UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MSG 284 – Introduction to Geometric Modelling
[Pengenalan kepada Pemodelan Geometri]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

1. (a) Show that $\sum_{i=0}^{2} L_i^2(u) = 1$, where

$$L_i^2(u) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{(u-u_j)}{(u_i-u_j)}, \qquad i = 0, 1, 2,$$

are the Lagrange polynomials of degree 2.

(b) Let

$$\boldsymbol{R}(u) = \begin{cases} \boldsymbol{P}(u), & u \in [0, 1], \\ \boldsymbol{Q}(u), & u \in [1, 2], \end{cases}$$

be composed of two adjacent curve segments $\boldsymbol{P}$ and $\boldsymbol{Q}$ which can be represented locally as two quadratic Bézier curves

$$\boldsymbol{P}(t) = \sum_{i=0}^{2} \boldsymbol{C}_i B_i^2(t), \qquad t \in [0, 1],$$

$$\boldsymbol{Q}(t) = \sum_{i=0}^{2} \boldsymbol{D}_i B_i^2(t), \qquad t \in [0, 1],$$

where

$$B_i^2(t) = \frac{2!}{i!\,(2-i)!} t^i (1-t)^{2-i}, \qquad i = 0, 1, 2,$$

are the Bernstein polynomials of degree 2. $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$, $i = 0, 1, 2$, are the Bézier points and t is the local parameter of each curve segment. If $\boldsymbol{C}_0 = (1, 0)$, $\boldsymbol{C}_1 = (0, 1)$ and $\boldsymbol{C}_2 = (-1, 0)$, find the points $\boldsymbol{D}_i$, $i = 0, 1, 2$, such that $\boldsymbol{R}$ is C^2 continuous at $u = 1$.

(c) Given polynomial

$$\boldsymbol{P}(t) = \boldsymbol{V}_0 H_0(t) + \boldsymbol{T}_0 H_1(t) + \boldsymbol{T}_1 H_2(t) + \boldsymbol{V}_1 H_3(t), \qquad t \in [0, 1]$$

which interpolates the points $\boldsymbol{V}_i$ and their tangent vectors $\boldsymbol{T}_i$, $i = 0, 1$. Find the cubic functions $H_i(t)$, $t \in [0, 1]$, $i = 0, 1, 2, 3$, such that

$$\begin{array}{llll} H_0(0) = 1, & H_0'(0) = 0, & H_0(1) = 0, & H_0'(1) = 0, \\ H_1(0) = 0, & H_1'(0) = 1, & H_1(1) = 0, & H_1'(1) = 0, \\ H_2(0) = 0, & H_2'(0) = 0, & H_2(1) = 0, & H_2'(1) = 1, \\ H_3(0) = 0, & H_3'(0) = 0, & H_3(1) = 1, & H_3'(1) = 0. \end{array}$$

Symbol ' ′ ' indicates the derivative with respect to parameter t.

[100 marks]

1. (a) *Tunjukkan bahawa* $\sum_{i=0}^{2} L_i^2(u) = 1$, *di mana*

$$L_i^2(u) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{(u-u_j)}{(u_i-u_j)}, \qquad i = 0, 1, 2,$$

ialah polinomial Lagrange berdarjah 2.

(b) *Katakan*

$$\boldsymbol{R}(u) = \begin{cases} \boldsymbol{P}(u), & u \in [0, 1], \\ \boldsymbol{Q}(u), & u \in [1, 2], \end{cases}$$

digubah dengan dua segmen lengkung yang bersebelahan $\boldsymbol{P}$ and $\boldsymbol{Q}$. *Dua lengkung segmen ini dapat diwakili secara setempat sebagai lengkung Bézier kuadratik*

$$\boldsymbol{P}(t) = \sum_{i=0}^{2} \boldsymbol{C}_i B_i^2(t), \qquad t \in [0, 1],$$

$$\boldsymbol{Q}(t) = \sum_{i=0}^{2} \boldsymbol{D}_i B_i^2(t), \qquad t \in [0, 1],$$

di mana

$$B_i^2(t) = \frac{2!}{i!\,(2-i)!} t^i (1-t)^{2-i}, \qquad i = 0, 1, 2,$$

ialah polinomial Bernstein berdarjah 2. $\boldsymbol{C}_i$ *dan* $\boldsymbol{D}_i$, $i = 0, 1, 2$, *merupakan titik-titik Bézier dan* t *sebagai parameter setempat bagi setiap segmen lengkung. Jika* $\boldsymbol{C}_0 = (1, 0)$, $\boldsymbol{C}_1 = (0, 1)$ *dan* $\boldsymbol{C}_2 = (-1, 0)$, *cari titik-titik* $\boldsymbol{D}_i$, $i = 0, 1, 2$, *supaya* $\boldsymbol{R}$ *adalah selanjar secara* C^2 *pada* $u = 1$.

(c) *Diberi polinomial*

$$\boldsymbol{P}(t) = \boldsymbol{V}_0 H_0(t) + \boldsymbol{T}_0 H_1(t) + \boldsymbol{T}_1 H_2(t) + \boldsymbol{V}_1 H_3(t), \qquad t \in [0, 1]$$

yang menginterpolasi titik-titik $\boldsymbol{V}_i$ *dan vektor-vektor tangen* $\boldsymbol{T}_i$, $i = 0, 1$. *Cari fungsi-fungsi kubik* $H_i(t)$, $t \in [0, 1]$, $i = 0, 1, 2, 3$, *supaya*

$$\begin{array}{llll} H_0(0) = 1, & H_0'(0) = 0, & H_0(1) = 0, & H_0'(1) = 0, \\ H_1(0) = 0, & H_1'(0) = 1, & H_1(1) = 0, & H_1'(1) = 0, \\ H_2(0) = 0, & H_2'(0) = 0, & H_2(1) = 0, & H_2'(1) = 1, \\ H_3(0) = 0, & H_3'(0) = 0, & H_3(1) = 1, & H_3'(1) = 0. \end{array}$$

Simbol ' ′ ' *menandakan terbitan terhadap parameter* t.

[100 markah]

2. (a) Suppose a Bézier curve of degree n is defined by

$$\boldsymbol{P}(t) = \sum_{i=0}^{n} \boldsymbol{C}_i B_i^n(t), \qquad t \in [0, 1],$$

where

$$B_i^n(t) = \frac{n!}{i!\,(n-i)!} t^i (1-t)^{n-i}$$

are the Bernstein polynomials of degree n and $\boldsymbol{C}_i$ are the Bézier points of the curve.

(i) Show that $\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$.

(ii) Show that $\boldsymbol{P}(1) = \boldsymbol{C}_n$.

(iii) Prove that the curve $\boldsymbol{P}(t)$ lies within the convex hull of the control polygon $\boldsymbol{C}_0\boldsymbol{C}_1 \ldots \boldsymbol{C}_n$.

(iv) Let $n = 3$, $\boldsymbol{C}_0 = (1, 1)$, $\boldsymbol{C}_1 = (1, 0)$, $\boldsymbol{C}_2 = (-1, 1)$ and $\boldsymbol{C}_3 = (-1, 0)$. If the curve $\boldsymbol{P}(t)$ is degree raised by one to

$$\boldsymbol{P}(t) = \sum_{i=0}^{4} \boldsymbol{C}_i^* B_i^4(t), \qquad t \in [0, 1],$$

evaluate the new control points $\boldsymbol{C}_i^*$, $i = 0, 1, \ldots, 4$.

(b) Given a tensor product Bézier surface of degree $(2, 2)$

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \boldsymbol{C}_{i,j} B_j^2(v) B_i^2(u), \qquad 0 \le u, v \le 1.$$

(i) Find the cross boundary derivative along the boundary $v = 0$.

(ii) Find the normal vector of surface $\boldsymbol{S}$ at $(u, v) = (0, 0)$.

[100 marks]

2. (a) *Andaikan lengkung Bézier berdarjah* n *ditakrifkan sebagai*

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{C}_i B_i^n(t), \qquad t \in [0, 1],$$

di mana

$$B_i^n(t) = \frac{n!}{i!\,(n-i)!} t^i (1-t)^{n-i}$$

ialah polinomial Bernstein berdarjah n *dan* $\mathbf{C}_i$ *adalah titik-titik kawalan Bézier.*

(i) *Tunjukkan bahawa* $\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$.

(ii) *Tunjukkan bahawa* $\mathbf{P}(1) = \mathbf{C}_n$.

(iii) *Buktikan bahawa lengkung* $\mathbf{P}(t)$ *terletak di dalam hul cembung bagi poligon kawalan* $\mathbf{C}_0\mathbf{C}_1 \ldots \mathbf{C}_n$.

(iv) *Katakan* $n = 3$, $\mathbf{C}_0 = (1, 1)$, $\mathbf{C}_1 = (1, 0)$, $\mathbf{C}_2 = (-1, 1)$ *dan* $\mathbf{C}_3 = (-1, 0)$. *Jika lengkung* $\mathbf{P}(t)$ *ditingkatkan darjahnya dengan satu kepada*

$$\mathbf{P}(t) = \sum_{i=0}^{4} \mathbf{C}_i^* B_i^4(t), \qquad t \in [0, 1],$$

kira titik-titik kawalan yang baru $\mathbf{C}_i^*$, $i = 0, 1, \ldots, 4$.

(b) *Diberi suatu permukaan hasil darab tensor Bézier berdarjah* (2, 2)

$$\mathbf{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{C}_{i,j} B_j^2(v) B_i^2(u), \qquad 0 \le u, v \le 1.$$

(i) *Cari terbitan silang sempadan pada sempadan* $v = 0$.

(ii) *Cari vektor normal bagi permukaan* $\mathbf{S}$ *pada* $(u, v) = (0, 0)$.

[100 markah]

3. (a) Given a knot vector $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots, u_{n+k})$, $n \geq k-1$. A B-spline curve of order k is defined by

$$\boldsymbol{P}(u) = \sum_{i=0}^{n} \boldsymbol{D}_i N_i^k(u), \qquad u \in [u_{k-1}, u_{n+1}],$$

where $\boldsymbol{D}_i$, $i = 0, 1, \ldots, n$, are the de Boor points and $N_i^k(u)$ are the normalized B-splines of order k defined recursively as follow

$$N_i^1(u) = \begin{cases} 1, & u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad k > 1.$$

(i) Describe the conditions on the knot vector $\boldsymbol{u}$ so that the B-spline curve interpolates the first and the last de Boor points.

(ii) Let $k = 3$ and $\boldsymbol{u} = (0, 1, 2, 3, 4, 5)$. Show that $\boldsymbol{P}(u)$, $u \in [2, 3]$, can be expressed locally by

$$\frac{1}{2}\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_0 \\ \boldsymbol{D}_1 \\ \boldsymbol{D}_2 \end{bmatrix},$$

where $t = (u - 2) \in [0, 1]$.

(iii) Let $\boldsymbol{u} = (0, 1, 2, 3, 4, 5)$, $\boldsymbol{D}_0 = (0, 1)$, $\boldsymbol{D}_1 = (1, 1)$ and $\boldsymbol{D}_2 = (1, 0)$. Use the de Boor algorithm to evaluate the point on the B-spline curve of order 3 at $u = 2.5$.

(b) Given a bilinearly blended Coons patch $\boldsymbol{F}(u, v)$, $0 \leq u, v \leq 1$, that interpolates four boundary curves as

$$\boldsymbol{F}(u, 0) = (2u, \ \ 0, \ \ 1-u), \qquad u \in [0, 1],$$

$$\boldsymbol{F}(u, 1) = (2u, \ \ 2, \ \ \sin(\tfrac{\pi}{2}u)), \qquad u \in [0, 1],$$

$$\boldsymbol{F}(0, v) = (0, \ \ 2v, \ \ 1-v), \qquad v \in [0, 1],$$

$$\boldsymbol{F}(1, v) = (2+v-v^2, \ \ 2v, \ \ v), \qquad v \in [0, 1].$$

Evaluate the point on the patch $\boldsymbol{F}(u, v)$ when $(u = 0.5, v = 0.5)$.

[100 marks]

3. (a) *Diberi vektor knot* $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots, u_{n+k})$, $n \geq k-1$. *Splin-B berperingkat* k *ditakrif sebagai*

$$\boldsymbol{P}(u) = \sum_{i=0}^{n} \boldsymbol{D}_i N_i^k(u), \qquad u \in [u_{k-1}, u_{n+1}],$$

di mana $\boldsymbol{D}_i$, $i = 0, 1, \ldots, n$, *merupakan titik-titik de Boor dan* $N_i^k(u)$ *ialah splin-B ternormal berperingkat* k *yang ditakrif seperti berikut*

$$N_i^1(u) = \begin{cases} 1, & u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

dan

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad k > 1.$$

(i) *Nyatakan syarat pada vektor knot* $\boldsymbol{u}$ *supaya lengkung splin-B menginterpolasi titik pertama dan titik akhir de Boor.*

(ii) *Andaikan* $k = 3$ *dan* $\boldsymbol{u} = (0, 1, 2, 3, 4, 5)$. *Tunjukkan bahawa* $\boldsymbol{P}(u)$, $u \in [2, 3]$, *boleh diungkapkan secara setempat sebagai*

$$\frac{1}{2}\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_0 \\ \boldsymbol{D}_1 \\ \boldsymbol{D}_2 \end{bmatrix},$$

di mana $t = (u-2) \in [0, 1]$.

(iii) *Katakan* $\boldsymbol{u} = (0, 1, 2, 3, 4, 5)$, $\boldsymbol{D}_0 = (0, 1)$, $\boldsymbol{D}_1 = (1, 1)$ *dan* $\boldsymbol{D}_2 = (1, 0)$. *Gunakan algoritma de Boor untuk menilai titik lengkung splin-B berperingkat 3 pada* $u = 2.5$.

(b) *Diberi satu tampalan Coons teraduan bilinear* $\boldsymbol{F}(u, v)$, $0 \leq u, v \leq 1$, *yang menginterpolasi empat sempadan*

$$\boldsymbol{F}(u, 0) = (2u, \ 0, \ 1-u), \qquad u \in [0, 1],$$

$$\boldsymbol{F}(u, 1) = (2u, \ 2, \ \sin(\tfrac{\pi}{2}u)), \qquad u \in [0, 1],$$

$$\boldsymbol{F}(0, v) = (0, \ 2v, \ 1-v), \qquad v \in [0, 1],$$

$$\boldsymbol{F}(1, v) = (2+v-v^2, \ 2v, \ v), \qquad v \in [0, 1].$$

Nilaikan titik pada tampalan $\boldsymbol{F}(u, v)$ *apabila* $(u = 0.5, v = 0.5)$.

[100 markah]

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