UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MAT 202 – Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all three** [3] questions.

*[**Arahan** : Jawab **semua tiga** [3] soalan.]*

1\. (a) (i) Define supremum and infimum for a nonempty set $S \subset \mathbb{R}$.
[$\mathbb{R}$ = set of all real numbers]

(ii) Find the supremum and infimum for the following sets if they exist.

(a) $A = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$.

(b) $A = \left\{ (-1)^n \dfrac{1}{n} : n \in \mathbb{N} \right\}$.

[$\mathbb{N}$ = set of natural numbers]

(b) Let S and T be nonempty sets of real numbers with T a bounded set and $S \subset T$.

(i) Prove that $\sup S \leq \sup T$ and $\inf T \leq \inf S$.

(ii) Give an example for the case when $\sup S = \sup T$ and $\inf S = \inf T$.

(c) Let A, B and C be nonempty sets. Then show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(d) Define a countable set. Show that the set of all integers is countable.

(e) Let $\{a_n\} = \left\{ \dfrac{3n+2}{n-1} : n \in \mathbb{N} \right\}$.

(i) Determine the limit of $\{a_n\}$.

(ii) Use the definition to verify the convergent of $\{a_n\}$.

(iii) Find the smallest integer N such that $\left| \dfrac{3n+2}{n-1} - 3 \right| < 0.01$ for all $n \geq \mathbb{N}$.

[100 marks]

1. (a) (i) *Takrifkan supremum dan infimum bagi suatu set tak kosong* $S \subset \mathbb{R}$.
[$\mathbb{R}$ *= set nombor nyata]*

(ii) *Tentukan supremum dan infimum set berikut jika wujud*

(a) $A = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$.

(b) $A = \left\{ (-1)^n \dfrac{1}{n} : n \in \mathbb{N} \right\}$.

[$\mathbb{N}$ *= set nombor asli]*

(b) *Andaikan* S *dan* T *adalah set nombor nyata dan tak kosong dengan* T *set terbatas dan* $S \subset T$.

(i) *Buktikan bahawa* $\sup S \le \sup T$ *dan* $\inf T \le \inf S$.

(ii) *Beri satu contoh bagi kes dimana* $\sup S = \sup T$ *dan* $\inf S = \inf T$.

(c) *Andaikan A, B dan C adalah set tak kosong. Tunjukkan bahawa*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(d) *Takrifkan set terbilangkan. Tunjukkan bahawa set semua integer adalah terbilangkan.*

(e) *Andaikan* $\{a_n\} = \left\{ \dfrac{3n+2}{n-1} : n \in \mathbb{N} \right\}$.

(i) *Tentukan had* $\{a_n\}$.

(ii) *Gunakan takrifan untuk menentusahkan penumpuan* $\{a_n\}$.

(iii) *Cari integer terkecil N supaya* $\left| \dfrac{3n+2}{n-1} - 3 \right| < 0.01$ *semua* $n \ge \mathbb{N}$.

[100 markah]

2. (a) Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and suppose that $\{x_n\}$ is a sequence such that $a_n \le x_n \le b_n$ for all *n*. If the sequence $\{a_n\}$ and $\{b_n\}$ both converges to *L*, then show that the sequence $\{x_n\}$ converges to *L*.

(b) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be a closed interval on $\mathbb{R}$. Given $I_n \supset I_{n+1}$, then prove that $\bigcap_{n=1}^{\infty} I_n \ne \phi$.

(c) Given the sequence $\{a_n\}$ converges to ℓ. Show that every subsequence of $\{a_n\}$ also converges to ℓ. Further more determine whether or not the sequence $\{a_n\} = \{(-1)^n\}$ is convergent. Justify your answer.

(d) Use the definition of Cauchy sequence, determine whether the sequence $\left\{\frac{2n^2+1}{n^2}\right\}$ is Cauchy.

[100 marks]

2. (a) *Andaikan* $\{a_n\}$ *dan* $\{b_n\}$ *adalah jujukan menumpu dan* $\{x_n\}$ *adalah suatu jujukan supaya* $a_n \le x_n \le b_n$ *untuk semua n. Jika jujukan* $\{a_n\}$ *dan* $\{b_n\}$ *kedua-duanya menumpu ke L, tunjukan bahawa jujukan* $\{x_n\}$ *menumpu ke L.*

   (b) *Untuk setiap* $n \in \mathbb{N}$, *biarkan* $\mathrm{I}_n = [a_n, b_n]$ *adalah selang tertutup pada* $\mathbb{R}$. *Diberi* $I_n \supset I_{n+1}$, *buktikan bahawa* $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

   (c) *Diberi jujukan* $\{a_n\}$ *menumpu ke* ℓ. *Tunjukkan bahawa setiap subjujukan bagi* $\{a_n\}$ *juga menumpu ke* ℓ. *Selanjutnya, tentukan samada jujukan* $\{a_n\} = \{(-1)^n\}$ *menumpu atau tidak. Jelaskan jawapan anda.*

   (d) *Gunakan takrifkan jujukan Cauchy, tentukan samada jujukan* $\left\{\frac{2n^2+1}{n^2}\right\}$ *adalah Cauchy.*

[100 markah]

3. (a) Given $\tau=\{(-n,n),n\in\mathbb{N}\}$, an open covering of a set $A\subseteq\mathbb{R}$, where A is a compact set. Show that A is bounded.

(b) Given a set $A=(0,1)$ and collection

$$\vartheta=\left\{\left(\frac{1}{3},1\right),\left(\frac{1}{4},\frac{1}{2}\right),\left(\frac{1}{5},\frac{1}{3}\right),\ldots,\left(\frac{1}{n},\frac{1}{n-2}\right),\ldots\right\}.$$

(i) Determine whether ϑ is an open covering for A. Give your reasons.

(ii) Use the definition of compactness (in terms of open covering) to show that A is not compact.

(c) Given a set $A=(0,1)\cap Q$. [Q = set of all rational numbers]
Find the interior points, accumulation points/limit point and the isolated points. Furthermore, determine whether A is open on closed or neither?

(d) Given a function $f(x)=\dfrac{x-1}{x+2}$. Using the definition, show that this function is uniformly continuous on $[0,\infty)$.

(e) Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on $\mathbb{R}$ by $f_n(x)=x^n$. Determine for what values of x on the given set, the sequence of functions converges point wise. Next find the point wise limit of the sequence $\{f_n\}$.

[100 marks]

3. (a) *Diberi* $\tau = \{(-n,n), n \in \mathbb{N}\}$, *adalah tudung terbuka bagi* set $A \subseteq \mathbb{R}$, *dimana* A *adalah set padat. Tunjukkan bahawa A terbatas.*

(b) *Diberi* set $A = (0,1)$ *dan pungutan*

$$\vartheta = \left\{\left(\frac{1}{3},1\right),\left(\frac{1}{4},\frac{1}{2}\right),\left(\frac{1}{5},\frac{1}{3}\right),\ldots,\left(\frac{1}{n},\frac{1}{n-2}\right),\ldots\right\}.$$

(i) *Tentukan samada* ϑ *adalah suatu tudung terbuka bagi A. Berikan alasan.*

(ii) *Gunakan takrifkan kepadatan set (dalam sebutan tudung terbuka) untuk menunjukkan bahawa A tidak padat.*

(c) *Diberi* set $A = (0,1) \cap Q$ [Q = set semua nombor nisbah]

Cari titik pedalaman, titik had/titik tumpukan dan titik terpencil.

Seterusnya, tentukan samada A *adalah terbuka, tertutup atau bukan kedua-duanya.*

(d) *Diberi fungsi* $f(x) = \dfrac{x-1}{x+2}$. *Menggunakan takrifkan, tunjukkan bahawa fungsi ini selanjar secara seragam pada* $[0,\infty)$.

(e) *Diberi suatu jujukan fungsi* $\{f_n\}_{n=1}^{\infty}$ *ditakrifkan sebagai* $f_n(x) = x^n$ *pada* $\mathbb{R}$. *Tentukan apakah nilai* x *di atas set yang diberi, jujukan fungsi adalah menumpu titik demi titik. Seterusnya, cari had titik demi titik bagi jujukan* $\{f_n\}$.

[100 markah]

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