathcal{U}$. Thus $f \in \mathcal{PST}(\alpha)$

if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\alpha + \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U}). \tag{1.11}$$

The class $\mathcal{PST}$, called *parabolic starlike functions*, was introduced by Rønning [167]. Analytically, $f \in \mathcal{PST}$ if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U}).$$

Rønning [167] also showed that

$$f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{PST}(1/2) = \mathcal{PST}.$$

Closely related is the class $\mathcal{UST}$ of uniformly starlike functions introduced by Goodman [63]. A function $f \in \mathcal{S}$ is *uniformly starlike* if it maps every circular arc γ contained in $\mathcal{U}$ with center $\zeta \in \mathcal{U}$ onto a starlike domain with respect to $f(\zeta)$. A two-variable analytic characterization of the class $\mathcal{UST}$ is given by

$$\mathcal{UST} := \left\{ f \in \mathcal{S} : \ \operatorname{Re}\left(\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right) > 0, \ \zeta, z \in \mathcal{U} \right\}. \tag{1.12}$$

Goodman [62] showed that the classical Alexander relation (Theorem 1.5) does not hold between $\mathcal{UST}$ and $\mathcal{UCV}$. Such a question between $\mathcal{UST}$ and $\mathcal{UCV}$ is in fact equivalent to $\mathcal{UST} = \mathcal{PST}$, and it was shown in [62, 168] that there is no inclusion between $\mathcal{UST}$ and $\mathcal{PST}$:

$$\mathcal{UST} \not\subset \mathcal{PST}, \quad \mathcal{PST} \not\subset \mathcal{UST}.$$

Several authors have studied the above classes, amongst which include the works of [62,102–104,165,179]; surveys on the classes $\mathcal{UCV}$, $\mathcal{UST}$ and $\mathcal{PST}$ can be found in [14] by Ali and Ravichandran, and in [166] by Rønning.

The class of meromorphic functions is yet another subclass of univalent functions that will be discussed in the thesis. Let Σ denote the class of normalized *meromorphic* functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.13}$$

that are analytic in the punctured unit disk $\mathcal{U}^* := \{z : 0 < |z| < 1\}$ except for a simple pole at 0. In 1914, Gronwall [65] proved the following Area Theorem.

Theorem 1.9 (Area Theorem) *If f is univalent function of the form*

$$f(\xi) = \xi + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{\xi^n} \quad (|\xi| > 1), \tag{1.14}$$

then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$

The interest of the class Σ arose from an application of the Area Theorem in the proof of the Bieberbach's conjecture for the second coefficient. A function f of the form (1.14) and $g \in \Sigma$ are related by the transformation $f(1/z) = g(z)$.

The transformation

$$f(\xi) = \frac{1}{g(1/\xi)} \quad (|\xi| > 1) \tag{1.15}$$

takes each g in $\mathcal{S}$ into a function f of the form (1.14). By the transformation (1.15), the Koebe function takes a particularly simple form

$$\phi(\xi) = \frac{1}{k(1/\xi)} = \xi - 2 + \frac{1}{\xi}$$

which maps the exterior of unit disk $\{\xi \in \mathbb{C} : 1 < |\xi| < \infty\}$ onto the domain consisting of the entire complex plane minus the slit $-4 \leq w \leq 0$.

A function $f \in \Sigma$ is said to be *starlike* if it is univalent and the complement of $f(\mathcal{U})$ is a starlike domain with respect to the origin where $f(z) \neq 0$ for $z \in \mathcal{U}$. Denote by Σ^{st} the class of meromorphically starlike functions. Analytically, it is known that $f \in \Sigma^{st}$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < 0 \quad (z \in \mathcal{U}).$$

Note that $f \in \Sigma^{st}$ implies $f(z) \neq 0$ for $z \in \mathcal{U}$. Similarly, a function $f \in \Sigma$ is *convex*, denoted by $f \in \Sigma^{cv}$, if it is univalent and the complement of $f(\mathcal{U})$ is a convex domain. Analytically, $f \in \Sigma^{cv}$ if and only if

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} < 0 \quad (z \in \mathcal{U}).$$

In general, for $0 \leq \alpha < 1$, the classes of *meromorphic starlike functions of order* α and *meromorphic convex functions of order* α respectively are defined by

$$\Sigma^{st}(\alpha) := \left\{ f \in \Sigma : \ \operatorname{Re} \frac{zf'(z)}{f(z)} < \alpha \right\},$$
$$\Sigma^{cv}(\alpha) := \left\{ f \in \Sigma : \ \operatorname{Re} \frac{(zf'(z))'}{f'(z)} < \alpha \right\}.$$

These classes have been studied by several authors [23, 24, 88, 116, 117, 191, 192, 205]. We assembled geometric features and analytic expressions of the well-known subclasses of univalent functions to apply for future convenience.

1.3 Function with Negative Coefficients

The following simple result follows from an application of the Noshiro-Warschawski Theorem (Theorem 1.6).

Theorem 1.10 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, *and* $\sum_{n=2}^{\infty} |a_n| \leq 1$. *Then* $f \in \mathcal{S}$.

If $a_n \le 0$ for all n, then the condition above is also a necessary condition for f to be univalent. In 1961, Merkes *et al.* [105] obtained a sufficient condition for $f \in \mathcal{A}$ to be starlike of order α, which is also necessary in the event $a_n \le 0$.

Theorem 1.11 [105, Theorem 2, p. 961] *Let* $0 \le \alpha < 1$, *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. *Then* $f \in \mathcal{ST}(\alpha)$ *if*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \le 1-\alpha. \tag{1.16}$$

If $a_n \le 0$ *for all* n*, then* (1.16) *is a necessary condition for* $f \in \mathcal{ST}(\alpha)$.

This motivated the investigation of functions whose coefficients are negative. The class of functions with negative coefficients in $\mathcal{A}$, denoted by $\mathcal{T}$, consists of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0). \tag{1.17}$$

Denote by $\mathcal{TST}(\alpha)$ and $\mathcal{TCV}(\alpha)$ the respective subclasses of functions with negative coefficients in $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$. For starlike and convex functions of order α with negative coefficients, Silverman [182] determined the distortion theorem, covering theorem, and coefficients inequalities and extreme points. Silverman [182] also provided a survey, some open problems, and conjectures on analytic functions with negative coefficients. In 2003, the classes $\mathcal{TST}$ and $\mathcal{TCV}$ were generalized in terms of subordination by Ravichandran [158]. The subordination concept and its applications will be treated in Section 1.8.

As in the case with the Bieberbach's conjecture, there are several easily stated questions related to the class $\mathcal{T}$ that appear difficult to solve. Related works to analytic functions with negative coefficients include [10, 11, 26, 89, 118, 119, 136, 155, 156, 175]. Merkes *et al.* [105] proved Theorem 1.11 based on a method used by Clunie and Keogh [46], which was later applied to obtain sufficient conditions for functions f to be in certain subclasses of analytic functions. For instance, the following lemma is a sufficient coefficient condition for functions $f \in \mathcal{A}$ to satisfy

$$\operatorname{Re}\left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \beta \quad (\alpha \geq 0,\ \beta < 1,\ z \in \mathcal{U}). \tag{1.18}$$

Lemma 1.1 [97] *Let $\beta < 1$, and $\alpha \geq 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq 1 - \beta, \tag{1.19}$$

then f satisfies (1.18)*. If $a_n \leq 0$ for all n, then* (1.19) *is a necessary condition for functions f to satisfy* (1.18)*.*

Geometric properties of analytic functions satisfying (1.18) will be investigated in Chapter 5. Sălăgean [176] obtained several interesting implications for analytic functions with negative coefficients. Motivated by the investigation of Sălăgean [176], several implications are investigated for functions $f \in \mathcal{A}$ satisfying (1.18). The largest bound β for analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$ are determined that will ensure f to be either starlike or convex of some positive order. For $f \in \mathcal{TST}(\alpha)$, and $f \in \mathcal{TCV}(\alpha)$, the largest value is obtained that bounds each coefficient inequality of the form $\sum n a_n$, $\sum n(n-1)a_n$, $\sum (n-1)a_n$ and $\sum n^2 a_n$. The results obtained will be applied to ensure the hypergeometric functions $zF(a,b;c;z)$ satisfy (1.18). The hypergeometric functions will be treated in Section 1.9.

1.4 Univalent Functions with Fixed Second Coefficient

Certain properties of analytic functions are influenced by their second coefficient. In 1920, Gronwall [66] extended the distortion and growth theorems for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with a pre-assigned second coefficient. Corresponding results for convex functions with a pre-assigned second coefficient were also obtained [66].

Let the class $\mathcal{A}_b$ consist of functions $f \in \mathcal{A}$ with a fixed second coefficient a_2 with $|a_2| = 2b$, $0 \le b \le 1$. Each $f \in \mathcal{A}_b$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|a_2| = 2b).$$

Let $\mathcal{CV}_b(\alpha)$ denote the class of convex functions of order α and $\mathcal{ST}_b(\alpha)$ denote the class of starlike functions of order α where $f \in \mathcal{A}_b$. Also denote by $\mathcal{ST}_b := \mathcal{ST}_b(0)$ and $\mathcal{CV}_b := \mathcal{CV}_b(0)$ the class of starlike functions and the class of convex functions with $|a_2| = 2b$ respectively. Finkelstein [52] obtained distortion and growth theorems for the classes $\mathcal{ST}_b$ and $\mathcal{CV}_b$. The results obtained in [52] were generalized to the class $\mathcal{ST}_b(\alpha)$ by Tepper [199] and the class $\mathcal{CV}_b(\alpha)$ by Padmanabhan [140]. Later in 2001, Padmanabhan [141] investigated the problem for general classes of functions defined by subordination.

Silverman [181] investigated the influence of the second coefficient on the class of close-to-convex functions. Here, a function $f \in \mathcal{A}_b$ is close-to-convex of order β and type α, denoted by $f \in \mathcal{CCV}_b(\alpha, \beta)$, if there is a function $\psi \in \mathcal{CV}_b(\alpha)$ such that

$$\operatorname{Re} \frac{f'(z)}{\psi'(z)} > \beta \quad (\beta \ge 0).$$

Silverman [181] proved distortion and covering theorems for $f \in \mathcal{CCV}_b(\alpha, \beta)$. The theory of differential subordination for functions $f \in \mathcal{A}_b$ was discussed in [13,122]. Ali *et al.* provided a brief history of these works in [9].

Lewandowski *et al.* [92] proved that an analytic function f satisfying

$$\operatorname{Re} \left(\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \right) > 0 \quad (z \in \mathcal{U}) \tag{1.20}$$

is starlike. The class of such functions was extended to the form (1.18) and has subsequently been investigated by Ramesha *et al.* [157], Nunokawa *et al.* [133], Obradović and Joshi [134], Padmanabhan [142], Ravichandran [160, 162], and Liu *et al.* [97]. For $-\alpha/2 \leq \beta < 1$, Li and Owa [93] proved that functions satisfying (1.18) are starlike. In 2002, the class of analytic functions satisfying

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < \beta \quad (\beta > 1,\ z \in \mathcal{U})$$

was considered by Owa and Nishiwaki [128], while its subclasses were earlier investigated by Uralegaddi *et al.* [204, 206], Owa and Srivastava [138]. Liu *et al.* [96] investigated the class of functions satisfying

$$\operatorname{Re} \left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right) < \beta \quad (\alpha \geq 0,\ \beta > 1,\ z \in \mathcal{U}). \tag{1.21}$$

In Chapter 7, the class of functions satisfying (1.18) and (1.21) will be put in a general form

$$\mathcal{L}(\alpha, \beta) \cap \mathcal{A}_b := \left\{ f \in \mathcal{A}_b : \alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\beta)z}{1 - z},\ \beta \in \mathbb{R} \setminus \{1\},\ \alpha \geq 0 \right\}. \tag{1.22}$$

Also, the well-known class of analytic functions introduced by Janowski [73] defined by

$$\mathcal{ST}[A, B] \cap \mathcal{A}_b = \left\{ f \in \mathcal{A}_b : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1 \right\}$$

will be considered. The radius properties for functions $f \in \mathcal{L}(\alpha, \beta) \cap \mathcal{A}_b$ and $f \in \mathcal{ST}[A, B] \cap \mathcal{A}_b$ are investigated in Chapter 7. The radius problems will be treated in the next section.

1.5 Radius Problems

Let $\mathcal{M}$ be a set of functions and $\mathcal{P}$ be a property which functions in $\mathcal{M}$ may or may not possess in a disk $|z| < r$. The least upper bound of all numbers r such that every function $f \in \mathcal{M}$ has the property $\mathcal{P}$ in the disk $\mathcal{U}_r = \{z : |z| < r\}$ is the radius for the property $\mathcal{P}$ in the set $\mathcal{M}$. Every univalent analytic function is univalent, but every univalent function is not always convex. However, every univalent analytic mapping maps a sufficiently small disk into a convex domain. The largest radius of the disk with this property is the radius of convexity. It is known that the radius of convexity for the set $\mathcal{S}$ is $2 - \sqrt{3}$ and is attained by the Koebe function [127]. Grunsky [67] proved that the radius of starlikeness for the set $\mathcal{S}$ is $\tanh(\pi/4)$. The radius of close-to-convexity for the set $\mathcal{S}$ was determined by Krzyż [87]. A list of such radius problems was provided by Goodman [61, Chapter 13].

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, de Branges [37] proved the Bieberbach's conjecture that $|a_n| \leq n$ $(n \geq 2)$ (Theorem 1.2). However, the inequality $|a_n| \leq n$ $(n \geq 2)$ does not imply f is univalent; for example, $f(z) = z + 2z^2$ satisfies the coefficient inequality but f is not a member of $\mathcal{S}$ as $f'(-1/4) = 0$. In view of this, it is interesting to investigate the radius of univalence, starlikeness, and other geometric properties of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ when the Taylor coefficients of f satisfy $|a_n| \leq cn + d$ $(n \geq 2)$.

The inequality $|a_n| \leq M$ holds for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|f(z)| \leq M$, and for these functions, Landau [90] proved that the radius of univalence is $M - \sqrt{M^2 - 1}$. For functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying the inequality $|a_n| \leq n$ $(n \geq 2)$, Gavrilov [58] showed that the radius of univalence is the real root $r_0 \approx 0.1648$ of the equation $2(1-r)^3 - (1+r) = 0$, and for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|a_n| \leq M$ $(n \geq 2)$, the radius of univalence is $1 - \sqrt{M/(1+M)}$. Yamashita [209] showed that the radius of univalence obtained by Gavrilov is the radius of starlikeness as well. Indeed, Gavrilov [58, Theorem 1] estimated the radius of univalence to be $0.125 < r_0 < 0.130$, while Yamashita [209] obtained $r_0 \approx 0.1648$. Yamashita also determined the radius of convexity for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|a_n| \leq M$ $(n \geq 2)$ to be the real root of $(M+1)(1-r)^3 - M(1+r) = 0$.

Recently Kalaj *et al.* [74] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying similar coefficient inequalities.

In [161], Ravichandran obtained the sharp radii of starlikeness and convexity of order α for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$ $(M > 0)$, $n \geq 3$. The radius constants for uniform convexity and parabolic starlikeness for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$, $n \geq 3$ were also obtained. Ravichandran [161] determined the radius of positivity for the real part of the functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ satisfying the inequality $|c_n| \leq 2M$ $(M > 0)$, $n \geq 3$ with $|c_2| = 2b$, $0 \leq b \leq 1$.

Let $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfy either $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$ for $n \geq 3$. In Chapter 7, sharp $\mathcal{L}(\alpha, \beta)$-radius and sharp $\mathcal{ST}[A, B]$- radius for these classes are obtained. The radius constants obtained by Ravichandran [161] and Yamashita [209] are shown to be special cases of the results obtained in Chapter 7.

1.6 Convolution

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in the unit disk $\mathcal{U}$. The Hadamard product of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

The alternative representation as a convolution integral

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f\left(\frac{z}{\zeta}\right) g(\zeta) \frac{d\zeta}{\zeta} \quad (|z| < \rho < 1),$$

is the reason $f * g$ is also called the convolution of f and g where R_f and R_g are the radii of convergence for f and g respectively [172, p. 11]. Since f and g are analytic in $\mathcal{U}$, $R_f \geq 1$ and $R_g \geq 1$. Thus,

$$\frac{1}{R_{f*g}} = \limsup |a_n b_n|^{\frac{1}{n}} \leq \left(\limsup |a_n|^{\frac{1}{n}}\right) \left(\limsup |b_n|^{\frac{1}{n}}\right) = \frac{1}{R_f} \frac{1}{R_g} \leq 1,$$

where R_{f*g} is the radius of convergence for $f * g$. Hence $f * g$ is analytic in $|z| < R_f R_g$. Mandelbrojt and Schiffer [150] conjectured univalence is preserved under integral convolution; namely if $f, g \in \mathcal{S}$, then

$$G(z) = \int_0^z \frac{(f * g)(t)}{t} dt \in \mathcal{S}.$$

Epstein and Schöenberg [50], Hayman [70], and Loewner and Netanyahu [100] proved counterexamples to the Mandelbrojt and Schiffer conjecture. In 1958, Pólya and Schöenberg [150] conjectured that

$$\mathcal{CV} * \mathcal{CV} \subset \mathcal{CV}.$$

Suffridge [195] proved that the convolution of every pair of convex functions is close-to-convex. In 1973, the Polya and Schöenberg's conjecture was proved by Ruscheweyh and Sheil-Small [173]. They also proved that the class of starlike functions and close-to-convex functions are closed under convolution with convex functions. However, it turns out that the class of univalent functions is not closed under convolution. In fact, $\mathcal{ST} * \mathcal{ST}$ is not even contained in the family $\mathcal{S}$. For example, let $f = g = k \in \mathcal{ST}$, where k is the Koebe function. Then $f * g \notin \mathcal{S}$

because $a_n = n^2 > n$. Further details about related works can be found in [48].

A subclass of analytic functions considered by Ruscheweyh [172] known as prestarlike functions was applied to the basic convolution results.

For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of *prestarlike functions of order* α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} : \; f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{ST}(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$. In particular,

$$f \in \mathcal{R}_{1/2} \Leftrightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2} \quad (z \in \mathcal{U}),$$

$$f \in \mathcal{R}_0 \Leftrightarrow \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathcal{U}). \tag{1.23}$$

Therefore, $\mathcal{R}_{1/2} = \mathcal{ST}_{1/2}$ and $\mathcal{R}_0 = \mathcal{CV}$. It is a known result [172] that the classes of starlike functions of order α and convex functions of order α are closed under convolution with prestarlike functions of order α. Prestarlike functions have a number of interesting geometric properties. Ruscheweyh [172] and Sheil-Small [180] investigated the significance of prestarlike functions. The results and techniques of Ruscheweyh and Sheil-Small developed in [173] in connection with their proof of the Polya-Schöenberg conjecture have been applied in many convolution articles. The convex hull method is based on the following convolution result for prestarlike and starlike functions.

Theorem 1.12 [172, Theorem 2.4] *Let* $\alpha \leq 1$, $\phi \in \mathcal{R}_\alpha$ *and* $f \in \mathcal{ST}(\alpha)$. *Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\operatorname{co}}(H(\mathcal{U})),$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$, *where* $\overline{\operatorname{co}}(H(\mathcal{U}))$ *denotes the closed convex* *hull of $H(\mathcal{U})$.*

In Chapter 2, the classical subclasses of starlike, convex, close-to-convex and quasi-convex functions are extended to new subclasses of analytic functions. Using the method of convex hull and the theory of differential subordinations discussed later in Section 1.8, convolution properties of these newly defined subclasses of analytic functions are investigated. It is shown that these classes are closed under convolution with prestarlike functions. Also, new subclasses for meromorphic functions are similarly introduced, and the convolution features of these subclasses are investigated. It is proved that these classes are also closed under convolution with prestarlike functions. It is shown that the Bernardi-Libera-Livingston integral operator preserve all these subclasses of analytic and meromorphic functions. It would be evident that various earlier works, for example those of [3, 35, 44, 120, 148, 159], are special instances of the results obtained.

1.7 Dual Set and Duality for Convolution

Let $\mathcal{A}_0$ be the set of all functions $f \in \mathcal{H}(\mathcal{U})$ satisfying $f(0) = 1$. For $V \subset \mathcal{A}_0$, define the dual set

$$V^* := \left\{ f \in \mathcal{A}_0 : \ (f * g)(z) \neq 0 \text{ for all } g \in V, z \in \mathcal{U} \right\}.$$

The second dual V^{**} is defined as $V^{**} = (V^*)^*$. It is of interest to investigate the relations between V and V^{**}. In general, V^{**} is much bigger than V, but many properties of V remain valid in V^{**}. Let Λ be the set of continuous linear functionals on $\mathcal{H}(\mathcal{U})$ and $\lambda(V) := \{\lambda(f) : f \in V\}$. In 1975, Ruscheweyh [170] proved the following fundamental result, known as the *Duality Principle*.

Theorem 1.13 (Duality Principle) [170] *Let $V \subset \mathcal{A}_0$ have the following properties:*

(1) V is compact,

(2) $f \in V$ implies $f(xz) \in V$ for all $|x| \leq 1$.

Then $\lambda(V) = \lambda(V^{**})$ for all $\lambda \in \Lambda$ on $\mathcal{A}$, and $\overline{\mathrm{co}}(V) = \overline{\mathrm{co}}(V^{**})$.

The Duality Principle has numerous applications to the class of functions possessing certain geometric properties like bounded real part, convexity, starlikeness, close-to-convexity and univalence. The monograph of Ruscheweyh [172], and also the paper [170] in which many of the results of this topic were first published have become basic references for duality theory. As an application of Duality Principle, the following corollary was shown by Ruscheweyh [172]. The result is false with V^{**} replaced by $\overline{\mathrm{co}}(V)$.

Corollary 1.1 [172, Corollary 1.1. p. 17] *Let $V \subset \mathcal{A}_0$ satisfy the conditions in Theorem 1.13. Let $\lambda_1, \lambda_2 \in \Lambda$ with $0 \notin \lambda_2(V)$. Then for any $f \in V^{**}$ there exists a function $g \in V$ such that*

$$\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(g)}{\lambda_2(g)}.$$

Ruscheweyh determined a big class of sets in $\mathcal{A}_0$ in which the above result was applicable.

Theorem 1.14 [170, Theorem 1, p. 68] *If*

$$V_\beta = \left\{(1-\beta)\frac{1+xz}{1+yz} + \beta : \ |x| = |y| = 1, \ \beta \in \mathbb{R}, \ \beta \neq 1\right\},$$

then

$$V_\beta^* = \left\{f \in A_0 : \mathrm{Re}\, f(z) > \frac{1-2\beta}{2(1-\beta)}\right\},$$

and

$$V_\beta^{**} = \left\{f \in A_0 : \exists\phi \in \mathbb{R} \text{ with } \mathrm{Re}\, e^{i\phi}(f(z) - \beta) > 0, \ z \in \mathcal{U}\right\}.$$

Singh and Singh [187] proved the Bernardi integral operator

$$F_c(z) = (c+1)\int_0^1 t^{c-1} f(tz)dt \quad (c > -1)$$

is starlike for $-1 < c \leq 0$, where $\operatorname{Re} f'(z) > 0$ in $\mathcal{U}$. In 1986, Mocanu proved that

$$\operatorname{Re} f'(z) > 0 \Rightarrow F_1 \in \mathcal{ST},$$

and the result was later improved by Nunokawa [131]. Singh and Singh [186] also proved

$$\operatorname{Re} f'(z) > -\frac{1}{4} \Rightarrow F_0 \in \mathcal{ST}.$$

Such problems were earlier handled using the theory of subordination which will be discussed in Section 1.8. In 1975, Fournier and Ruscheweyh [53] used the Duality Principle [172] to find the sharp bound for β such that $F_c(\mathcal{P}(\beta)) \subset \mathcal{ST}$ where $\mathcal{P}(\beta)$ is given by

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0, \quad z \in \mathcal{U} \right\}, \tag{1.24}$$

and $-1 < c \leq 2$.

Indeed, Fournier and Ruscheweyh [53] investigated starlikeness properties of a general operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt. \tag{1.25}$$

over functions f in the class $\mathcal{P}(\beta)$ given by (1.24), where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. Ali and Singh [21] found a sharp estimate of the parameter β that ensures $V_\lambda(f)$ is convex over $\mathcal{P}(\beta)$.

The duality theory of convolutions developed by Ruscheweyh [172] is now popularly used by several authors to discuss similar problems, among which include the works of [27–31, 45, 47, 83, 152–154]. As a consequence of these works, several interesting results on integral operators for special choices of λ were derived. A survey on integral transforms in geometric function theory was provided by Kim [81]. Integral operators will be treated again in Section 1.9.

The class $\mathcal{W}_\beta(\alpha,\gamma)$ defined by

$$\begin{aligned}\mathcal{W}_\beta(\alpha,\gamma) := \Big\{ & f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with} \\ & \operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\right) > 0,\ z \in \mathcal{U}\Big\},\end{aligned} \tag{1.26}$$

for $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$ was recently introduced by Ali *et al.* [12]. Ali *et al.* [7] investigated the starlikeness of integral transform (4.1) over the class $\mathcal{W}_\beta(\alpha,\gamma)$ by applying the Duality Principle.

In Chapter 4, the Duality Principle is used to determine the best value of $\beta < 1$ that ensures the integral operator $V_\lambda(f)$ in (1.25) maps the class $\mathcal{W}_\beta(\alpha,\gamma)$ defined in (1.26) into the class of convex functions. Simple necessary and sufficient condition for $V_\lambda(f)$ to be convex are obtained. For specific choices of the admissible function λ, several applications are investigated. As an important consequence, it is shown that a function f satisfying the third-order differential equation

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta$$

is convex in $\mathcal{U}$ where $\beta > -0.629445$. Also, the smallest value of $\beta < 1$ is obtained such that the generalized integral operator of the form $\rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$, over the class of $\mathcal{W}_\beta(\alpha,\gamma)$ is starlike. Corresponding result for $\rho z + (1-\rho)V_\lambda(f)$,

$\rho < 1$, to be convex is also derived.

1.8 Differential Subordination

In this section, the basic definitions and theorems in the theory of subordination and certain applications of differential subordinations are described. A function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. If g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The following concepts and terminologies were introduced by Miller and Mocanu in [111].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$, and h be univalent in $\mathcal{U}$. If an analytic function p satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.27}$$

then p is called a *solution* of the differential subordination. The univalent function q is called a *dominant of the solution* of the differential subordination, or more simply, *dominant*, if $p(z) \prec q(z)$ for all p satisfying (1.27). A dominant q_1 satisfying $q_1(z) \prec q(z)$ for all dominants q of (1.27) is said to be the *best dominant* of (1.27). The best dominant is unique up to a rotation of $\mathcal{U}$. Miller and Mocano provided a comprehensive discussion on differential subordination in [111].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$, and $h(z)$ be analytic in $\mathcal{U}$. Let p and $\psi(p(z), zp'(z), z^2p''(z); z)$ be univalent in $\mathcal{U}$. If p satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.28}$$

then p is called a *solution* of the differential superordination. An analytic function q is called a *subordinant* of the solution of the differential superordination, or