UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MGM563 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

[Arahan : *Jawab* ***semua empat*** *[4] soalan].*

1. (a) Let X and Y be two random variables with joint probability mass function (pmf) $f(x,y) = \left(\frac{1}{2}\right)^{x+y}$, for $x = 1, 2, \ldots, y = 1, 2, \ldots$, and zero elsewhere.

   (i) Determine the joint moment generating function (mgf) of X and Y.

   (ii) Show that $M(t_1,t_2) = M(t_1,0)M(0,t_2)$.

[20 marks]

(b) Let $f(x,y) = 21x^2y^3$, $0<x<y<1$, and zero elsewhere, be the joint probability density function (pdf) of X and Y.

   (i) Find the conditional mean of X, given $Y = y$, for $0<y<1$.

   (ii) Find the distribution of $Z = E(X \mid Y)$.

[30 marks]

(c) Let X and Y be two independent random variables, whose marginal pdfs are given as follows:

$$f_X(x) = 1, \ 0 \le x \le 1$$

and

$$f_Y(y) = 1, \ 0 \le y \le 1.$$

Find the pdf of $W = X + Y$.

[30 marks]

(d) Assume that $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a $N(0,1)$ distribution and $Y_1, Y_2, \ldots, Y_m$ denote a random sample of size m from a $N(0,1)$ distribution. If the two samples are independent, find the distribution of the following statistics:

   (i) $\overline{X} - \overline{Y}$

   (ii) $\frac{1}{2}(X_n - Y_m)^2$

   (iii) $\dfrac{\sqrt{2mn}\left(\overline{X} - \overline{Y}\right)}{\sqrt{m+n}\left(X_n - Y_m\right)}$

[20 marks]

1. (*a*) *Biarkan X dan Y sebagai dua pembolehubah rawak dengan fungsi jisim kebarangkalian (fjk)* $f(x,y)=\left(\frac{1}{2}\right)^{x+y}$, *untuk* $x = 1, 2, \ldots, y = 1, 2, \ldots$, *dan sifar di tempat lain.*

   (i) *Cari fungsi penjana momen (fpm) tercantum untuk X dan Y.*

   (ii) *Tunjukkan bahawa* $M(t_1,t_2)=M(t_1,0)M(0,t_2)$.

   [20 markah]

   (*b*) *Biarkan* $f(x,y)=21x^2y^3$, $0<x<y<1$, *dan sifar di tempat lain, sebagai fungsi ketumpatan kebarangkalian (fkk) tercantum untuk X dan Y.*

   (i) *Cari min bersyarat X, diberi* $Y=y$, *untuk* $0<y<1$.

   (ii) *Cari taburan untuk* $Z=E(X\mid Y)$.

   [30 markah]

   (*c*) *Biarkan X dan Y sebagai dua pembolehubah rawak tak bersandar dengan fkk sut masing-masing diberikan seperti berikut:*

   $$f_X(x)=1,\ 0\le x\le 1$$

   dan

   $$f_Y(y)=1,\ 0\le y\le 1.$$

   Cari fkk untuk $W=X+Y$.

   [30 markah]

   (*d*) *Andaikan bahawa* $X_1,X_2,\ldots,X_n$ *mewakili suatu sampel rawak saiz n daripada taburan N(0,1) dan* $Y_1,Y_2,\ldots,Y_m$ *mewakili suatu sampel rawak saiz m daripada taburan N(0,1). Jika kedua-dua sampel itu adalah tak bersandar, cari taburan untuk statistik berikut:*

   (i) $\overline{X}-\overline{Y}$

   (ii) $\frac{1}{2}(X_n-Y_m)^2$

   (iii) $\dfrac{\sqrt{2mn}\left(\overline{X}-\overline{Y}\right)}{\sqrt{m+n}\left(X_n-Y_m\right)}$

   [20 markah]

2. (a) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x) = 2x$, $0 < x < 1$, and zero elsewhere.
   (i) Find the joint pdf of Y_3 and Y_4.
   (ii) Find the conditional pdf of Y_3, given $Y_4 = y_4$.
   (iii) Find $E(Y_3 \mid y_4)$.

[30 marks]

(b) Let W_n represent a random variable with mean μ and variance b/n^p, where $p > 0$, μ and b are constants (not functions of n). Prove that W_n converges in probability to μ.
Hint: Use the Chebyshev's inequality

[20 marks]

(c) Find a formula for the method of moments estimate for the parameter θ in the following pdf:

$$f_Y(y;\theta) = \theta k^{\theta}\left(\frac{1}{y}\right)^{\theta+1} ; y \geq k,\ \theta \geq 1$$

Assume that k is known and the data consist of a random sample of size n.

[20 marks]

(d) Let $X_1, X_2, \ldots, X_n$ represent a random sample from the Be(θ) distribution. If the prior distribution of Θ is given by $g_{\Theta}(\theta) = I_{(0,1)}(\theta)$, find the posterior Bayes estimator of $\theta(1-\theta)$ with respect to the prior pdf $g_{\Theta}(\theta)$.

[30 marks]

2. (a) *Biarkan* $Y_1 < Y_2 < Y_3 < Y_4$ *sebagai statistik tertib untuk suatu sampel rawak saiz* $n = 4$ *daripada taburan dengan fkk* $f(x) = 2x$, $0 < x < 1$, *dan sifar di tempat lain.*

   (i) *Cari fkk tercantum* Y_3 *dan* Y_4.

   (ii) *Cari fkk bersyarat* Y_3, *diberi* $Y_4 = y_4$.

   (iii) *Cari* $E(Y_3 \mid y_4)$.

[30 markah]

(b) *Biarkan* W_n *mewakili suatu pembolehubah rawak dengan min* μ *dan varians* b/n^p, *yang mana* $p > 0$, μ *dan* b *adalah pemalar (bukan fungsi n). Buktikan bahawa* W_n *menumpu secara kebarangkalian kepada* μ.
Petua: Gunakan ketaksamaan Chebyshev

[20 markah]

(c) *Cari suatu formula untuk anggaran kaedah momen bagi parameter* θ *dalam fkk berikut:*

$$f_Y(y;\theta) = \theta k^{\theta} \left(\frac{1}{y}\right)^{\theta+1} \; ; y \geq k, \; \theta \geq 1$$

Andaikan bahawa k diketahui dan data terdiri daripada suatu sampel rawak saiz n.

[20 markah]

(d) *Biarkan* $X_1, X_2, \ldots, X_n$ *mewakili suatu sampel rawak daripada taburan* $Be(\Theta)$. *Jika taburan prior bagi* Θ *diberi oleh* $g_{\Theta}(\theta) = I_{(0,1)}(\theta)$, *cari penganggar Bayes posterior untuk* $\theta(1-\theta)$ *terhadap fkk prior* $g_{\Theta}(\theta)$.

[30 markah]

3. (a) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0$.

(i) Find the Cramer-Rao lower bound for the variance of unbiased estimators of θ.

(ii) Is $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ a uniformly minimum variance unbiased estimator (UMVUE) of θ?

[30 marks]

(b) Show that $\prod_{i=1}^{n} X_i$ is a sufficient statistic for θ if the random sample is taken from the gamma distribution with the following pdf:

$$f(x_i;\theta) = \frac{x_i^{\theta-1} e^{-x_i/6}}{\Gamma(\theta) 6^{\theta}}$$

[20 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample of size *n* having the following pdf:

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta}, & 0 < x < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

(i) Is $f(x;\theta)$ a member of the exponential class of pdfs? Explain.

(ii) Find a complete sufficient statistic for θ.

[20 marks]

(d) Let $X_1, X_2, ..., X_n$ be independently and identically distributed (iid) random variables with common pdf

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta; \ -\infty < \theta < \infty \\ 0 & \text{, elsewhere} \end{cases}.$$

Let $Y_1 = \min(X_1, X_2, ..., X_n)$. Show that Y_1 is a consistent estimator of θ.

[30 marks]

3. (a) Biarkan $X_1, X_2, \ldots, X_n$ sebagai suatu sampel rawak daripada taburan dengan $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$.

(i) Cari batas bawah Cramer-Rao untuk varians penganggar saksama θ.

(ii) Adakah $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ suatu penganggar saksama bervarians minimum secara seragam (PSVMS) bagi θ?

[30 markah]

(b) Tunjukkan bahawa $\prod_{i=1}^{n} X_i$ ialah suatu statistik cukup untuk θ jika sampel rawak diambil daripada taburan gama dengan fkk berikut:

$$f(x_i;\theta) = \frac{x_i^{\theta-1} e^{-x_i/6}}{\Gamma(\theta) 6^{\theta}}$$

[20 markah]

(c) Andaikan bahawa $X_1, X_2, \ldots, X_n$ ialah suatu sampel rawak saiz n dengan fkk berikut:

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta}, 0 < x < 1 \\ 0 \quad , \text{di tempat lain} \end{cases}$$

(i) Adakah $f(x;\theta)$ fkk ahli kelas eksponen? Jelaskan.

(ii) Cari suatu statistik cukup lengkap bagi θ.

[20 markah]

(d) Biarkan $X_1, X_2, \ldots, X_n$ sebagai pembolehubah rawak yang tak bersandar dan bertaburan secara secaman dengan fkk sepunya

$$f(x) = \begin{cases} e^{-(x-\theta)}, x > \theta; \ -\infty < \theta < \infty \\ 0 \quad , \text{di tempat lain} \end{cases}.$$

Biarkan $Y_1 = \min(X_1, X_2, \ldots, X_n)$. Tunjukkan bahawa Y_1 ialah penganggar konsisten bagi θ.

[30 markah]

4. (a) Let $\overline{X}$ denote the mean of a random sample of size 25 from a gamma, $G(\alpha, \beta)$ distribution with the following pdf:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & 0 < x < \infty \\ 0 & , \text{elsewhere} \end{cases}$$

Here, $E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$. Solve the following questions if $\alpha = 4$ and $\beta > 0$.

(i) Use the central limit theorem to find an approximate pivotal quantity.

(ii) Use the approximate pivotal quantity in no. (i) to find an 95.4% approximate confidence interval for β.

[20 marks]

(b) Let X have a normal distribution with mean θ and standard deviation 5000. Consider the simple hypothesis $H_0: \theta = 30000$ and the alternative composite hypothesis $H_1: \theta > 30000$. We shall observe n independent values of X, say $x_1, x_2, \ldots, x_n$, and we shall reject H_0 if and only if $\bar{x} \geq c$.

(i) Find the power function $\pi(\theta)$ of the test.

(ii) Determine n and c so that $\pi(\theta)$ has the values $\pi(30000) = 0.01$ and $\pi(35000) = 0.98$.

[30 marks]

(c) Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a Bernoulli distribution with pmf $f(x;\theta) = \theta^x (1-\theta)^{1-x}$, x = 0, 1, and zero elsewhere. Consider testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$. Find the uniformly most powerful (UMP) test for testing the above hypotheses.

[20 marks]

(d) Let $X_1, X_2, \ldots, X_{10}$ be a random sample from an exponential pdf $f(x;\lambda) = \lambda e^{-\lambda x}$, $x > 0$, and zero elsewhere. Find the generalized likelihood-ratio test for testing $H_0: \lambda = \lambda_0$ vs. $H_1: \lambda \neq \lambda_0$. Explain how is the critical value determined if the size of the Type-I error, α is 0.05.

[30 marks]

4. (a) *Biarkan* $\overline{X}$ *mewakili min suatu sampel rawak saiz 25 daripada taburan gama,* $G(\alpha, \beta)$ *dengan fkk berikut:*

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}, & 0<x<\infty\\ 0 & \text{, di tempat lain}\end{cases}$$

Di sini, $E(X) = \alpha\beta$ *dan* $Var(X)=\alpha\beta^2$. *Selesaikan soalan berikut jika* $\alpha = 4$ *dan* $\beta >$ 0.

(i) *Gunakan teorem had memusat untuk mencari suatu kuantiti pangsian hampiran.*

(ii) *Gunakan kuantiti pangsian hampiran dalam no. (i) untuk mencari selang keyakinan hampiran 95.4% untuk* β.

[20 markah]

(b) *Biarkan X mempunyai taburan normal dengan min* θ *dan sisihan piawai 5000. Pertimbangkan hipotesis ringkas* $H_0:\theta=30000$ *dan hipotesis gubahan alternatif* $H_1:\theta>30000$. *Kita akan memperhatikan n nilai tak bersandar untuk X, katakan* $x_1,x_2,\ldots,x_n$, *dan kita akan menolak* H_0 *jika dan hanya jika* $\overline{x}\geq c$.

(i) *Cari fungsi kuasa* $\pi(\theta)$ *untuk ujian tersebut.*

(ii) *Tentukan nilai-nilai n dan c supaya* $\pi(30000) = 0.01$ *dan* $\pi(35000) = 0.98$.

[30 markah]

(c) *Biarkan* $X_1,X_2,\ldots,X_n$ *mewakili suatu sampel rawak saiz n daripada taburan Bernoulli dengan fjk* $f(x;\theta)=\theta^x(1-\theta)^{1-x}$, $x = 0, 1$, *dan sifar di tempat lain. Pertimbangkan pengujian* $H_0:\theta=\theta_0$ *lawan* $H_1:\theta>\theta_0$. *Cari ujian paling berkuasa secara seragam (PBS) bagi menguji hipotesis di atas.*

[20 markah]

(d) *Biarkan* $X_1,X_2,\ldots,X_{10}$ *sebagai suatu sampel rawak daripada fkk eksponen* $f(x;\lambda)=\lambda e^{-\lambda x}, x>0$, *dan sifar di tempat lain. Cari ujian nisbah kebolehjadian teritlak bagi menguji* $H_0:\lambda=\lambda_0$ *lawan* $H_1:\lambda\neq\lambda_0$. *Jelaskan bagaimana nilai genting dicari jika saiz ralat Jenis-I,* α *ialah 0.05.*

[30 markah]

Lampiran

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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