RAMSEY ALGEBRAS: A RAMSEYAN COMBINATORICS FOR UNIVERSAL ALGEBRAS

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by

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LIST OF ABBREVIATIONS

ZFC the Zermelo-Frankel axiomatization of set theory appended with the axiom of choice

UBR Unique Binary Representation (of the integers)

NAF Non-adjacent Form (representation of the integers)

LIST OF SYMBOLS

$\mathrm{Dom}(f)$	domain of the function f
$\mathrm{Cod}(f)$	codomain of the function f
$\mathrm{Rn}(f)$	range of the function f
$\|\|f\|\|$	arity of the function f or arity of the function symbol f
$\|\|\sigma\|\|$	length of the (finite) sequence σ, i.e. $\|\mathrm{Dom}(\sigma)\|$
$\|A\|$	cardinality of the set A; determinant of the matrix A
$\mathscr{P}(A)$	the powerset of A
ω	the set of natural numbers, also the first infinite ordinal
$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of all integers
$\aleph_0$	the first infinite cardinal number, i.e. the cardinality of ω
$[A]^\kappa$	the set of subsets of A with cardinality κ
$[A]^{<\omega}$	the set of finite subsets of A

ALJABAR RAMSEY: KOMBINATORIK BERTEMA RAMSEY UNTUK ALJABAR-ALJABAR SEJAGAT

ABSTRAK

Aljabar Ramsey merupakan teori bertema Ramsey untuk aljabar-aljabar. Perumusan yang tepat dari aljabar Ramsey didasarkan atas karya Carlson mengenai ruang Ramsey topologi, dari mana pelbagai hasil kombinatorik klasik seperti Teorem Ellentuck dan Teorem Hindman dapat diperolehi. Selepas kerja-kerja terobosannya mengenai ruang Ramsey, Carlson mencadangkan bahawa, bagi ruang-ruang yang dihasilkan oleh aljabar-aljabar, kajian bersifat kombinatorik tulen boleh dijalankan atas ruang-ruang tersebut, di mana hasil yang bersifat topologi asli dapat diperoleh dari hasil kajian kombinatorik itu. Arah pengajian sedemikian turut dikenali sebagai Aljabar Ramsey. Cadangan itu pertama kali diburu oleh Teh dalam kajian doktor falsafal beliau dan beberapa hasil asas mengenai aljabar homogen telah diperolehi. Tesis ini dimulai dengan pengenalan yang diperlukan untuk perbincangan yang berpadu mengenai subjek aljabar Ramsey. Kami juga menetapkan simbol-simbol yang diperlukan sebelum melangkah ke hasil-hasil penyelidikan asli. Kami juga memberi serba sedikit motivasi perkenalan aljabar Ramsey ke dalam kesusasteraan matematik. Selepas itu, kami memperluaskan konsep aljabar Ramsey ke tetapan yang lebih umum yang merangkumi aljabar-aljabar heterogen. Ini dilakukan bukan sahaja demi keumuman, tetapi juga struktur-struktur yang bersifat heterogen boleh dijumpai dalam karya Carlson dan ia hanyalah semulajadi untuk mengkaji aljabar-aljabar sedemikian. Kami juga membentangkan beberapa hasil asas yang berkaitan dengan aljabar-aljabar heterogen sebelum melangkah ke contoh-contoh konkrit. Bagi contoh-contoh konkrit, kami mengkaji tiga aljabar, iaitu oktonion nyata di bawah pendaraban, ruang vektor, dan pelbagai aljabar

matriks. Kami menunjukkan bahawa oktonion nyata tidak membentuk aljabar Ramsey di bawah pendaraban. Kajian sedemikian didorong oleh persoalan mengenai apakah peranan ciri bersekutu (associative property) dalam penentuan semi-kumpulan itu adalah aljabar Ramsey, sama ada ia diperlukan untuk sistem perduaan (binary systems) untuk menjadi Ramsey atau tidak. Bagaimanapun, kami tidak dapat memberi jawapan yang lengkap untuk soalan ini, tetapi keputusan yang kami perolehi menunjukkan bahawa ia adalah mungkin kerana oktonion nyata, di bawah pendaraban, membentuk apa yang dikenali sebagai aljabar diassociative dan aljabar seperti ini mempunyai hampir semua ciri-ciri kumpulan. Bagi kajian ruang vektor dalam konteks aljabar (heterogen) Ramsey, kami mendapat suatu teorem klasifikasi ruang vektor berdasarkan sifat aljabar Ramsey mereka. Selepas mengkaji aljabar-aljabar matriks, kita mengalih tumpuan kepada sifat-sifat aljabar Ramsey dari segi aljabar-aljabar yang mempunyai bahasa yang sama. Kajian bertema ini merangkumi aljabar-aljabar homomorfik dan isomorphik, manakala kemuncak dalam tema ini adalah suatu teorem mengenai hasil penggabungan mana-mana koleksi aljabar-aljabar Ramsey. Akhirnya, kami menyimpulkan penyelidikan doktoral ini dengan suatu kajian tentang aljabar-aljabar yang " setara " dalam beberapa segi, tetapi tidak semestinya dari segi bahasa yang sama.

RAMSEY ALGEBRAS: A RAMSEYAN COMBINATORICS FOR UNIVERSAL ALGEBRAS

ABSTRACT

The study of Ramsey algebras is a Ramseyan-type study on algebras. The precise formulation of a Ramsey algebra is based on the work of Carlson on topological Ramsey spaces, from which a wide array of classical combinatorial results such as the Ellentuck theorem and Hindman's theorem can be derived. After his groundbreaking work on Ramsey spaces, Carlson suggested that, for spaces that are generated by algebras, one may pursue a purely combinatorial study of these spaces, where results of topological nature can be derived from their associated combinatorial results. Such a direction of study would then be known as Ramsey algebra. The suggestion was first pursued by Teh in his doctoral work and some basic results concerning homogeneous algebras were obtained. We begin the thesis with a preliminary, introductory section required for a cohesive discussion of the subject as well as setting up the required symbols. We introduce the motivating notions and results for the introduction of Ramsey algebras into the literature. We then extend the notion of a Ramsey algebra to the more general setting that encompasses heterogeneous algebras. This is done not only for the sake of generality, but also heterogeneous structures are ubiquitous in the said work of Carlson and it is only natural that we consider heterogeneous algebras as well. We also present some basic results related to heterogeneous algebras before studying some concrete examples. The concrete examples that we study are the real octonions under multiplication, vector spaces, and various matrix algebras. We show that the real octonions do not form a Ramsey algebra under multiplication. Such a study was motivated by the question as to what role associativity plays in determining semigroups being

Ramsey algebras, whether it is indeed essential for a binary system to be Ramsey. We do not have a complete answer to this question, but our result in this chapter shows that it is likely to be the case since, under multiplication, the real octonions form what is known as a diassociative algebra and such an algebra has almost all the properties of a group. A detailed study of vector spaces in the context of (heterogeneous) Ramsey algebras then ensues, leading up to a classification of vector spaces based on their Ramsey algebraic properties. This doctoral work also contains a study of a different theme, namely a focusing of the Ramsey properties of algebras of the same language; under this theme, properties involving homomorphic and isomorphic algebras are explored while a highlight in this theme is a result on the amalgamation of an arbitrary family of Ramsey algebras. In contrast, we conclude the doctoral research with a chapter dedicated to algebras that are "equivalent" in some sense, but not necessarily of the same language.

CHAPTER 1

INTRODUCTION

The origin of Ramsey algebras can be traced back to the work of Carlson on topological Ramsey spaces [3]. When a topological space is generated by algebras, it was noted by Carlson himself that one can embark on the study of the space by purely combinatorial means. In turn, one can view the study of Ramsey algebras as a Ramsey-type combinatorics for algebras. In this introductory chapter, we will give a precise definition of a Ramsey algebra as well as giving a detailed account of the origins of the subject. We will make a connection between topological Ramsey spaces and Ramsey algebras. Basic results about Ramsey algebras will also be given in this chapter.

1.1 Terminology and Notation

Following set theoretic convention, the natural numbers are defined to be the nonnegative integers and the set of these numbers will be denoted by ω. The Greek alphabet ω will also be used to mean the cardinal $\aleph_0$ or the least infinite ordinal as is the custom of set theory. We will often have the occasion of dealing with the nonzero natural numbers and, for that reason, we reserve the notation $\mathbb{N}$ for the set $\omega \setminus \{0\}$. The set of real numbers will be denoted by $\mathbb{R}$. The cardinality of a set A will be denoted by $|A|$. The power set of A is denoted by $\mathscr{P}(A)$ as usual and $[A]^{<\omega}$ denotes the collection of finite subsets of A while $[A]^{\kappa}$ denotes the subsets of A of cardinality exactly κ. A partition of a set A has its usual meaning, namely a collection of pairwise disjoint subsets of A whose union is A. We abuse terminology and identify a partition with the function c from A into some cardinal κ whose collection of preimages is the partition in question. A partition will also be known as a *coloring* on A. A partition or coloring is said to be *finite* if the cardinal κ is finite.

Cartesian products and functions will carry the usual meanings. If f is a function, the symbol $\mathrm{Dom}(f)$ denotes the domain of f, $\mathrm{Cod}(f)$ the codomain of f, and $\mathrm{Rn}(f)$ the range of f. If A is a set, we say that f is a function *on* A if $\mathrm{Dom}(f) = A^n$ for some $n \in \mathbb{N}$ and $\mathrm{Cod}(f) = A$. If $A_1, \ldots, A_n, A_{n+1}$ are nonempty sets, $f : A_1 \times \cdots \times A_n \to A_{n+1}$, and there exist $i, j \in \{1, \ldots, n, n+1\}$ such that $A_i \neq A_j$, then we say that f is a *heterogeneous* function. Otherwise, f is said to be a homogeneous function. Following algebraic terminology, we will also often speak of *operations* in place of functions. Thus, we will speak of f being an operation on A when f is a function on A or f being a heterogeneous operation when f is a heterogeneous function. If $X \subseteq A$, $Y \subseteq B$, and $f : A \to B$, then $f[X]$ denotes the set $\{f(x) : x \in X\}$ and $f^{-1}[Y]$ denotes the set $\{x \in A : f(x) \in Y\}$. The arity of an operation f has its usual meaning and will be denoted by $||f||$.

If A is a set, a *finite sequence of* A is a function from a proper initial segment of ω into A; a proper initial segment of ω is a set of the form $\{0, \ldots, N\}$ for some $N \in \omega$. An infinite *infinite sequence of* A is a function from ω into A. A sequence is often denoted by an arrow over an alphabet such as $\vec{x}$. If $n \in \omega$ is in the domain of $\vec{x}$, then $\vec{x}(n)$ will often be known as the n-th term of the sequence $\vec{x}$. The *length* of a finite sequence $\vec{x}$, denoted $||\vec{x}||$, is the cardinality of its domain. When it is called for, sequences will be enclosed within angled brackets $\langle \cdots \rangle$. For instance, the sequence of natural numbers in its natural order may be explicitly denoted by $\langle 0, 1, 2, \ldots \rangle$.

The set of all functions from A into B is denoted by ${}^{A}B$. Thus, if A is a set, then the set of all infinite sequences of A is denoted by ${}^{\omega}A$. The restriction of a function $f : A \to B$ to a subset A' of A is denoted by $f \restriction A'$. An initial segment of a sequence $\vec{x}$ is a restriction of $\vec{x}$ to an initial segment of the domain of $\vec{x}$; if $N \in \omega$ is in the domain of $\vec{x}$, then $\vec{x} \restriction (N+1)$ denotes the restriction $\vec{x} \restriction \{0, \ldots, N\}$. The concatenation symbol for (finite) sequences is denoted by $*$. Thus, $\vec{x}_1 * \vec{x}_2 * \cdots$ is the concatenation of the sequences $\vec{x}_1, \vec{x}_2, \ldots$. For example, if $\vec{x} = \langle x_1, \ldots, x_N \rangle$ and $\vec{y} = \langle y_1, \ldots, y_M \rangle$, then $\vec{x} * \vec{y} = \langle x_1, \ldots, x_N, y_1, \ldots, y_M \rangle$. The notation $\vec{x} - (N+1)$ is shorthand for $\vec{x} \restriction (\mathrm{Dom}(\vec{x}) \setminus \{0, \ldots, N\})$ and it is the unique sequence $\vec{y}$ such that $(\vec{x} \restriction (N+1)) * \vec{y} = \vec{x}$. If $\vec{y}$ is a sequence such that $(\vec{x} \restriction (N+1)) * \vec{y} = \vec{x}$ for some $N \in \omega$, then $\vec{y}$ is called a *tail* of $\vec{x}$.

If a Cartesian product has n components, i.e. $A_1 \times \cdots \times A_n$ for some $n \in \mathbb{N}$, then an element of the Cartesian product is called an n-tuple. A 2-tuple is also called an *ordered pair*. An n-tuple is denoted in the parenthesis form $(x_1, \ldots, x_n)$, where $x_i \in A_i$ for each $i \in \{1, \ldots, n\}$. Due to the nature of the subject of Ramsey algebra, we will often identify an n-tuple with its associated sequence. Specifically, $(x_1, \ldots, x_n)$ will be identified with the sequence $\langle x_1, \ldots, x_n \rangle$. Tuples are often denoted with a bar such as $\bar{x} = (x_1, \ldots, x_n)$ while sequences are denoted with an arrow such as $\vec{x} = \langle x_1, \ldots, x_n \rangle$. If f is a function with input $(x_1, \ldots, x_n)$, we may indicate this relationship as $f(x_1, \ldots, x_n)$, $f(\bar{x})$, or $f(\vec{x})$. Such an identification is expedient when we come to the notion of a reduction.

An *indexed* collection of sets is a collection $\mathscr{C}$ of sets where there exist some nonempty set I and some function $F : I \to \mathscr{C}$ such that, for each $A \in \mathscr{C}$, there exists a $\xi \in I$ for which $F(\xi) = A$. The set I is called the *indexing set* of the collection.

In such a case, we often write the indexed collection as $(A_\xi)_{\xi\in I}$ or $\{A_\xi : \xi \in I\}$, where $A_\xi = F(\xi)$ for each $\xi \in I$.

Definition 1.1.1 (Algebra)**.** *An algebra is a structure of the form $((A_\xi)_{\xi\in I}, \mathscr{F})$, where $(A_\xi)_{\xi\in I}$ is an indexed collection of nonempty sets and $\mathscr{F}$ is a collection of operations, each having as domain a Cartesian product of some finitely many members of $(A_\xi)_{\xi\in I}$ and codomain a member of $(A_\xi)_{\xi\in I}$.*

If $((A_\xi)_{\xi\in I}, \mathscr{F})$ is an algebra, each member of the collection $(A_\xi)_{\xi\in I}$ is called a *phylum*. The plural of phylum is *phyla*. An algebra with exactly one phylum will be referred to as a *homogeneous* algebra if emphasis is needed; otherwise, the algebra is *heterogeneous*.

Assumption 1.1.1. *For the purposes of Ramsey algebras, we will assume that the phyla of any given algebra are* pairwise disjoint.

Whenever the number of phyla of an algebra is finite and few, we will often list them out instead of enclosing them within brackets as is done formally. The same abuse of notation is adopted when the family of operations are but a few members. Thus, for instance, the additive semigroup of the positive integers would have been written as $(\{\mathbb{N}\}, \{+\})$ if we were to abide to strict formality, but such a notation is most definitely pedantic and cumbersome. As such, we will write the structure concisely as $(\mathbb{N}, +)$ as is customary in the literature. Similarly, the real field is conveniently expressed as $(\mathbb{R}, +, \times)$.

1.2 Hindman's Theorem

Infinite Ramsey theory arguably begins with the following theorem:

Theorem 1.2.1 (Ramsey)**.** *For each $r,n \in \omega$ and each coloring $c : [\omega]^n \to \{1,\ldots,r\}$, there exists an infinite subset $H \subseteq \omega$ such that c is constant on $[H]^n$.*

An H with the property stated in the theorem is often known as a *homogeneous* set for the coloring. Theorems concerning the existence of certain homogeneous sets are often known as Ramsey-type theorems. The Ramsey-type theorem that motivated a series of questions leading up to the introduction of Ramsey algebras owes itself to Hindman.

Theorem 1.2.2 (Hindman [11])**.** *For each $m \in \mathbb{N}$ and each coloring $f : \mathbb{N} \to \{1,\ldots,m\}$, there exists an infinite $S \subseteq \mathbb{N}$ such that f is constant on the set $\{\sum_{i\in F} i : F \in [S]^{<\omega},\ F \neq \varnothing\}$.*

Hindman's theorem can be cast in terms of unions of finite sets as per the observation made by Graham and Rothschild (cf. [9]) and worked out by Milliken [16]:

Theorem 1.2.3. *For each $m \in \mathbb{N}$ and each coloring $f : \omega \to \{1,\ldots,m\}$, there exists $r \in \{1,\ldots,m\}$ and an infinite $X \subseteq [\omega]^{<\omega}$ such that the members of X are pairwise disjoint and, whenever $A \in [X]^{<\omega}$, we have $f(\bigcup A) = r$.*

Following the union-of-sets version of Hindman's theorem, it was then natural to ask if $[\omega]^{<\omega}$ can be replaced by $[\omega]^{\omega}$, the set of infinite subsets of the natural numbers. Erdös and Rado's answer was "no" [6]. Let $S \subseteq [\omega]^{\omega}$. We say that S is *Ramsey* if there exists an infinite $X \subseteq \omega$ such that $[X]^{\omega} \subseteq S$ or $[X]^{\omega} \subseteq [\omega]^{\omega} \setminus S$.

Theorem 1.2.4 (Erdös-Rado [6])**.** *There exists an $S \subseteq \mathscr{P}(\omega)$ that is not Ramsey.*

The set S exhibited by Erdös and Rado is nonconstructive with a use of the axiom of Choice. Due to this nonconstructive nature, Dana Scott had suggested that sufficiently constructive sets could well be Ramsey [8]. Galvin-Prikry and Silver showed that such is indeed the case. To state these results precisely, we equip $\mathscr{P}(\omega)$ with the usual product topology. Scott's notion of sufficient constructivity can now be construed as definability in the sense of descriptive set theory–a set is definable whenever it is in the Borel hierarchy or the projective hierarchy. For the definitions of the Borel, analytic, and projective sets and hierarchies, the reader is referred to [13].

Theorem 1.2.5 (Galvin-Prikry [8])**.** *Every Borel set is Ramsey.*

Theorem 1.2.6 (Silver [21])**.** *Every analytic set is Ramsey.*

The proof that Silver gave for his theorem uses the metamathematical method of *forcing*. In avoiding metamathematical methods, Ellentuck gave a proof of a topological nature.

1.3 The Ellentuck Space and Ramsey Spaces

In this section, we will describe the Ellentuck topology [5] and introduce the notion of a Ramsey space. Knowledge of basic topology is assumed and a good reference is [15]; a good discussion of the property of Baire, which appears in Ellentuck's Theorem (Theorem 1.3.1), can be found in [18].

For each $n \in \omega$ and each $A \in [\omega]^{\omega}$, let

$$[n,A]^{\omega} = \{B \in [\omega]^{\omega} : B \subseteq A \text{ and } B \text{ contains the first } n \text{ elements of } A\}. \tag{1.1}$$

The Ellentuck topology $\mathscr{T}_E$ is defined to be the topology generated by the sets of the form given by Eq. 1.1. The Ellentuck topology refines the product topology on $[\omega]^\omega$. We call $([\omega]^\omega, \mathscr{T}_E)$ the Ellentuck space. In the previous section, we have defined a set $S \in [\omega]^\omega$ to be Ramsey if there exists an infinite $X \subseteq \omega$ such that $[X]^\omega \subseteq S$ or $[X]^\omega \subseteq [\omega]^\omega \setminus S$. A set $S \in [\omega]^\omega$ is said to be *completely Ramsey* if, for each $n \in \omega$ and for each infinite $A \subseteq \omega$, there exists a $B \in [n,A]^\omega$ such that $[n,B]^\omega \subseteq S$ or $[n,B]^\omega \subseteq [\omega]^\omega \setminus S$.

Theorem 1.3.1 (Ellentuck [5])**.** *$S \in [\omega]^\omega$ is completely Ramsey if and only if S has the property of Baire under the Ellentuck topology.*

Silver's theorem now follows from Ellentuck's theorem as every analytic set has the property of Baire under the Ellentuck topology.

1.4 Topological Ramsey Spaces

Ellentuck's topological proof of Silver's theorem led Carlson into the development of a comprehensive combinatorial theory now called topological Ramsey spaces (cf. [3]). Carlson originally called these spaces simply as Ramsey spaces; the modern usage of the term Ramsey space is reserved to a more general framework for which topological Ramsey space is a special case. Nevertheless, we will adopt this shorter terminology *Ramsey space* throughout this dissertation for brevity and in concert with Carlson's terminology in [3].

In this section, we will give an account of Carlson's work on Ramsey space. We will give a rather detailed formulation of the terminologies for completeness sake, but which is otherwise not required for the rest of the dissertation. The key idea is encompasses in Theorem 1.4.1, the Abstract Ellentuck Theorem, on which the relation between Ramsey spaces and our objects of interest–Ramsey algebras–hinges upon and which is given in Theorem 1.5.1 in the next section.

We begin with the notion of a pre-ordering with approximations due to Carlson [3]. Suppose that R is a nonempty set, p is a function on $\omega \times R$, and $\leq$ is pre-ordering on R. A pre-order is a binary relation that is reflexive and transitive. For what follows in 1–3, let $A, B \in R$. We call $\mathscr{R} = (R, \leq, p)$ a *pre-ordering with approximations* if:

1. Approximations for every A, B start alike: $p(0,A) = p(0,B)$.
2. If $A \neq B$, then their approximations differ at some point: $p(n,A) \neq p(n,B)$ for some $n \in \omega$.
3. Once the approximations differ, they differ thereafter: If $p(n,A) = p(m,B)$, then $n = m$ and $p(i,A) = p(i,B)$ for each $i < n$.

Each pre-ordering with approximations $\mathscr{R}$ can be canonically identified with a structure consisting of infinite sequences, also equipped with a pre-ordering. Namely, for each $A \in R$, the corresponding infinite sequence $\vec{a}$ is the sequence of approximations of A: $p(0,A), p(1,A), \ldots$. The corresponding pre-ordering $\preceq$ is such that, if $\vec{a}$ corresponds to A and $\vec{b}$ corresponds to B, then $\vec{a} \preceq \vec{b}$ if and only if $A \leq B$. Therefore, $\mathscr{R} = (R, \leq, p)$ is isomorphic to a pre-ordering on a set of infinite sequences $(R^*, \preceq)$. Due to this isomorphic correspondence, we will formulate the notion of a Ramsey space in terms of infinite sequences as this has direct relevance to the notion of a Ramsey algebra, which we will formulate shortly in the next section.

Definition 1.4.1 (Natural Topology)**.** *Let R be a set of infinite sequences and let $\leq$ be* *a pre-ordering on R. For all $\vec{b} \in R$ and all $n \in \omega$, the sets*

$$[n, \vec{b}] = \{\vec{a} \in R : \vec{a} \leq \vec{b} \text{ and } \vec{a} \restriction n = \vec{b} \restriction n\}$$

form a neighborhood basis of what is known as the natural topology *on R.*

Definition 1.4.2. *Let R and $\leq$ be as defined in Definition 1.4.1 and let X be a subset of R. Then X is said to be* Ramsey *if, for each $\vec{b} \in R$ and for each $n \in \omega$, there exists an $\vec{a} \leq \vec{b}$ such that $[n, \vec{a}] \subseteq X$ or $[n, \vec{a}] \cap X = \varnothing$. In the event there exists an $\vec{a} \leq \vec{b}$ such that $[n, \vec{a}] \subseteq X^C$ whenever $\vec{b} \in R$ and $n \in \omega$, we say that X is* Ramsey null.

Definition 1.4.3 (Ramsey space)**.** *Let R and $\leq$ be as defined in Definition 1.4.1 and let R be equipped with the natural topology. Then R is said to be a Ramsey space if every set possessing the Baire property is Ramsey and every meager set is Ramsey null.*

Obviously, every Ramsey null set is Ramsey. In addition, in the presence of the Axiom of Choice, the latter property in the definition of a Ramsey space is redundant.

In order to state Carlson's necessary and sufficient condition for a space being Ramsey, we need the concept of finitization.

Definition 1.4.4 (Finitization)**.** *Let R and $\leq$ be as defined in Definition 1.4.1 and suppose that $\trianglelefteq$ is a pre-ordering on the set R' of initial segments of sequences in R. Then $\trianglelefteq$ is said to be a* finitization *of R if the following holds:*

for each $\vec{a}, \vec{b} \in R$, we have $\vec{a} \leq \vec{b}$ if and only if, for each $n \in \omega$, there exists an $N \in \omega$ such that $\vec{a} \restriction n \trianglelefteq \vec{b} \restriction N$.

Let R and $\leq$ again be as defined in Definition 1.4.1 and let $\trianglelefteq$ be a finitization on R. Let R' be the set of initial segments of the members of R. We single out two desirable conditions for finitizations:

(A1) For each $\vec{a} \in R$ and each $n \in \omega$, the set $\{\tau \in R' : \tau \trianglelefteq (\vec{a} \restriction n)\}$ is finite.

(A2) If $\vec{a} \in [n, \vec{b}]$ and $(\vec{b} \restriction n) \trianglelefteq (\vec{b} \restriction N)$ for some $N \in \omega$ but $(\vec{b} \restriction n) \trianglelefteq (\vec{b} \restriction i)$ for all $i < N$, then there exists an $\vec{a}' \in [N, \vec{b}]$ such that $[n, \vec{a}'] \subseteq [n, \vec{b}]$.

A third property is key to Carlson's necessary and sufficient condition:

(A3) Whenever $\vec{b} \in R$ and X is a set of initial segments of members of R of length $n+1$, there exists $\vec{a} \in [n, \vec{b}]$ such that the set consisting of all initial segments of sequences in $[n, \vec{a}]$ of length $n+1$ is either a subset of X or is disjoint from X.

Collectively, A1, A2, and A3 is an abstraction of the salient features appearing in the proof given by Ellentuck. We may now state the theorem due to Carlson, which also appears in [3].

Theorem 1.4.1 (Abstract Ellentuck Theorem)**.** *Let R and $\leq$ be as defined in Definition 1.4.1 and endow R with the natural topology. If R is closed and has finitization satisfying A1 and A2, then R is a Ramsey space if and only if A3 holds.*

1.5 Ramsey Algebra

The initial study of Ramsey algebras involves algebras of the homogeneous nature. This section marks the beginning of the doctoral study, which is to formulate the notion of a Ramsey algebra to include heterogeneous algebras.

We refer the definition of an algebra to Definition 1.1.1. The notion of a reduction is of utmost importance in Ramsey theory. For an algebra, it is given by an application of operations, which we now give precisely.

Definition 1.5.1 (Orderly Composition & Orderly Terms)**.** *Let $\mathscr{C}$ be a family of operations on $(A_\xi)_{\xi\in I}$. An operation F on $(A_\xi)_{\xi\in I}$ is said to be an* orderly composition *of $\mathscr{C}$ if there exists $f, f_1, \ldots, f_{||f||} \in \mathscr{C}$ such that*

1. *f_j is an n_j-ary operation for each $j \in \{1, \ldots, ||f||\}$,*
2. *$\sum_{j=1}^{||f||} n_j = ||F||$, and*
3. *if $\bar{x}_1 = (x_1, \ldots, x_{n_1})$ and $\bar{x}_j = \left(x_{\sum_{i=1}^{j-1} n_i+1}, \ldots, x_{\sum_{i=1}^{j} n_i}\right)$ for each $j = \{2, \ldots, ||f||\}$, then $F(x_1, \ldots, x_n) = f(f_1(\bar{x}_1), \ldots, f_{||f||}(\bar{x}_{||f||}))$.*

If $\mathscr{F}$ is a family of operations on $(A_\xi)_{\xi\in I}$, then the collection $\mathrm{OT}(\mathscr{F})$ *of* orderly terms *over $\mathscr{F}$ is the* smallest *collection $\mathscr{C}$ of operations containing $\mathscr{F} \cup \left\{\mathrm{id}_{A_\xi}\right\}_{\xi\in I}$ and is closed under orderly compositions.*

The definition of $\mathrm{OT}(\mathscr{F})$ can be given equivalently by recursion on F. We defer a precise statement of this recursion until Subsection 1.5.1.

Example 1.5.1. *Consider the addition $+$ and multiplication $\times$ operations on the integers. The composition $f(x_1,x_2,x_3,x_4) = +(\times(x_1,x_2), \times(x_3,x_4)) = x_1x_2 + x_3x_4$ is an orderly composition over $\{+,\times\}$. Another example is given by $g(x_1,x_2,x_3,x_4) = +(\times(+(x_1,x_2),x_3),x_4) = ((x_1+x_2)x_3)+x_4$. However, $h(x_1,x_2,x_3) = +(x_2,\times(x_1,x_3)) = x_2+x_1x_3$ is not an orderly composition.*

Definition 1.5.2 (Reduction $\le_{\mathscr{F}}$)**.** *Let $((A_\xi)_{\xi\in I},\mathscr{F})$ be an algebra and let $\vec{a}$ and $\vec{b}$ be members of ${}^{\omega}\left(\bigcup_{\xi\in I}A_\xi\right)$. Then $\vec{a}$ is said to be a* reduction *of $\vec{b}$ if there exist orderly terms f_j over $\mathscr{F}$ and finite subsequences $\vec{b}_j$ of $\vec{b}$ such that*

1. *$\vec{a}(j) = f_j(\bar{b}_j)$ for each $j \in \omega$ and*
2. *$\vec{b}_0 * \vec{b}_1 * \cdots$ forms a subsequence of $\vec{b}$.*

We write $\vec{a} \le_{\mathscr{F}} \vec{b}$ to mean $\vec{a}$ is a reduction of $\vec{b}$ with respect to $\mathscr{F}$.

If the family of operations $\mathscr{F}$ is clear from context, we will omit reference to $\mathscr{F}$, e.g. $\vec{a} \le \vec{b}$, and simply speak of $\vec{a}$ as being a reduction of $\vec{b}$.

It is easy to see that every subsequence $\vec{a}$ of a sequence $\vec{b}$ is a reduction of $\vec{b}$. In particular, the relation $\le_{\mathscr{F}}$ is reflexive. We note that the inclusion of the identity functions in the set of orderly terms is needed to ensure this property. In addition, $\le_{\mathscr{F}}$ is also transitive, hence $\le_{\mathscr{F}}$ is a pre-order on $R = {}^{\omega}\left((A_\xi)_{\xi\in I}\right)$. To see that $\le_{\mathscr{F}}$ is transitive, first let $\vec{a},\vec{b},\vec{c} \in {}^{\omega}\left((A_\xi)_{\xi\in I}\right)$ and suppose that $\vec{b} \le_{\mathscr{F}} \vec{c}$ and $\vec{a} \le_{\mathscr{F}} \vec{b}$. For each $i \in \omega$, let $\vec{c}_i$ be a subsequence of $\vec{c}$ and $f_i \in \mathrm{OT}(\mathscr{F})$ such that $\vec{b}(i) = f_i(\vec{c}_i)$ and $\vec{c}_0 * \vec{c}_1 * \cdots$ forms a subsequence of $\vec{c}$. Also, for each $n \in \omega$, let $\vec{b}_n$ be a finite subsequence of $\vec{b}$ and $F_n \in \mathrm{OT}(\mathscr{F})$ such that $\vec{a}(n) = F_n(\vec{b}_n)$ and $\vec{b}_0 * \vec{b}_1 * \cdots$ forms a subsequence of $\vec{b}$. Now, suppose that $\vec{b}_n = \langle \vec{b}(n_1),\ldots,\vec{b}(n_N)\rangle$, it follows that

$$\vec{a}(n) = F_n(\vec{b}_n) = F_n(\vec{b}(n_1),\ldots,\vec{b}(n_N)) = F_n(f_{n_1}(\vec{c}_{n_1}),\ldots,f_{n_N}(\vec{c}_{n_N})),$$

from which one can easily deduce that $\vec{a} \le_{\mathscr{F}} \vec{c}$.

Example 1.5.2. *Consider the ring* $(\mathbb{Z},+,\times)$ *of integers. If* $\vec{b}=\langle 1,2,3,\ldots\rangle$*, then the sequence* $\vec{a}$ *of integers given by* $\vec{a}(n)=f(\vec{b}(4n),\vec{b}(4n+1),\vec{b}(4n+2),\vec{b}(4n+3))$ *for each* $n\in\omega$*, where* $f(x_1,x_2,x_3,x_4)=+(\times(x_1,x_2),\times(x_3,x_4))=x_1x_2+x_3x_4$*, is a reduction of* $\vec{b}$ *since* f *is an orderly term over* $\{+,\times\}$*. That is,* $\vec{a}=\langle 12,86,222,\ldots\rangle\leq_{\{+,\times\}}\vec{b}$*.*

Definition 1.5.3. *Suppose* $((A_\xi)_{\xi\in I},\mathscr{F})$ *is an algebra and* $\vec{e}\in{}^{\omega}I$*. We say that* $\vec{a}\in{}^{\omega}\left(\bigcup_{\xi\in I}A_\xi\right)$ *is* $\vec{e}$-sorted *if* $\vec{a}(n)\in A_{\vec{e}(n)}$ *for each* $n\in\omega$ *and that* $\vec{e}$ *is the* sort *of* $\vec{a}$*.*

Under the assumption that the phyla $(A_\xi)_{\xi\in I}$ are pairwise disjoint, the sort of any sequence in ${}^{\omega}\left(\bigcup_{\xi\in I}A_\xi\right)$ is *unique*.

Definition 1.5.4. *If* $\vec{b}$ *is an* $\vec{e}$*-sorted sequence, define*

$$\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{b})=\left\{\vec{a}(0):\vec{a}\leq_{\mathscr{F}}\vec{b}\text{ and }\vec{a}\text{ is }\vec{e}\text{-sorted}\right\}.$$

When the algebra in question is homogeneous, in which case I is a singleton, the sort associated with the algebra is unique. As such, every reduction involves a pair of sequences of the same unique sort. Therefore, we will drop any reference to the sort in any discussion involving homogeneous algebras. Thus, for such an algebra $(A,\mathscr{F})$, Definition 1.5.4 has a simple characterization:

Definition 1.5.5. *Let* $(A,\mathscr{F})$ *be a homogeneous algebra. If* $\vec{b}\in{}^{\omega}A$*, then*

$$\begin{aligned}\mathrm{FR}_{\mathscr{F}}(\vec{b}) &= \left\{\vec{a}(0):\vec{a}\leq_{\mathscr{F}}\vec{b}\right\}\\ &= \{f(\tau):f\in\mathrm{OT}(\mathscr{F}),\tau\text{ is a finite subsequence of }\vec{b}\}.\end{aligned}$$

Example 1.5.3. *Consider again the ring of integers as in Example 1.5.2 and now* *let $\vec{b}$ be the sequence $\langle 1,1,\ldots\rangle$. Then the set $\mathrm{FR}_{\{+,\times\}}(\vec{b})$ equals the set of positive integers. For, the positive integers is closed under $+$ and $\times$, and $f(1,\ldots,1)$ is a positive integer for every $f \in \mathrm{OT}(\{+,\times\})$. Conversely, if p is a positive integer, then $p = f(1,\ldots,1)$, where f is the orderly term of the sum of p 1's. Similarly, if $\vec{c} = \langle 2,2,\ldots\rangle$, then $\mathrm{FR}_{\{+,\times\}}(\vec{c})$ is the set of even positive integers.*

We can finally state the definition of a Ramsey algebra.

Definition 1.5.6 ($\vec{e}$-Ramsey Algebra). *Let $\vec{e}$ be a sort. An algebra $((A_\xi)_{\xi\in I},\mathscr{F})$ is said to be an $\vec{e}$-Ramsey algebra if, for each $\vec{e}$-sorted sequence $\vec{b}$ and each $X \subseteq A_{\vec{e}(0)}$, there exists an $\vec{e}$-sorted reduction $\vec{a}$ such that $\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a}) \subseteq X$ or $\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a}) \cap X = \varnothing$.*

Such a sequence $\vec{a}$ is said to be homogeneous *for X (with respect to $\mathscr{F}$).*

Remark 1.5.1. *As is the case with most Ramsey theoretic results, the statement of Definition 1.5.6 can be stated in terms of finite coloring. Namely, for any given sort $\vec{e}$, the algebra $((A_\xi)_{\xi\in I},\mathscr{F})$ is an $\vec{e}$-Ramsey algebra if and only if, for each finite coloring of $A_{\vec{e}(0)}$ and each $\vec{e}$-sorted $\vec{b}$, there exists $\vec{a} \leq_{\mathscr{F}} \vec{b}$ such that $\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a})$ is monochromatic. This equivalence can be proved by a simple induction to the number of coloring.*

We now give a case for the relation between the notion of a Ramsey algebra and the notion of a Ramsey space, which appears in Section 4 of [28]:

Theorem 1.5.1. *Suppose $\left((A_\xi)_{\xi\in I},\mathscr{F}\right)$ is an algebra, where $\mathscr{F}$ is a finite family of* nonunary *operations. Then $\mathfrak{R}^{\vec{e}}\left((A_\xi)_{\xi\in I},\mathscr{F}\right)$ is a topological Ramsey space if and only if $((A_\xi)_{\xi\in I},\mathscr{F})$ is an $(\vec{e}-m)$-Ramsey algebra for each $m \in \omega$.*

In particular, if $((A_\xi)_{\xi\in I},\mathscr{F})$ is homogeneous, then $\mathfrak{R}^{\vec{e}}\left((A_\xi)_{\xi\in I},\mathscr{F}\right)$ is a topological Ramsey space if and only if $((A_\xi)_{\xi\in I},\mathscr{F})$ is a Ramsey algebra. For further details, the reader is referred to [28].

We end this subsection with a very important result of Carlson.

Example 1.5.4 (Ramsey space of variable words)**.** *The spaces of certain variable words over some finite alphabets are very important examples of Ramsey spaces given by Carlson himself in his unifying paper [3]. The operations involved in these example are "substitution" and "evaluation," which we will not phrase precisely and the interested readers are referred to Carlson's own article above. These examples are very important because, by choosing appropriate set of alphabets in each case, many classical combinatorial results such as Hindman's theorem and the Hale-Jewette's theorem [10] can be derived as corollaries.*

In the language of Ramsey algebras, the result on variable words can be stated as:

The algebra of variable words with finite alphabets equipped with the operations of "substitution" and "evaluation" is a Ramsey algebra.

1.5.1 Induction on the Generation of Orderly Terms

The collection of orderly terms over $\mathscr{F}$ can be stratified by the formation of orderly terms. As we mentioned in the paragraph following Definition 1.5.1, this stratification can be done in a more concrete and recursive manner. The collection $\mathrm{OT}(\mathscr{F})$ is, in fact, the collection of operations on $(A_\xi)_{\xi\in I}$ which can be generated by an application of finitely many of the following rules:

1. for each $\xi \in I$, the identity function id_{A_ξ} is an orderly term,

2. every operation in $\mathscr{F}$ is an orderly term,

3. if F is an operation on $(A_\xi)_{\xi\in I}$ given by $F(\bar{x}_1,\ldots,\bar{x}_k) = f(f_1(\bar{x}_1), \ldots, f_k(\bar{x}_k))$ for some $f \in \mathscr{F}$ and some orderly terms $f_1,\ldots,f_k$, then F is an orderly term.

We state this as a theorem below. It is important because almost every proof concerning properties of orderly terms hinges upon it. It allows us to prove properties about orderly terms by an induction on the generation of the orderly terms based on a concrete, recursive hierarchy. We call the theorem "Induction on the Generation of Orderly Terms" because this is the phrase we will cite when the method is invoked.

Theorem 1.5.2 (Induction on the Generation of Orderly Terms)**.** *Let $\mathscr{F}$ be a family of operations on $(A_\xi)_{\xi\in I}$. Every orderly term $F \in \mathrm{OT}(\mathscr{F})$ can be given by*

$$F(\bar{x}_1,\ldots,\bar{x}_k) = f(f_1(\bar{x}_1),\ldots,f_2(\bar{x}_k)) \tag{1.2}$$

for some k-ary $f \in \mathscr{F} \cup \{\mathrm{id}_{A_\xi} : \xi \in I\}$ and some $f_1,\ldots,f_k \in \mathrm{OT}(\mathscr{F})$.

Proof. We begin with two cases of F, the first being when $F = \mathrm{id}_{A_\xi}$ for some $\xi \in I$ and the other when $F(\bar{x}_1,\ldots,\bar{x}_k) \in \mathscr{F}$. In the first case, we take both f, f_1 to be the appropriate identity function so that $F(x) = f(f_1(x))$. In the second case, supposing that $\mathrm{Dom}(F) = A_{\xi_1} \times \cdots \times A_{\xi_k}$, then we let $f = F$ and $f_1(x_1) = \mathrm{id}_{A_{\xi_1}}, \ldots, f_k(x_k) = \mathrm{id}_{A_{\xi_k}}$. Thus, in such cases, we see that the conclusion of the theorem is satisfied.

Now, suppose $F' \in \mathrm{OT}(\mathscr{F})$ is a k-ary operation satisfying the conclusion of the theorem and F is given by

$$F(\bar{z}_1 * \cdots * \bar{z}_k) = F'(F_1(\bar{z}_1), \ldots, F_N(\bar{z}_k)), \tag{1.3}$$

where $F_1, \ldots, F_k \in \mathrm{OT}(\mathscr{F})$ also satisfying the conclusion of the theorem. Thus, let $f \in \mathscr{F} \cup \{\mathrm{id}_{A_\xi} : \xi \in I\}$ be n-ary and let $h_1, \ldots, h_n \in \mathrm{OT}(\mathscr{F})$ be such that

$$F'(y_1, \ldots, y_k) = f(h_1(\bar{y}_1), \ldots, h_n(\bar{y}_n)),$$

whereby, $n \leq k$. For clarity, $\bar{y}_1 = \left(y_1 * \cdots * y_{||h_1||}\right)$, $\bar{y}_2 = \left(y_{||h_1||+1}, \ldots, y_{||h_1||+||h_2||}\right)$, ..., $\bar{y}_n = \left(y_{\sum_{i=1}^{n-1}||h_i||+1}, \ldots, y_k\right)$. Therefore, the composition Eq. 1.3 is such that

$$\begin{aligned} & F'(F_1(\bar{z}_1), \ldots, F_N(\bar{z}_k)) \\ = \; & f\left(h_1\left(F_1(\bar{z}_1), \ldots, F_{||h_1||}\left(\bar{z}_{||h_1||}\right)\right), \ldots, h_n\left(F_{\sum_{i=1}^{n-1}||h_i||+1}\left(\bar{z}_{\sum_{i=1}^{n-1}||h_i||+1}\right), \ldots, F_k(\bar{z}_k)\right)\right) \end{aligned}$$

But, $h_1\left(F_1(\bar{z}_1), \ldots, F_{||h_1||}\left(\bar{z}_{||h_1||}\right)\right)$, ..., $h_n\left(F_{\sum_{i=1}^{n-1}||h_i||+1}\left(\bar{z}_{\sum_{i=1}^{n-1}||h_i||+1}\right), \ldots, F_k(\bar{z}_k)\right)$ are each a member of $\mathrm{OT}(\mathscr{F})$ and so it follows that $F(\bar{z}_1 * \cdots * \bar{z}_k)$ satisfies the conclusion of the theorem. □

The difference between Definition 1.5.1 and Theorem 1.5.2 is that, now, we may take f to be a member of $\mathscr{F}$ instead of some "previously" defined $f \in \mathrm{OT}(\mathscr{F})$. Thus, Theorem 1.5.2 furnishes a hierarchy $\mathscr{F}_N$ ($N \in \omega$) that is more concrete to work with:

Definition 1.5.7 (A Standard Stratification of Orderly Terms). *Let* $((A_\xi)_{\xi \in I}, \mathscr{F})$ *be an* *algebra. Define* $\mathscr{F}_0 = \mathscr{F} \cup \{\mathrm{id}_{A_\xi} : \xi \in I\}$ *and, whenever* $\mathscr{F}_N$ *is defined, set*

$$\mathscr{F}_{N+1} = \left\{ f(f_1(\bar{x}_1), \ldots, f_{||f||}(\bar{x}_{||f||})) : f \in \mathscr{F} \cup \{\mathrm{id}_{A_\xi} : \xi \in I\}, f_1, \ldots, f_{||f||} \in \mathscr{F}_N \right\}.$$

The statement of Theorem 1.5.2 implies that $\mathrm{OT}(\mathscr{F}) \subseteq \bigcup_{N\in\omega} \mathscr{F}_N$ while the reverse inclusion holds by the definition of $\mathrm{OT}(\mathscr{F})$. Therefore, $\mathrm{OT}(\mathscr{F}) = \bigcup_{N\in\omega} \mathscr{F}_N$.

1.5.2 Examples & Literature Review

Hindman's theorem is the classic example of a Ramsey algebra. On the other hand, we have the following:

Theorem 1.5.3. *No infinite field is a Ramsey algebra. In fact, no infinite ring without zero divisors can be a Ramsey algebra.*

The theorem can be found in [25]. The proof of the theorem hinges upon the following lemma, which is Lemma 5.4 of the same article.

Lemma 1.5.1. *Suppose* $(\mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}})$ *is an infinite field. There exists a sequence* $\vec{\beta} \in {}^{\omega}\mathbb{F}$ *such that for every orderly terms* f, g, f', g' *over* $\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}$ *and for every finite subsequences* $\vec{\beta}_0 * \vec{\beta}_1$ *and* $\vec{\beta}_2 * \vec{\beta}_3$ *of* $\vec{\beta}$*, the following holds:*

$$f(\bar{\beta}_0) +_{\mathbb{F}} g(\bar{\beta}_1) \neq f'(\bar{\beta}_2) \times_{\mathbb{F}} g'(\bar{\beta}_3). \tag{1.4}$$

The sequence $\vec{\beta}$ can be constructed by recursion. The fact that the procedure works has to do with the fact that the set Y below is always a *proper* subset of $\mathbb{F}$:

$$Y = \left\{ f(\bar{\beta}_0) +_{\mathbb{F}} g(\bar{\beta}_1) : \Psi(f,g) \right\}, \tag{1.5}$$

where $\Psi(f,g)$ is the statement "$f,g \in \mathrm{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$ and $\vec{\beta}_0 * \vec{\beta}_1$ is a finite subsequence of $\vec{\beta}$." For the specifics, the interested reader is referred to [25].

One may now arrive at the proof of the theorem using a sequence $\vec{\beta}$ given by the lemma and the associated subset Y. For, a moment's reflection reveals that, for each $f,g \in \mathrm{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$ and each finite subsequence $\vec{\beta}_0 * \vec{\beta}_1$ of $\vec{\beta}$, we have

$$f(\bar{\beta}_0) \times_{\mathbb{F}} g(\bar{\beta}_1) \notin Y.$$

Thus, given an arbitrary $\vec{a} \leq_{\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}} \vec{\beta}$, we have $\vec{a}(0) +_{\mathbb{F}} \vec{a}(1) \in Y$ and $\vec{a}(0) \times_{\mathbb{F}} \vec{a}(1) \notin Y$. Hence, no $\vec{a} \leq_{\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}} \vec{\beta}$ is homogeneous for Y, establishing the theorem.

We will make use of this theorem in the proofs of some results later.

The study of Ramsey algebras is a fairly recent endeavor. As mentioned above, Ramsey algebra is introduced as a purely combinatorial study of Carlson's work on topological Ramsey spaces. At its inception, the work on Ramsey algebras focused mainly on homogeneous algebras. There were a few general themes to the study during the initial stage of the subject. The first theme would be an attempt to understand the basic properties of Ramsey algebras and that includes an attempt to classify algebras according to their Ramsey properties. This dissertation is, for the most part, an extension of this theme to include also heterogeneous algebras; we will elaborate in the next section. The main results from previous works for this theme is [25], some of which are presented above as examples. [26] gives further examples and a formal first-order logical formulation of the subject, which we briefly cover in Section 2.3 of Chapter 2.

A second theme to the previous studies of Ramsey algebras has to do with a recurring theme in infinitary set theory, namely the relation between combinatorial objects and various ultrafilters. This theme is not pursued in the doctoral work, but existing work on the topic can be found in [23] and [25]. Such a theme is also featured in Carlson's work [3]. A third direction for the subject is an attempt to approach Ramsey algebras locally (a term that we will not elaborate) and this approach appears in [27]. We will have no occasion in the dissertation to treat this topic.

1.6 Objectives

The doctoral research can be broadly summarized as further developing the initiating work done on Ramsey algebras by Teh [22]. It has the following objectives:

1. Extend the notion of a Ramsey algebra to accommodate for heterogeneous algebras and to derive some basic results thereafter. Previous works on the subject only considered homogeneous or one-sorted algebras.
2. Classify algebras based on their Ramsey algebraic properties.
3. Construct new Ramsey algebras from given ones.

Below is a more detailed description.

We began the thesis by developing the notion of a heterogeneous Ramsey algebras, which has been done above in the previous subsections. Carlson's work on Ramsey spaces was broad enough to include algebras of the heterogeneous nature. The first study of Teh on Ramsey algebras was mainly concerned with algebras of the homogeneous nature. It was, therefore, natural that an extension of the work to heterogeneous algebras be carried out. Thus, following this introductory chapter on the formulation of a heterogeneous Ramsey algebra, the next chapter is dedicated to some elementary results of the topic, both pertaining to heterogeneous as well as homogeneous algebras.

In the third chapter, we put our focus on a very specific topic of how the property of being an associative algebra plays a role in determining whether a binary system is Ramsey or not. Our objective is to understand what role associativity plays in the said question. Associativity is a crucial property needed in the proof of Hindman's theorem. While we do not have a complete answer to the question, we showed that associativity is indeed somehow essential.

In the next two chapters (Chapters 4 & 5), we return to the heterogeneous setting, this time we study concrete algebraic structures, namely vector spaces and some matrix algebras. The purpose is to obtain a better idea of the nature of heterogeneous Ramsey algebras in a very specific sense. Concrete structures offer a glimpse into the general property of heterogeneous structures. A fortuitous outcome of this research presents us with a classification of vector spaces based on their Ramsey algebraic properties.

Chapter 6 is a continuation of the theme of Chapter 3, where we further explore the general properties of Ramsey algebras. In particular, we looked at some homomorphism theorems and the Ramsey algebraic properties of new algebras formed from old. In addition, we look at other ways to obtain new Ramsey algebras from existing ones and the highlight is an amalgam algebra obtained from any arbitrary family of existing Ramsey algebras. We also conclude the research with a study of algebras with some equivalence defined on them and that comprises the content of the penultimate chapter.

CHAPTER 2

SOME DEVELOPMENT OF THE GENERAL THEORY

In this chapter, we concern ourselves with some miscellaneous results of foundational nature. This chapter also sets the tone for arguments in the study of Ramsey algebras. A formulation of the subject from formal logic is also introduced in the final section. Some of the results of this section are presented in the paper [28] along with the results on vector spaces of Chapter 4.

2.1 Elementary Results

Throughout our study of heterogeneous algebras, a subset of the set of sorts of a given indexing set will be of particular interest. Given any algebra $\mathscr{A}$ with the indexing set I, we single out the following class of sorts:

$$\Omega = \{\vec{e} \in {}^{\omega}I : \text{if } \vec{e}(i) = \xi \text{ for some } i, \text{ then } |\{i : \vec{e}(i) = \xi\}| = \aleph_0\}.$$

The first term of a sort is of critical importance in reductions, we therefore further break the set Ω down such that, for each $\xi \in I$,

$$\Omega_\xi = \{\vec{e} \in \Omega : \vec{e}(0) = \xi\}.$$

As we will see, sequences of sorts belonging in Ω bear resemblance to properties familiar from homogeneous Ramsey algebras. For example, if $\vec{e} \in \Omega$, then

$$c \in \mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{b}) \Longleftrightarrow c = f(\tau) \tag{2.1}$$

for some $f \in \mathrm{OT}(\mathscr{F})$ with $\mathrm{Cod}(f) = A_{\vec{e}(0)}$ and some finite subsequence τ of $\vec{b}$. Eqv. 2.1 can be easily shown to hold. This characterization has as a special case that of a homogeneous algebra, namely Eq. 1.5.5.

Proposition 2.1.1. *Suppose that $\vec{e} \in \Omega$, $\vec{e}'$ is any sort such that $\vec{e}\,'(0) = \vec{e}(0)$, $\vec{a}$ is $\vec{e}$-sorted, $\vec{a}\,'$ is $\vec{e}\,'$-sorted, and $\vec{a}\,' \leq_{\mathscr{F}} \vec{a}$. Then, $\mathrm{FR}^{\vec{e}\,'}_{\mathscr{F}}(\vec{a}\,') \subseteq \mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a})$.*

Proof. Let $\vec{c}\,' \leq_{\mathscr{F}} \vec{a}\,'$ be $\vec{e}\,'$-sorted; we want to show that $\vec{c}\,'(0)$ is also a member of $\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a})$. To do this, note that $\vec{c}\,' \leq_{\mathscr{F}} \vec{a}$ by the transitivity of $\leq_{\mathscr{F}}$. Now, apply Eq. 2.1 and we see that $\vec{c}\,'(0)$ is indeed a member of $\mathrm{FR}^{\vec{e}}_{\mathscr{F}}(\vec{a})$. □

Let Ω^J be the set of those $\vec{e} \in \Omega$ such that all indices $\xi \in I$ appearing in $\vec{e}$ form the subset $J \subseteq I$ and let $\Omega^J_\eta = \left\{\vec{e} \in \Omega^J : \vec{e} \in \Omega_\eta\right\}$.

Theorem 2.1.1. *For any family $\mathscr{F}$ of operations, if $\mathscr{A} = \left(\bigcup_{\xi \in I} A_\xi, \mathscr{F}\right)$ is an $\vec{e}$-Ramsey algebra for an $\vec{e} \in \Omega^J_\eta$, then it is an $\vec{e}$-Ramsey algebra for all $\vec{e} \in \Omega^J_\eta$.*

Proof. Let $\vec{e}, \vec{e}\,' \in \Omega^J_\eta$ and suppose that $\mathscr{A}$ is an $\vec{e}$-Ramsey algebra. It suffices to prove that if $\mathscr{A}$ is an $\vec{e}$-Ramsey algebra, then it is an $\vec{e}\,'$-Ramsey algebra.

Let the $\vec{e}\,'$-sorted sequence $\vec{b}'$ and $X \subseteq A_\eta$ be given. We obtain an $\vec{e}$-sorted subsequence $\vec{b}$ of $\vec{b}'$, which is possible since $\vec{e}\,' \in \Omega^J$. Then, choose an $\vec{e}$-sorted $\vec{a} \leq_{\mathscr{F}} \vec{b}$