UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

# MAT 516 – Curve and Surface Methods for CAGD
# *[Kaedah Lengkung dan Permukaan untuk RGBK]*

Duration : 3 hours
*[Masa : 3 jam]*

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Please ensure that this examination paper contains NINE printed pages before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions**: Answer **all three** [3] questions.

*[**Arahan**: Jawab **semua tiga** [3] soalan.]*

1. The $n$th degree Bezier curve is defined as $P(t) = \sum_{i=0}^{n} V_i B_i^n(t),\quad 0 \le t \le 1$ with $B_i^n(t) = \frac{n!}{(n-i)!i!} t^i (1-t)^{n-i}$ and $V_i$, its Bezier control point.

(a) Show that

i. $B_i^n(t) = \frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{i+1}{n+1} B_{i+1}^{n+1}(t).$

ii. $(1-t) B_i^{n-1}(t) + t B_{i-1}^{n-1}(t) = B_i^n(t).$

iii. $\frac{dB_i^n(t)}{dt} = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$, thus the hodograph curve is $P'(t) = n \sum_{i=0}^{n-1} (V_{i+1} - V_i) B_i^{n-1}(t).$

(b) i. Prove that the control points for the corresponding Bezier curve of degree $n+1$ are $V_0' = V_0, V_{n+1}' = V_n$, and $V_i' = \frac{i}{n+1} V_{i-1} + (1 - \frac{i}{n+1}) V_i$, for $i = 1, 2, \cdots, n$. Write each $V_i'$ in term of $V_i$ and $V_{i-1}$ when $n = 4$.

ii. Discuss the parametric and geometric continuity of Bezier curves. Write the respective equations and sketch the position of the control points of two adjacent cubic Bezier curves which meet at the common control point with $C^1$, $C^2$, $G^1$ and $G^2$ continuity.

[100 marks]

1. *Lengkung Bezier berdarjah n ditakrif sebagai* $P(t) = \sum_{i=0}^{n} V_i B_i^n(t), \quad 0 \le t \le 1$ *dengan* $B_i^n(t) = \dfrac{n!}{(n-i)!i!} t^i (1-t)^{n-i}$ *dan* $V_i$ *adalah titik kawalan Bezier.*

(a) *Tunjukkan bahawa*

i. $B_i^n(t) = \dfrac{n+1-i}{n+1} B_i^{n+1}(t) + \dfrac{i+1}{n+1} B_{i+1}^{n+1}(t).$

ii. $(1-t) B_i^{n-1}(t) + t B_{i-1}^{n-1}(t) = B_i^n(t).$

iii. $\dfrac{dB_i^n(t)}{dt} = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$, *dengan demikian lengkung hodograf ialah* $P'(t) = n\sum_{i=0}^{n-1} (V_{i+1} - V_i) B_i^{n-1}(t).$

(b) i. *Buktikan bahawa titik kawalan lengkung Bezier sepadan berdarjah* $n+1$ *adalah* $V_0' = V_0, V_{n+1}' = V_n$, *dan* $V_i' = \dfrac{i}{n+1} V_{i-1} + (1 - \dfrac{i}{n+1}) V_i$, *untuk* $i = 1, 2, \cdots, n$. *Tulis setiap* $V_i'$ *dalam sebutan* $V_i$ *dan* $V_{i-1}$ *apabila* $n = 4$.

ii. *Bincang keselanjaran berparameter dan keselanjaran geometri lengkung Bezier. Tulis persamaan dan lakarkan kedudukan titik-titik kawalan dua lengkung Bezier kubik bersebelahan yang masing-masingnya memenuhi keselanjaran* $C^1$, $C^2$, $G^1$ *and* $G^2$ *pada titik kawalan sepunya.*

*[100 markah]*

2. (a) Write the Aitken algorithm to construct a polynomial **p** which satisfies the interpolatory constraints $\mathbf{p}(t_i) = \mathbf{p}_i;\ i = 0,\cdots,n$ starting from data points $\mathbf{p}_i = p_i^0$ with corresponding parameter values $t_i$. Arrange the points $p_i^r;\ r = 1,\cdots,n;\ i = 0,\cdots,n-r$ in a systolic array for the case $n = 4$.

(b) A quadratic rational Bezier curve is defined as $r(t) = \dfrac{\sum_{i=0}^{2} w_i p_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}$, $0 \le t \le 1$ with the control points, $p_i$ and corresponding weights $w_i$, $i = 0,1,2$.

i. Show that when every $w_i = 1$, the curve reduces to a quadratic Bezier curve.

ii. Let the set of weights be $\{w_0 = 1, w_1 = 0, w_2 = 1\}$. Show that the curve is just the straight line between $p_0$ and $p_1$.

iii. The ratio $\dfrac{w_1^2}{w_0 w_2}$ which is known as the conic shape factor will determine a specific type of conic curve. State the appropriate interval and value of this ratio in order to get an ellipse, a parabola and a hyperbola. Then state the conditions which must prevail in order to obtain a circular arc.

[100 marks]

2. (a) *Tulis algoritma Aitken bagi membina satu polinomial* $\mathbf{p}$ *yang memenuhi kekangan interpolasi* $\mathbf{p}(t_i) = \mathbf{p}_i;\ i = 0,\cdots,n$ *bermula dari titik data* $\mathbf{p}_i = p_i^0$ *dengan nilai parameter sepadan* $t_i$. *Susun titik-titik* $p_i^r;\ r = 1,\cdots,n;\ i = 0,\cdots,n-r$ *tersebut dalam tatasusunan sistol bagi kes* $n = 4$.

(b) *Lengkung Bezier nisbah kuadratik ditakrifkan sebagai*

$$r(t) = \frac{\sum_{i=0}^{2} w_i p_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad 0 \le t \le 1$$

*dengan* $p_i$, *titik kawalan dan* $w_i$, *pemberat untuk* $i = 0,1,2$.

i. *Tunjukkan bahawa apabila kesemua* $w_i = 1$, *lengkung tersebut terturun kepada lengkung Bezier kuadratik.*

ii. *Andaikan set pemberat adalah* $\{w_0 = 1,\ w_1 = 0,\ w_2 = 1\}$. *Tunjukkan bahawa lengkung adalah hanya garis lurus di antara* $p_0$ *dan* $p_1$.

iii. *Nisbah* $\frac{w_1^2}{w_0 w_2}$ *yang dikenali sebagai faktor bentuk kon akan menentukan jenis lengkung kon tertentu. Nyatakan selang atau nilai nisbah yang sesuai untuk memperoleh suatu elips, parabola dan hiperbola. Seterusnya, nyatakan syarat yang mesti dipenuhi untuk mendapatkan satu lengkok membulat.*

*[100 markah]*

3. (a) A B-spline curve of order $k$ is defined by $P(t) = \sum_{i=0}^{n} V_i N_{i,k}(t), t \in [t_{k-1}, t_{n+1}]$. $V_i, i = 0,1,\cdots,n$ are its control points, $T = \{t_0, t_1, \cdots, t_{n+k}\}$ is a knot vector $(n \geq k-1)$ and $N_{i,k}(t)$ are the normalized B-spline basis functions of order $k$ defined recursively by

$$N_{i,1}(t) = \begin{cases} 1 , & \text{if} \quad t \in [t_i, t_{i+1}) \\ 0 , & \text{otherwise} \end{cases}$$

and

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t), \text{ where } 0 \leq i \leq n.$$

i. Explain the conditions on $\{t_0, t_1, t_2, \ldots, t_{n+k}\}$ so that the B-spline curve of order $k$ (or degree $k$-1) interpolates the first and last control points.

ii. If {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is the given knot vector, sketch the basis functions and state the maximum number of control points and the interval for $t$ in order to generate a B-spline curve of order 4 (or of degree 3).

iii. If {0, 1, 2, 3, 4, 5} is the knot vector, find $N_{i,3}(t)$ for i= 0, 1, 2. By using an appropriate transformation of $N_{i,3}(t)$, show that $P(t) = \sum_{i=0}^{2} V_i N_{i,3}(t)$, can be expressed locally by

$$P(t) = \frac{1}{2}\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} , \ t \in [0, 1).$$

Then, show that $P(t)$ and $Q(t) = \sum_{i=1}^{3} V_i N_{i,3}(t), \quad t \in [0,1)$ are connected with $C^1$ continuity.

3. (a) *Satu lengkung splin-B peringkat k ditakrifkan oleh* $P(t) = \sum_{i=0}^{n} V_i N_{i,k}(t), t \in [t_{k-1}, t_{n+1}]$. $V_i, i = 0,1,\cdots,n$ *adalah titik kawalannya,* $T = \{t_0, t_1, \cdots, t_{n+k}\}$ *ialah vektor simpulan* $(n \geq k-1)$ *dan* $N_{i,k}(t)$ *adalah fungsi asas splin-B ternormal peringkat k ditakrif secara rekursi oleh*

$$N_{i,1}(t) = \begin{cases} 1 & , \quad jika \quad t \in [t_i, t_{i+1}) \\ 0 & , \quad selainnya \end{cases}$$

*dan*

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t), \text{ dengan } 0 \leq i \leq n.$$

i. *Terangkan syarat-syarat ke atas* $\{t_0, t_1, t_2, \ldots, t_{n+k}\}$ *supaya lengkung splin-B peringkat k (atau darjah k-1) menginterpolasi titik kawalan yang pertama dan titik kawalan yang terakhir.*

ii. Jika $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *adalah vektor simpulan, lakar fungsi asas dan nyata bilangan maksimum titik kawalan dan selang bagi t untuk menjana lengkung splin-B peringkat* 4 *(atau darjah* 3*).*

iii. *Jika* $\{0, 1, 2, 3, 4, 5\}$ *adalah vektor simpulan, cari* $N_{i,3}(t)$ *bagi* $i = 0, 1, 2$. *Dengan menggunakan transformasi yang sesuai untuk* $N_{i,3}(t)$, *tunjukkan bahawa* $P(t) = \sum_{i=0}^{2} V_i N_{i,3}(t)$, *boleh diwakili secara setempat oleh*

$$P(t) = \frac{1}{2}\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix}, \quad t \in [0,1).$$

*Seterusnya, tunjukkan bahawa P(t) dan* $Q(t) = \sum_{i=1}^{3} V_i N_{i,3}(t)$, $t \in [0,1)$ *bersambung dengan keselanjaran* $C^1$.

(b) The bicubic B-spline patch is defined by 16 control points, $V_{ij}$, $i = 0,1,2,3;\ j = 0,1,2,3$ and is given by

$P(u,v) = (\frac{1}{6})^2 UMCM^T V^T$ where $U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$,

$$V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix},\ M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

and $C = \begin{bmatrix} V_{00} & V_{01} & V_{02} & V_{03} \\ V_{10} & V_{11} & V_{12} & V_{13} \\ V_{20} & V_{21} & V_{22} & V_{23} \\ V_{30} & V_{31} & V_{32} & V_{33} \end{bmatrix}$ for $0 \le u,\ v \le 1$.

Write its four corners $P(0,0), P(0,1), P(1,0)$ and $P(1,1)$ as a barycentric sum of nine control points.

[100 marks]

(b) *Tampalan splin-B bikubik ditakrif oleh 16 titik kawalan,* $V_{ij}$, $i = 0,1,2,3$; $j = 0,1,2,3$ dan ditulis sebagai

$P(u,v) = (\frac{1}{6})^2 UMCM^T V^T$ *dengan* $U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$,

$$V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

*dan* $C = \begin{bmatrix} V_{00} & V_{01} & V_{02} & V_{03} \\ V_{10} & V_{11} & V_{12} & V_{13} \\ V_{20} & V_{21} & V_{22} & V_{23} \\ V_{30} & V_{31} & V_{32} & V_{33} \end{bmatrix}$ *untuk* $0 \le u,\ v \le 1$.

*Tulis empat penjurunya,* $P(0,0), P(0,1), P(1,0)$ *dan* $P(1,1)$ *sebagai hasil tambah baripusat sembilan titik kawalan.*

*[100 markah]*

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