UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

# MAT 102 – *Advanced Calculus*
# *[Kalkulus Lanjutan]*

Duration : 3 hours
*[Masa : 3 jam]*

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions :** Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

1. (a) Let

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

(i) Show that $f_x$ and $f_y$ exist at $(0,0)$.

(ii) Show that $D_v f(0,0)$ exists in all directions $v$.

(iii) Show that $f$ is discontinuous at $(0,0)$.

(b) Let $w=x^2+y^2+z^2$ and $x=e^t\cos t$, $y=e^t\sin t$, $z=e^t$.

(i) Write $w$ as a function of the variable $t$, then find $\frac{dw}{dt}$.

(ii) By using the Chain Rule, find $\frac{dw}{dt}$ in terms of the variable $t$.

(c) (i) Find the point where the local extremum of $f(x,y)=x^2+xy+3y^2$ occurs.

(ii) Find the nearest point and the furthest point on the surface $z=xy+1$ from the origin.

[100 marks]

2. (a) Evaluate the following integrals:

(i) $\iint_D (x+y)\,dxdy$, where $D$ is the region on the $xy$-plane bounded by the line $y=x$ and the curve $y=x^2$.

(ii) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$.

(b) Let $D$ be the region on the $xy$-plane bounded by the parabola $y=4-x^2$ and the line $y=3x$. Find the volume of the solid bounded above by the surface $z=x^2+4$ and below by the region $D$.

(c) Check the convergence of the following improper integrals:

(i) $\int_1^5 \frac{1}{\sqrt{(5-x)(x-1)}}\,dx$

(ii) $\int_0^\pi \frac{\sin x}{x^3}\,dx$

(iii) $\int_1^\infty \frac{1-\cos x}{x^2}\,dx$

[100 marks]

1. (a) *Andaikan*

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

(i) *Tunjukkan bahawa* $f_x$ *and* $f_y$ *wujud pada* $(0,0)$.

(ii) *Tunjukkan bahawa* $D_v f(0,0)$ *wujud dalam semua arah* $v$.

(iii) *Tunjukkan bahawa* $f$ *adalah tak selanjar pada* $(0,0)$.

(b) *Andaikan* $w=x^2+y^2+z^2$ *dan* $x=e^t\cos t$, $y=e^t\sin t$, $z=e^t$.

(i) *Tulis* $w$ *sebagai suatu fungsi dalam pembolehubah* $t$, *kemudian cari* $\dfrac{dw}{dt}$.

(ii) *Dengan menggunakan Petua Rantai, cari* $\dfrac{dw}{dt}$ *dalam sebutan* $t$.

(c) (i) *Cari titik-titik berlakunya ekstremum tempatan untuk* $f(x,y)=x^2+xy+3y^2$.

(ii) *Cari titik terdekat dan titik terjauh dari asalan pada permukaan* $z=xy+1$.

*[100 markah]*

2. (a) *Nilaikan kamiran berikut:*

(i) $\iint\limits_D (x+y)\,dxdy$, *D ialah rantau pada satah xy yang dibatasi oleh garis* $y=x$ *dan lengkung* $y=x^2$.

(ii) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$.

(b) *Biar* $D$ *rantau pada satah xy yang dibatasi oleh parabola* $y=4-x^2$ *dan garis* $y=3x$. *Cari isipadu bongkah yang dibatasi dari atas oleh* $z=x^2+4$ *dan dari bawah oleh rantau* $D$.

(c) *Semak penumpuan kamiran tak wajar berikut:*

(i) $\displaystyle\int_1^5 \frac{1}{\sqrt{(5-x)(x-1)}}\,dx$

(ii) $\displaystyle\int_0^\pi \frac{\sin x}{x^3}\,dx$

(iii) $\displaystyle\int_1^\infty \frac{1-\cos x}{x^2}\,dx$

*[100 markah]*

3. (a) Find the following limits of sequences or functions:

(i) $\lim\limits_{n\to\infty}(an+b)^{\frac{1}{n}},\ \ a,b>0$

(ii) $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

(iii) $\lim\limits_{x\to\infty}\frac{x(\cos x-1)}{\sin x-x}$

(iv) $\lim\limits_{x\to 1}\left(\frac{1}{x-1}\ell n\frac{x}{2x-1}\right)$

($n$ is a positive integer and $x$ is a real number)

(b) Determine whether each of the following series converges:

(i) $\sum\limits_{k=1}^{\infty}k\left(\frac{2}{3+k}\right)^k$

(ii) $\sum\limits_{k=1}^{\infty}k\sin\left(\frac{1}{k}\right)$

(iii) $\sum\limits_{k=1}^{\infty}\frac{k^2-7}{k\sqrt{k^2+8}}$

(c) (i) Suppose that $\sum\limits_{k=1}^{\infty}a_k$ converges and $\sum\limits_{k=1}^{\infty}b_k$ diverges. Prove that $\sum\limits_{k=1}^{\infty}(a_k-b_k)$ diverges.

(ii) Hence determine whether $\sum\limits_{k=1}^{\infty}\frac{k-4^k}{k4^k}$ converges or diverges.

[100 marks]

3. (a) *Cari had untuk jujukan atau fungsi berikut:*

(i) $\lim_{n\to\infty}(an+b)^{\frac{1}{n}},\quad a,b>0$

(ii) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

(iii) $\lim_{x\to\infty}\frac{x(\cos x-1)}{\sin x-x}$

(iv) $\lim_{x\to 1}\left(\frac{1}{x-1}\ell n\frac{x}{2x-1}\right)$

*($n$ ialah integer positif dan $x$ ialah nombor nyata)*

(b) *Tentukan samada siri berikut menumpu:*

(i) $\sum_{k=1}^{\infty}k\left(\frac{2}{3+k}\right)^k$

(ii) $\sum_{k=1}^{\infty}k\sin\left(\frac{1}{k}\right)$

(iii) $\sum_{k=1}^{\infty}\frac{k^2-7}{k\sqrt{k^2+8}}$

(c) (i) *Andaikan* $\sum_{k=1}^{\infty}a_k$ *menumpu dan* $\sum_{k=1}^{\infty}b_k$ *mencapah. Buktikan bahawa* $\sum_{k=1}^{\infty}(a_k-b_k)$ *mencapah.*

(ii) *Dengan itu tentukan samada* $\sum_{k=1}^{\infty}\frac{k-4^k}{k4^k}$ *menumpu atau mencapah.*

*[100 markah]*

4. (a) The sequence $\{a_n\}$ is defined as $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$, $\forall\, n \geq 1$.

(i) Show that $\dfrac{a_{n+1}}{a_n} < 1$.

Hence determine whether the sequence $\{a_n\}$ is increasing or decreasing.

(ii) Is $\{a_n\}$ bounded?

(iii) Does $\lim\limits_{n\to\infty} a_n$ exist? Give your reason.

(b) Suppose that $f(x) = \ell n\, x$, $x > 0$.

Show that the Taylor series of $f$ about 1 is $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k}(x-1)^k$ and find its interval of convergence.

(c) (i) Prove that if $a_n \geq 0$, $\forall n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} a_n = \alpha$, then $\alpha \geq 0$.

(ii) Hence show that if $b_n \geq c_n, \forall n \in \mathbb{N}$, $\lim\limits_{n\to\infty} b_n = \beta$ and $\lim\limits_{n\to\infty} c_n = \delta$, then $\beta \geq \delta$.

[100 marks]

4. (a) *Jujukan* $\{a_n\}$ *ditakrif sebagai* $a_n = \dfrac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}$, $\forall n \geq 1$.

(i) *Tunjukkan bahawa* $\dfrac{a_{n+1}}{a_n} < 1$.

*Dengan itu tentukan samada jujukan* $\{a_n\}$ *adalah menumpu atau mencapah.*

(ii) *Adakah* $\{a_n\}$ *terbatas?*

(iii) *Adakah* $\lim_{n\to\infty} a_n$ *wujud? Beri alasan anda.*

(b) *Andaikan* $f(x) = \ell n\, x$, $x > 0$.
*Tunjukkan bahawa siri Taylor untuk* $f$ *di sekitar 1 ialah* $\sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k}(x-1)^k$ *dan cari selang penumpuannya.*

(c) (i) *Buktikan bahawa jika* $a_n \geq 0, \forall n \in \mathbb{N}$, *dan* $\lim_{n\to\infty} a_n = \alpha$, *maka* $\alpha \geq 0$.

(ii) *Dengan itu tunjukkan bahawa jika* $b_n \geq c_n, \forall n \in \mathbb{N}$, $\lim_{n\to\infty} b_n = \beta$ *dan* $\lim_{n\to\infty} c_n = \delta$, *maka* $\beta \geq \delta$.

*[100 markah]*

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