UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FOURTEEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi EMPAT BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all five** [5] questions.

*[**Arahan** : Jawab **semua lima** [5] soalan.]*

1. (a) Let X be a random variable with p.d.f.

$$f(x)=\begin{cases} 0 & \text{if} \quad x<0 \ , \\ c & \text{if} \quad 0\le x\le 1, \\ \dfrac{c}{x^2} & \text{if} \quad 1<x<\infty. \end{cases}$$

(i) Find the value of c.

(ii) Obtain the distribution function of X.

(iii) Find the p.d.f of $Y=\dfrac{1}{X}$.

[30 marks]

(b) A random variable X has a c.d.f.

$$F(x)=\begin{bmatrix} 0 & ; & c<0 \\ \dfrac{c}{4} & ; & 0\le c<1 \\ \dfrac{1}{2}+\dfrac{c-1}{4} & ; & 1\le c<2 \\ \dfrac{11}{12} & ; & 2\le c<3 \\ 1 & ; & 3\le c \end{bmatrix}$$

(i) Find the p.d.f. of X. Using the p.d.f., calculate $P\left(\frac{1}{2}<X<\frac{3}{2}\right)$.

(ii) Obtain the moment generating function of X.

(iii) Using the result in (ii), find the mean and variance of X.

[50 marks]

(c) Prove that for any two events A and B,

$$P(A\cap B)\ge 1-P(\bar{A})-P(\bar{B}).$$

Hence, letting $\{A_j\}$; $j=1,2,\cdots$ be a countable sequence of events, show that

$$P\left(\bigcap_{j=1}^{\infty}A_j\right)\ge 1-\sum_{j=1}^{\infty}P(\bar{A}_j)$$

[20 marks]

1. (a) *Katakan X adalah suatu pembolehubah rawak dengan f.k.k.*

$$f(x) = \begin{cases} 0 & jika \quad x < 0 \;, \\ c & jika \quad 0 \le x \le 1, \\ \frac{c}{x^2} & jika \quad 1 < x < \infty. \end{cases}$$

(i) *Cari nilai c.*

(ii) *Dapatkan fungsi taburan bagi X.*

(iii) *Cari f.k.k bagi* $Y = \frac{1}{X}$

[30 markah]

(b) *Suatu pembolehubah rawak X mempunyai f.t.l.*

$$F(x) = \begin{bmatrix} 0 & ; & c < 0 \\ \frac{c}{4} & ; & 0 \le c < 1 \\ \frac{1}{2} + \frac{c-1}{4} & ; & 1 \le c < 2 \\ \frac{11}{12} & ; & 2 \le c < 3 \\ 1 & ; & 3 \le c \end{bmatrix}$$

(i) *Dapatkan f.k.k. bagi X. Dengan menggunakan f.k.k., hitung* $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$.

(ii) *Dapatkan fungsi penjana momen bagi X.*

(iii) *Dengan keputusan yang diperolehi di (ii), dapatkan min dan varians bagi X.*

[50 markah]

(c) *Bagi sebarang dua peristiwa A dan B, buktikan*

$$P(A \cap B) \ge 1 - P(\bar{A}) - P(\bar{B}).$$

Seterusnya, katakan $\{A_j\}$; $j = 1,2,\cdots$ *adalah peristiwa jujukan terhingga. Tunjukkan bahawa*

$$P\left(\bigcap_{j=1}^{\infty} A_j\right) \ge 1 - \sum_{j=1}^{\infty} P(\bar{A}_j)$$

[20 markah]

2. (a) The values that can be taken on by a random variable X are 0,1 and 2 with probabilities a, b and c, respectively. The first and second moments of X are $\frac{9}{10}$ and $\frac{17}{10}$, respectively.

(i) Determine the values of a, b and c.

(ii) Suppose $X_i, i = 1, 2, 3$ are independent random variables that have a common distribution as X. If $Y = X_1 + X_2 + X_3$, find the probability distribution and moment generating function of Y.

[40 marks]

(b) The mean number of automobiles entering a mountain tunnel per 3-minute period is one. An excessive number of cars entering the tunnel during a brief period of time produces a hazardous situation.

(i) Find the probability that the number of automobiles entering the tunnel during a 3-minute period exceeds three.

(ii) Assume that the tunnel is observed during ten 3-minute intervals, thus giving ten independent observations, $Y_1, Y_2, \cdots, Y_{10}$. Find the probability that $Y > 3$ during at least one of the ten 3-minute intervals.

(iii) Suppose that there is a toll plaza right before the tunnel. Find the probability that the toll operator has to wait at least 3 minutes before the first car arrives after 9.00 am.

(iv) Find the probability that the toll operator has to wait less than 15 minutes before the eighth car arrives after 9.00 am.

[60 marks]

2. (a) *Nilai-nilai yang boleh diambil oleh suatu pembolehubah rawak X adalah $0, 1$ dan 2 dengan kebarangkalian masing-masing adalah a, b dan c. Momen pertama dan kedua bagi X adalah masing-masing $\frac{9}{10}$ dan $\frac{17}{10}$.*

(i) *Tentukan nilai a, b dan c.*

(ii) *Katakan $X_i, i = 1, 2, 3$ adalah pembolehubah rawak tak bersandar dan mempunyai taburan sepunya yang sama seperti X. Jika $Y = X_1 + X_2 + X_3$, cari taburan kerangkalian dan fungsi penjana momen bagi Y.*

[40 markah]

(b) *Min bilangan kenderaan yang memasuki sebuah terowong gunung dalam setiap satu tempoh 3 minit adalah satu. Bilangan kenderaan berlebihan yang memasuki terowong pada suatu tempoh pendek menimbulkan suatu situasi yang merbahaya.*

(i) *Cari kebarangkalian bahawa bilangan kenderaan yang memasuki terowong dalam suatu tempoh 3-minit melebihi tiga.*

(ii) *Andaikan terowong ini diperhatikan selama sepuluh selang 3-minit, dengan itu memberikan 10 cerapan tak bersandar $Y_1, Y_2, \cdots, Y_{10}$. Cari kebarangkalian bahawa $Y > 3$ pada sekurang-kurangnya satu daripada sepuluh selang 3-minit.*

(iii) *Katakan terdapat suatu plaza tol sebelum terowong. Cari kebarangkalian bahawa operator tol terpaksa menunggu sekurang-kurangnya 3 minit sebelum kereta pertama tiba selepas 9.00 pagi.*

(iv) *Cari kebarangkalian bahawa operator tol terpaksa menunggu kurang dari 15 minit sebelum kereta kelapan tiba selepas 9.00 pagi.*

[60 markah]

3. (a) Let (X, Y) have the joint p.d.f.

$$f(x,y) = xy + \frac{x^2}{2}; \quad 0 < x < 1, \quad 0 < y < 2.$$

(i) Show that $f(x,y)$ is a p.d.f..

(ii) Find $P\left(Y < 1 \middle| X < \frac{1}{2}\right)$ and $P(Y > X)$.

[30 marks]

(b) Suppose that the joint p.d.f. of X_1 and X_2 is given by

$$f(1,1) = \frac{1}{8} \qquad f(2,1) = \frac{1}{8}$$

$$f(1,2) = \frac{1}{4} \qquad f(2,2) = \frac{1}{2}$$

Are X_1 and X_2 independent? Justify your answer.

[30 marks]

(c) Let X be a random variable with a uniform distribution on the interval $(0,1)$. Find the probability distribution function of $Y = e^X$.

[40 marks]

3. (a) *Katakan* (X, Y) *mempunyai f.k.k. tercantum*

$$f(x,y) = xy + \frac{x^2}{2}; \quad 0 < x < 1, \quad 0 < y < 2.$$

(i) *Tunjukkan bahawa* $f(x, y)$ *adalah suatu f.k.k..*

(ii) *Dapatkan* $P\left(Y < 1 \middle| X < \frac{1}{2}\right)$ *dan* $P(Y > X)$.

[30 markah]

(b) *Katakan f.k.k. tercantum bagi* X_1 *dan* X_2 *diberi sebagai*

$$f(1,1) = \frac{1}{8} \qquad f(2,1) = \frac{1}{8}$$

$$f(1,2) = \frac{1}{4} \qquad f(2,2) = \frac{1}{2}$$

Adakah X_1 *dan* X_2 *tak bersandar? Huraikan jawapan anda.*

[30 markah]

(c) *Katakan* X *adalah suatu pembolehubah rawak dengan suatu taburan seragam pada selang* $(0,1)$. *Cari fungsi taburan kebarangkalian bagi* $Y = e^X$.

[40 markah]

4. (a) Suppose that

$$f(x,y) = \begin{cases} \frac{3}{16}(4-2x-y) & ; \quad x>0, y>0, 2x-y<4 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Find

(i) $E(Y|x)$,

(ii) $Var(Y|x)$,

(iii) ρ.

[40 marks]

(b) (i) Show how the Chebyshev's inequality was obtained.

(ii) If X is a random variable with $E(X) = 3$ and $E(X^2) = 13$, use Chebyshev's inequality to find $P(-2 < X < 8)$.

(iii) Suppose that $\bar{X}$ is the mean of a random sample of size 6 from a distribution in (ii). Use the Central Limit Theorem to find $P(-2 < \bar{X} < 8)$.

[30 marks]

(c) Suppose that $X_1, \cdots, X_k$ are i.i.d. geometric random variables.

(i) What is the distribution of $Y = \sum_{i=1}^{k} X_i$? Justify your answer.

(ii) What is the mean and variance of Y?

[30 marks]

4. (a) *Andaikan bahawa*

$$f(x,y) = \begin{cases} \frac{3}{16}(4-2x-y) & ; \quad x>0, y>0, 2x-y<4 \\ 0 & ; \quad di\ tempat\ lain. \end{cases}$$

Cari

(i) $E(Y|x)$,

(ii) $Var(Y|x)$,

(iii) ρ.

[40 markah]

(b) (i) *Tunjukkan bagaimana ketaksamaan Chebyshev diperolehi.*

(ii) *Jika* X *adalah suatu pembolehubah rawak dengan* $E(X) = 3$ *dan* $E(X^2) = 13$, *gunakan ketaksamaan Chebyshev untuk mendapatkan* $P(-2 < X < 8)$.

(iii) *Katakan bahawa* $\bar{X}$ *adalah min bagi suatu sampel rawak bersaiz 6 dari suatu taburan di (ii). Dengan menggunakan Teorem Had Memusat, cari* $P(-2 < \bar{X} < 8)$.

[30 markah]

(c) *Katakan* $X_1, \cdots, X_k$ *adalah pembolehubah rawak geometrik i.i.d..*

(i) *Apakah taburan bagi* $Y = \sum_{i=1}^{k} X_i$? *Huraikan jawapan anda.*

(ii) *Apakah min dan varians bagi Y?*

[30 markah]

5. (a) If X is a Poisson, $P(\lambda)$ and the conditional distribution of Y, given $X = x$ is a Binomial, $B(n,p)$, show that Y is a $P(\lambda p)$ random variable.

[30 marks]

(b) Suppose $X_1,\cdots,X_8$ and $Y_1,\cdots,Y_5$ are random samples from independent normal distributions $N(3,8)$ and $N(3,15)$, respectively.

(i) Find $P(\bar{X} > \bar{Y} + 1)$

(ii) Find a constant c such that $P\left(\frac{\bar{X}-2}{S} < c\right) = 0.90$ where S is a sample standard deviation of X.

(iii) Find a constant c such that $\mathrm{P}(\mathrm{S} < c) = 0.90$, where S is as in (ii).

[40 marks]

(c) Let X and Y be i.i.d. random variables with common p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \quad ; \quad -\infty < x < \infty$$

(i) Find the joint p.d.f. of $U = \left(\frac{X+Y}{2}\right)$ and $V = \left(\frac{X-Y}{2}\right)^2$

(ii) Are U and V independent? Justify your answer.

[30 marks]

5. (a) *Jika* X *adalah tertabur Poisson,* $P(\lambda)$ *dan taburan bersyarat bagi* Y, *diberi* $X = x$ *adalah tertabur Binomial,* $B(n, p)$, *tunjukkan bahawa* Y *adalah suatu pembolehubah rawak* $P(\lambda p)$.

[30 markah]

(b) *Katakan* $X_1, \cdots, X_8$ *dan* $Y_1, \cdots, Y_5$ *adalah sampel-sampel rawak masing-masing dari taburan normal tak bersandar* $N(3,8)$ *dan* $N(3,15)$.

(i) *Cari* $P(\bar{X} > \bar{Y} + 1)$

(ii) *Cari nilai* c *supaya* $P\left(\frac{\bar{X}-2}{S} < c\right) = 0.90$ *yang mana* S *adalah suatu sisihan piawai sampel bagi* X.

(iii) *Dapatkan nilai* c *supaya* $P(S < c) = 0.90$ *yang mana* S *adalah sama seperti di (ii).*

[40 markah]

(c) *Katakan* X *dan* Y *adalah pemboleh ubah rawak i.i.d. dengan f.k.k. sepunya*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad ; \quad -\infty < x < \infty$$

(i) *Dapatkan f.k.k. tercantum bagi* $U = \left(\frac{X+Y}{2}\right)$ *dan* $V = \left(\frac{X-Y}{2}\right)^2$

(ii) *Adakah* U *dan* V *tak bersandar? Huraikan jawapan anda.*

[30 markah]

LAMPIRAN

DISCRETE DISTRIBUTIONS	
Bernoulli	$f(x) = p^x(1-p)^{1-x}, \quad x = 0,1$ $M(t) = 1 - p + pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial	$f(x) = \frac{n!}{x!\,(n-x)!}p^x(1-p)^{n-x}, \quad x = 0,1,2,\cdots,n$ $M(t) = 1 - p + pe^t$ $\mu = p, \quad \sigma^2 = np(1-p)$
Geometric	$f(x) = (1-p)^x p, \quad x = 0,1,2,\cdots$ $M(t) = \frac{n!}{1-(1-p)e^t}, t < -\text{In}\,(1-p)$ $\mu = \frac{1-p}{p}, \quad \sigma^2 = \frac{(1-p)}{p^2}$
Negative Binomial	$f(x) = \frac{(x+r-1)!}{x!\,(r-1)!}p^r(1-p)^x, \quad x = 0,1,2,\cdots$ $M(t) = \frac{p^r}{[1-(1-p)e^t]^r}, t < -\text{In}\,(1-p)$ $\mu = \frac{r(1-p)}{p}, \quad \sigma^2 = \frac{r(1-p)}{p^2}$
Poisson	$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\cdots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Hipergeometric	$f(x) = \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{r}}, \quad x \leq r, x \leq n_1, r - x \leq n_2,$ $\mu = \frac{rn_1}{n}, \quad \sigma^2 = \frac{rn_1n_2(n-r)}{n^2(n-1)}$

CONTINUOUS DISTRIBUTION	
Uniform	$f(x)=\frac{1}{b-a}, a \le x \le b$ $M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)}, t \ne 0,\ M(0)=1$ $\mu=\frac{a+b}{1}, \sigma^2=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\quad 0 \le x \le \infty$ $M(t)=\frac{1}{1-\theta t},\quad t<1/\theta$ $\mu=\theta,\quad \sigma^2=\theta^2$
Gamma	$f(x)=\frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}},\quad 0 \le x < \infty$ $M(t)=\frac{1}{(1-\theta t)^\alpha},\quad t<1/\theta$ $\mu=\alpha\theta,\quad \sigma^2=\alpha\theta^2$
Chi Square	$f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2},\quad 0 \le x < \infty$ $M(t)=\frac{1}{(1-2t)^{r/2}},\quad t<\frac{1}{2}$ $\mu=r,\quad \sigma^2=2r$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2},\quad -\infty < x < \infty$ $M(t)=e^{\mu t+\frac{\sigma^2 t^2}{2}}$ $E(X)=\mu,\quad \text{Var}(X)=\sigma^2$
Beta	$f(x)=\frac{1}{B(\alpha,\beta)},\quad x^{\alpha-1}(1-x)^{\beta-1}, 0<x<1$ $\mu=\frac{\alpha}{\alpha+\beta},\quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+)(\alpha+\beta)^2}$

FORMULAE	
1.	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
2.	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
3.	$\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = (a+b)^n$
4.	$\sum_{x=0}^{n} \binom{n_1}{x}\binom{r-n_1}{r-x} = \binom{n}{r}$
5.	$\sum_{k=0}^{\infty} \binom{n+k-1}{k} w^k = (1-w)^{-n}, \lvert w \rvert < 1$
6.	$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx, \quad \Gamma(\alpha) = (\alpha-1)!$
7.	$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$
8.	$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

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