MULTIPLE SIMILARITY SOLUTIONS OF STEADY AND UNSTEADY CONVECTION BOUNDARY LAYER FLOWS IN VISCOUS FLUIDS AND NANOFLUIDS

AZIZAH BINTI MOHD ROHNI

UNIVERSITI SAINS MALAYSIA

2013

MULTIPLE SIMILARITY SOLUTIONS OF STEADY AND UNSTEADY CONVECTION BOUNDARY LAYER FLOWS IN VISCOUS FLUIDS AND NANOFLUIDS

by

AZIZAH BINTI MOHD ROHNI

Thesis submitted in fulfillment of the requirement for the degree of Doctor of Philosophy

September 2013

ACKNOWLEDGEMENTS

BISMILLAHIRRAHMANIRRAHIM

In the name of Allah, the Most Gracious, the Most Merciful. Peace and blessings be upon the Messenger of Allah. Alhamdulillah, all praises and thanks to Allah, the Lord of the worlds, Most Gracious, Most Merciful, Master of the Day of Judgement for blessing me with the will, determination, perseverance and strength to complete this task.

In completing this thesis, I am indebted to many people. First and foremost, I wish to express my deepest gratitude to my supervisor, Dr. Syakila Ahmad, my co-supervisor Prof. Ahmad Izani Md. Ismail and Prof. Ioan Pop from University of Cluj, Romania for their valuable guidance, assistance, concern, motivation and continuous support throughout my study. I would like to acknowledge with much appreciation to Prof. Roslinda Mohd Nazar, Prof. Zarita Zainuddin and Associate Prof. Norhashidah Hj. Mohd Ali for their constructive comments and suggestions to improve the quality of my thesis.

A great appreciation to Prof. Merkin from University of Leeds, United Kingdom for his generous collaboration in paper publication, to Dr. Mohd Zuki Salleh from Universiti Malaysia Pahang and Dr. Lok Yian Yian from Universiti Sains Malaysia for sharing their experiences and providing helpful suggestions on the numerical computations.

I also would like to thank the technical and administrative staff at the School of Mathematical Sciences USM and staffs in the Institute of Graduate Studies USM who have been very supportive and very helpful during my course of study here.

The financial support provided by Universiti Utara Malaysia, Ministry of Higher Education, Malaysia and Universiti Sains Malaysia under the incentive research grant (1001/PMATHS/822123) and RU grant (1001/PMATHS/811166) throughout the course of my study are gratefully acknowledged.

The crucial role of the editors and reviewers of all the journals that I have submitted my work throughout the preparation of various publications are also acknowledged and appreciated. Their valuable comments and helpful suggestions which consequently led to the improvement of my work are highly valued.

Last but not least, I thank my husband, my family, friends, colleagues and all those who have assisted me directly or indirectly towards the completion of this thesis.

I hope this thesis will always remind me of my parents, Allahyarham Mohd Rohni Mahmood and Allahyarhamah Rohani Kassim, my siblings, my close friend Sharifah Sakinah Syed Abdul Mutalib and all the significant people who came into my life and into my mind during hard times of this journey which leads me to the remembrance of Allah.

May Allah shower His blessings upon all of you.

TABLE OF CONTENTS

CHAPTER 2 - LITERATURE REVIEW

CHAPTER 3 - BOUNDARY LAYER EQUATIONS

CHAPTER 4 – STEADY MIXED CONVECTION BOUNDARY LAYER FLOW OVER AN EXPONENTIALLY SHRINKING VERTICAL SHEET WITH SUCTION

CHAPTER 5 – STEADY FREE CONVECTION OVER NON-LINEARLY SHRINKING SHEET WITH SUCTION AND VISCOUS DISSIPATION: CORTELL'S MODEL

CHAPTER 6 – UNSTEADY FREE CONVECTION BOUNDARY LAYER FLOW OVER A HORIZONTAL SHRINKING SHEET WITH SUCTION IN NANOFLUIDS

CHAPTER 7 - UNSTEADY MIXED CONVECTION BOUNDARY LAYER FLOW WITH SUCTION AND TEMPERATURE SLIP EFFECTS NEAR THE STAGNATION POINT ON A VERTICAL PERMEABLE SURFACE EMBEDDED IN A POROUS MEDIUM

CHAPTER 8 – STEADY MIXED CONVECTION BOUNDARY LAYER FLOW ALONG A VERTICAL CYLINDER EMBEDDED IN POROUS MEDIUM FILLED BY A NANOFLUID

CHAPTER 9 - CONCLUSIONS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Roman letters

A	unsteadiness parameter
C_f	skin friction coefficient
C_p	specific heat at constant temperature
f	non-dimensional stream function
g	acceleration due to gravity
Gr	Grashof number
$(\rho C_p)_{\mathrm{nf}}$	heat capacitance of the nanofluid
k	thermal conductivity
k_{nf}	thermal conductivity of the nanofluid
L	characteristic length of the sheet
Nu	Nusselt number
p	fluid pressure
p_∞	pressure of the ambient fluid
Pe	Péclet number
$\Pr$	Prandtl number
Re	Reynolds number
Re_x	local Reynolds number
s	wall mass transfer/suction parameter, $s > 0$
T	fluid temperature
T_∞	temperature of the ambient fluid
T_0	characteristic temperature of the sheet
$T_w(x)$	surface temperature

U	constant free stream velocity of the nanofluid
U_w	characteristic velocity of the sheet, $U_w > 0$
u, v	velocity components along the x and y axes
u_w	velocity of the shrinking sheet
v_w	velocity of the mass transfer/suction
x, y	Cartesian coordinates along the plate and normal to it, respectively

Greek letters

α	thermal diffusivity of the fluid
α_{nf}	effective thermal diffusivity of the nanofluid
β	coefficient of thermal expansion
η	similarity variable
λ	mixed convection/buoyancy parameter
μ	dynamic viscosity of the fluid
μ_{nf}	effective dynamic viscosity of the nanofluid
ν	kinematic viscosity of the fluid, $\nu = \mu/\rho$
φ	nanoparticle volume fraction
ψ	stream function
ρ	density of the fluid
ρ_{nf}	effective density of the nanofluid
τ_w	skin friction at the shrinking sheet
θ	dimensionless temperature

Subscripts

f	fluid fraction
s	solid fraction
nf	nanofluid fraction
w	condition at the surface
∞	ambient/free stream condition

PENYELESAIAN KESERUPAAN BERBILANG BAGI ALIRAN LAPISAN SEMPADAN OLAKAN MANTAP DAN TAK MANTAP DALAM BENDALIR LIKAT DAN NANOBENDALIR

ABSTRAK

Dalam kebanyakan masalah kompleks bagi aliran lapisan sempadan dan pemindahan haba, penyelesaian berbilang mungkin wujud disebabkan oleh ketaklinearan persamaan pembezaan, perubahan parameter geometri atau mekanikal bendalir. Adalah sukar untuk menggambarkan kewujudan penyelesaian berbilang secara uji kaji dan kerana itu pengiraan bermatematik adalah penting untuk menyediakan struktur aliran terperinci dan untuk melihat kewujudan penyelesaian berbilang. Tesis ini bertujuan untuk mengkaji penyelesaian keserupaan berbilang yang mungkin wujud dalam aliran lapisan sempadan dan pemindahan haba. Ini dilakukan dengan mempertimbangkan lima masalah yang berbeza iaitu dua masalah dalam bendalir likat, satu masalah dalam nanobendalir dan dua lagi adalah masing-masing dalam medium berliang dan dalam medium berliang diisi dengan nanobendalir. Bagi masalah dalam bendalir likat dan nanobendalir, situasi berbeza bagi helaian mengecut dipertimbangkan. Selebihnya, plat menegak dalam medium berliang dan silinder menegak dalam medium berliang yang diisi oleh nanobendalir dipertimbangkan. Persamaan-persamaan menakluk asas dalam bentuk persamaan terbitan separa bagi setiap masalah dijelmakan kepada persamaan keserupaan dalam bentuk persamaan terbitan biasa melalui pendekatan keserupaan. Sistem yang terhasil kemudiaannya diselesaikan secara berangka menggunakan teknik tembakan dengan bantuan fungsi *shootlib* dalam perisian Maple. Teknik ini melibatkan kaedah

Runge-Kutta bersama-sama dengan pembetulan Newton-Raphson. Untuk mengesahkan keputusan berangka yang diperoleh dalam kajian ini, perbandingan dengan penyelesaian yang sedia ada dalam kajian lepas bagi kes-kes tertentu telah dibuat dan didapati keputusan perbandingan adalah sangat baik. Seterusnya, kesan parameter-parameter menakluk yang berbeza ke atas dinamik aliran dan pemindahan haba telah diperiksa bagi setiap masalah tertentu. Didapati penyelesaian ganda tiga wujud dalam masalah cecair likat, iaitu masalah helaian menegak mengecut secara eksponen dan helaian mendatar mengecut secara tak linear. Penyelesaian dual didapati wujud bagi masalah dalam nanofluid dan medium berliang yang melibatkan helaian mengecut, plat menegak di titik genangan dan silinder menegak. Bagi setiap kes yang dipertimbangkan, terdapat juga kawasan yang mempunyai penyelesaian unik dan tiada penyelesaian. Kewujudan penyelesaian berbilang sama ada tiga, dua, unik atau tiada penyelesaian adalah dipengaruhi oleh parameter yang dipertimbangkan.

MULTIPLE SIMILARITY SOLUTIONS OF STEADY AND UNSTEADY CONVECTION BOUNDARY LAYER FLOWS IN VISCOUS FLUIDS AND NANOFLUIDS

ABSTRACT

For many complex problems in convection boundary layer flow and heat transfer, multiple solutions may exist due to the nonlinearity of the differential equations, variation of geometric or fluid mechanical parameters. It is difficult to visualize the occurrence of multiple solutions experimentally, therefore mathematical computation is important to provide the details flow structure and to notice the occurrence of multiple solutions. This thesis aims to study the possible multiple similarity solutions that might exist in boundary layer flows and heat transfer. This is done by considering five different problems which are two problems in viscous fluid, one problem in nanofluid and the remaining two are in porous medium and in porous medium filled with nanofluid, respectively. For the problems in viscous fluid and nanofluid, different situations of shrinking sheet have been considered. On the other hand, vertical plate in porous medium and vertical cylinder in porous medium filled by nanofluid have also been considered. The basic governing equations in partial differential equations form for each problem are first transformed into similarity equations in nonlinear ordinary differential equations form by similarity approach. The resulting systems are then solved numerically using the shooting technique with the aid of *shootlib* fuction in Maple software. This technique involves Runge-Kutta method together with Newton-Raphson correction. To validate the numerical results obtained in this study, comparison with existing solutions in literature for specific cases have been made and it is found to be in very good agreement. Further, the impact of different governing parameters on both the flow and heat transfer dynamics has been examined for each of the specific problem at hands. It is found that triple solutions exist in the viscous fluid problems, i.e the problem of exponentially shrinking vertical sheet and nonlinearly shrinking horizontal sheet. Dual solutions are found to exist for the problems in nanofluid and porous medium which involves shrinking sheet, vertical plate at stagnation point and vertical cylinder. For each of the cases considered, there are also regions for unique and no solutions. The occurrences of multiple solutions either triple, dual, unique or no solutions are influenced by the considered parameter.

CHAPTER 1

INTRODUCTION

1.1 Basic Concepts

1.1.1 Convection

Convection (convective heat transfer) is one of the three different modes of heat transfer, besides conduction and radiation. Convection is a process whereby energy is transferred from a surface to a fluid flowing over it due to temperature difference between the surface and the fluid. Therefore, in convection there is always a surface, a fluid flowing relative to this surface, and a temperature difference between the surface and the fluid (Oosthuizen and Naylor, 1999) as illustrated in Figure 1.1.

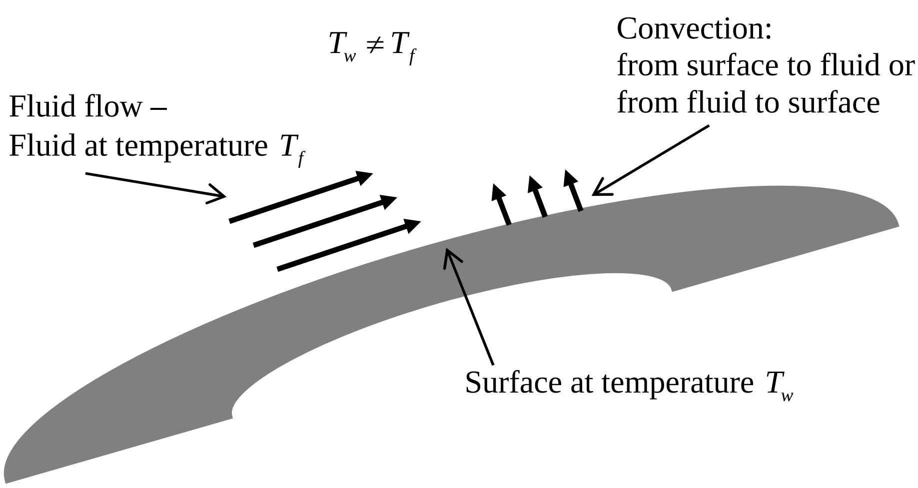

Figure 1.1 Convection.

Convection occurs extensively in practice such as the cooling of the cutting tool during machining operation, the cooling of the electronic components in a computer, the generation and condensation of steam in a thermal power plant, the heating and cooling of buildings, cooking and the thermal control of a re-entering spacecraft. Figure 1.2 shows some examples of situations in which convection is important.

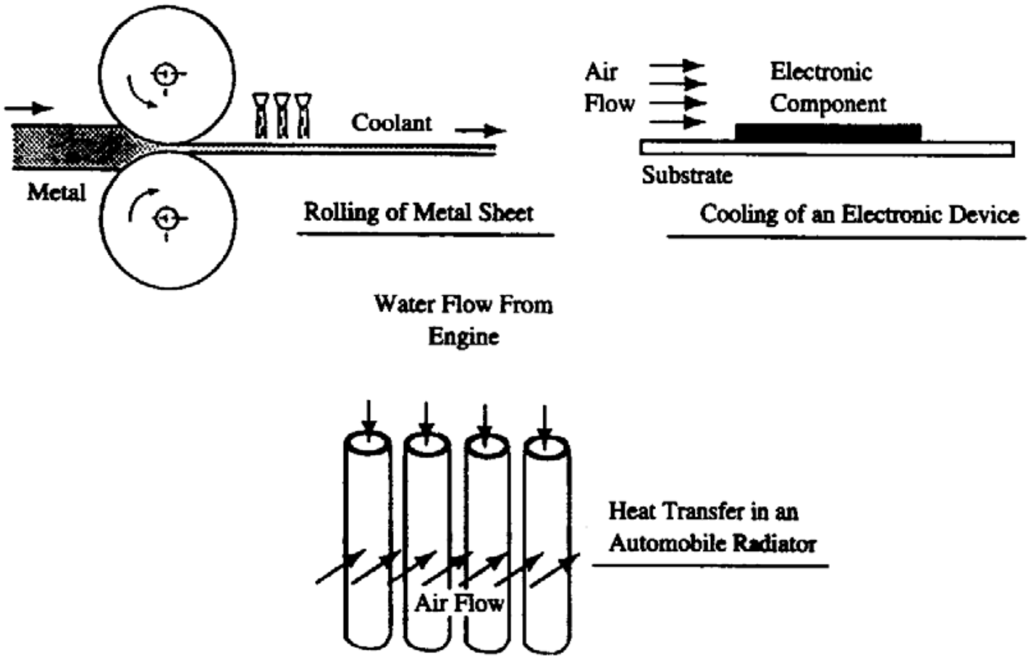

Figure 1.2 Some situations that involve convection (Oosthuizen and Naylor, 1999).

Convection can be classified into two basic processes, either natural (or free) or forced convection. In the case of forced convection, the fluid motion is caused by some external means such as fan or pump (Oosthuizen and Naylor, 1999). In the case of free convection, the flow is generated by the body forces that occur as a result of the density changes arising from the temperature changes in the flow field. These body forces are actually generated by pressure gradients imposed on the whole fluid. The most common source of this imposed pressure field is gravity and the body forces in this case are usually termed buoyancy forces. Another source of imposed pressure gradients which can cause free convection flow are the centrifugal forces which arise when there is an overall rotary motion such that exists in a rotating machine. In this thesis we consider only buoyancy forces.

In all flows involving heat transfer, the changes of temperature will occur and there exist the buoyancy forces arising from the gravitational field. The term “forced convection” is only applied to flows in which the effects of these buoyancy forces are negligible. In some flows in which a forced velocity exists, the effects of these buoyancy forces, will, however, not be negligible and such flows are termed combined- or mixed- free and forced convection flows. The various types of convection are illustrated in Figure 1.3. Finally, it is worth mentioning that convection is also classified as external and internal, depending on whether the fluid is forced to flow over a surface or in a pipe (Çengel, 2007). In this study, our concern is only on external convection.

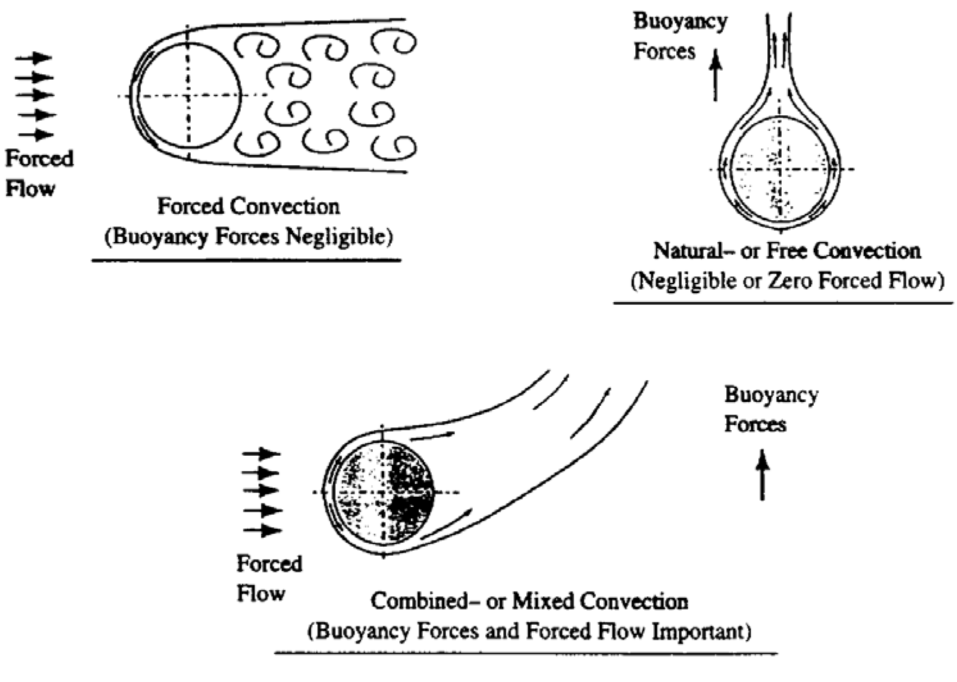

Figure 1.3 Forced, free and combined (mixed) convection (Oosthuizen and Naylor, 1999)

1.1.2 Viscous Fluid, Nanofluid and Porous Medium

1.1.2 (a) Viscous Fluid

The viscosity of a fluid is a measure of the resistance of a fluid deformed by shear stress and is one of the most important natures of fluid (Tanaka et al, 2012). Fluid flow can be classified as viscous or inviscid depending on the fluid idealization in terms of the presence of viscosity. Viscous fluids or real fluids are those, which have viscosity, whereas that having no viscosity is called inviscid fluid (ideal fluid). In reality, there is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Internal friction plays a vital role in viscous fluids during the motion of the fluid and viscous fluids are further classified into two categories (1) Newtonian fluids (2) Non- Newtonian fluids. A fluid that obeys the Newton’s law of viscosity i.e. shear stress increases linearly with strain rate ($\tau = \mu \dfrac{du}{dy}$; μ is constant dynamic viscosity of the fluid) is called a Newtonian fluid (Tanaka et al, 2012). Air, water, mercury are some of the examples of Newtonian fluids. In contrast, a fluid whose shear - strain rate relationship is not described with the Newton’s law of viscosity is called a non-Newtonian fluid. Many important industrial fluids are non-Newtonian in their flow characteristics. These include paints, coal tar, polymers, lubricants, plastics, printer ink and molecular materials etc.

1.1.2 (b) Nanofluid

Nanofluids can be defined as the dilution of nanometer-sized particles (smaller than 100nm) in a fluid (Das et al., 2007) as illustrated in Figure 1.4. The nanoparticles used in nanofluids are typically made of metals (Al, Cu) oxides (Al_2O_3,

TiO_2 and CuO), carbides (SiC), nitrides (AlN, SiN) or nonmetals (graphite, carbon nanotubes) and the base fluid is usually a conductive fluid, such as water or ethylene glycol (Wang and Mujumdar, 2007). Other base fluids are oil and other lubricants, bio-fluids and polymer solutions.

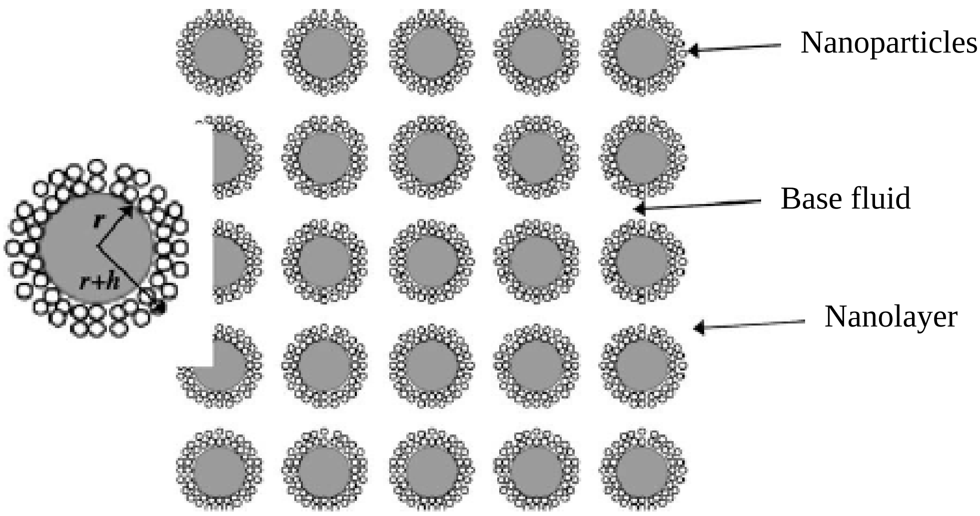

Figure 1.4 Physical model of nanofluid

Choi (1995) made the first attempt to introduce this innovative fluid. The mixture of a base fluid and nanoparticles has unique physical and chemical properties increases the thermal conductivity and therefore substantially enhances the heat transfer characteristics of the nanofluid (Aminossadati, 2009). Since the size of nanoparticles are in nanometer-sized, besides behaving similar as liquid molecules, they have the ability to flow smoothly through the microchannels easily (Khanafer et al., 2003), hence, nanofluids will enhance the thermal conductivity and convective heat transfer coefficient compared to the base fluid only (Kakac and Pramuanjaroenkij, 2009).

There are a few of nanofluid models for example Khanafer et al. (2003), Buongiorno (2006), Tiwari and Das (2007), Nield and Kuznetsov (2009) and Kuznetsov (2010). However, in this study, we only consider the model suggested by Tiwari and Das (2007) since we are interested to see how the nanoparticle volume fraction and types of nanofluid affected the fluid flow and heat transfer enhancement. The model suggested by Tiwari and Das (2007) takes into account the effect of nanoparticle volume fraction while the other model didn’t consider this factor. This model is a model that uses a single-phase model of Maxwell-Garnet model for thermal conductivity and Brinkman (1952) model for viscosity. This model has been succesfully applied by many researchers (Oztop and Abu Nada, 2008; Muthtamilselvan et al, 2010). These models are restricted to spherical nanoparticles where it does not account for other shapes of nanoparticles.

1.1.2 (c) Porous medium

A porous medium (porous material) is a solid which often called frame or matrix permeated by an interconnected network of pores (voids) that filled with a fluid (liquid or gas). Usually both the solid matrix and the pore network (also known as the pore space) are assumed to be continuous, so as to form two interpenetrating continua such as in a sponge. Many natural substances such as rocks, soils, biological tissues (e.g. bones), and man made materials such as cements, foams and ceramics can be considered as porous medium (Oosthuizen and Naylor, 1999; Straughan, 2008). Some of the well known porous medium can be seen in the Figures 1.5 (a)–(d)

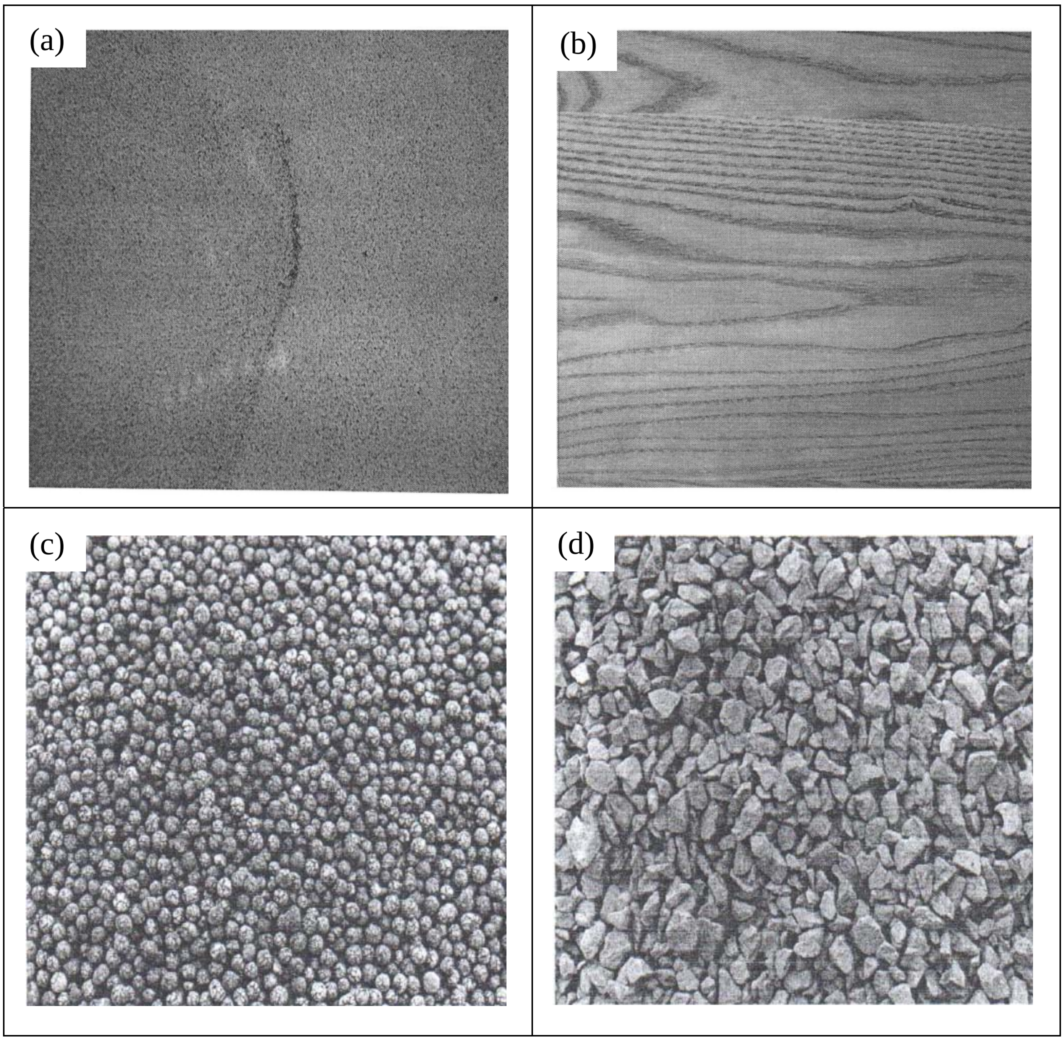

Figure 1.5 Some examples of porous medium (a) sandstone (b) wood (Straughan, 2008) (c) Liapor spheres (d) crushed limestone (Nield and Bejan, 2006)

There are several forms of the momentum equation which is the porous medium analog of the Navier-Stokes equations (Nield and Bejan, 2006). In this thesis, we use Darcy's law which states the volume-averaged velocity is proportion to the pressure gradient. In the Darcy model of flow through porous medium, it is assumed that the flow velocities are low and that the momentum changes and viscous forces in the fluid are consequently negligible compared to the drag force on the particles (Oosthuizen and Naylor, 1999).

1.1.3 Dimensionless Parameters

1.1.3 (a) Prandtl number

Prandtl number is a dimensionless parameter defined as (Cengel, 2007):

$$\Pr = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

It is named after Ludwig Prandtl and this parameter describes the relative thickness of the velocity/hydrodynamics and the thermal boundary layers. Table 1.1 shows that the Prandtl numbers of fluids range from less than 0.01 for liquids metals to more than 100, 000 for heavy oils.

Table 1.1 Prandtl number of different fluids (Cengel, 2007)

Fluid	Pr
Liquid metals	0.004 – 0.030
Gases	0.7 – 1.0
Water	1.7 – 13.7
Light organic fluids	5 – 50
Oils	50 – 100, 000
Glycerin	2000 – 100, 000

The Prandtl numbers of gases are about 1, which indicates that both momentum and heat dissipate through the fluid at about the same rate. Heat diffuses very quickly in liquid metals ($\Pr \ll 1$) and very slowly in oils ($\Pr \gg 1$) relative to momentum. Consequently the thermal boundary layer is much thicker for liquid metals and much thinner for oils relative to the velocity boundary layer.

1.1.3 (b) Nusselt number

The Nusselt number is named after Wilhelm Nusselt, and it is viewed as the dimensionless convection heat transfer coefficient. The Nusselt number is defined as

$$Nu = \frac{hL}{k} = \frac{h\Delta T}{k\Delta T/L} = \frac{\text{heat flux in convection}}{\text{heat flux in conduction}}$$

where h is heat transfer coefficient, k is the thermal conductivity of the fluid and L is the characteristic length. Heat transfer through fluid layer is by convection when the fluid involves some motion and by conduction when the fluid layer is motionless (Cengel, 2007). Therefore, the Nusselt number represents the enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid. The larger the Nusselt number, the more effective the convection. A Nusselt number of Nu = 1 for a fluid layer represents heat transfer across the layer by pure conduction (Cengel, 2007).

1.1.3 (c) Reynolds number

The transition from laminar to turbulent flow depends on the surface geometry, surface roughness, flow velocity, surface temperature and type of fluid, among other things (Cengel, 2007). In the 1880s, Osborn Reynolds discovered that the flow regime depends mainly on the ratio of the inertia forces to viscous forces in the fluid. This ratio is called the Reynolds number, which is a dimensionless quantity, and is expressed for external flow as (Cengel, 2007; Tanaka, 2012):

$$\mathrm{Re} = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{VL}{\nu} = \frac{\rho VL}{\mu}$$

where V is the upstream velocity (equivalent to the free-stream velocity for a flat plate), L is the characteristic length of the geometry, and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. At large Reynolds numbers, the inertia forces, which are proportional to the density and the velocity of the fluid, are large relative to the viscous forces, and thus the viscous forces cannot prevent the random and rapid fluctuations of the fluid. At small or moderate Reynolds numbers, however, the viscous forces are large enough to suppress these fluctuations and keep the fluid "in line". Thus the flow is turbulent in the first case and laminar in the second.

1.1.3 (d) Grashof Number

The Grashof number Gr, is the dimensionless parameter represents the natural convection effects. The Grashof number is defined as

$$Gr = \frac{g\beta\left(T_w - T_\infty\right)L^3}{\nu^2}$$

where g is the gravitational acceleration, β is coefficient of volume expansion, T_w is temperature of the surface, T_∞ the temperature of the fluid sufficiently far from the surface, L is characteristic length of the geometry and ν is kinematic viscosity of the fluid.

The flow regime in natural/free convection is governed by the dimensionless Grashof number, which represents the ratio of buoyancy force to the viscous force acting on the fluid. Grashof number provides the main criterion in determining whether the fluid flow is laminar or turbulent in natural convection.

When the surface is subjected to external flow, the problem involves both natural and forced convection. The relative importance of each mode of heat transfer is determined by the value of the coefficient Gr/Re^2. Natural convection effects are negligible if $Gr/\mathrm{Re}^2 \ll 1$, free convection dominates and the forced convection effects are negligible if $Gr/\mathrm{Re}^2 \gg 1$, and both effects are significant and must be considered if $Gr/\mathrm{Re}^2 \approx 1$.

1.1.3 (e) Eckert number

The Eckert number, Ec is a dimensionless quantity useful in determining the relative importance in a heat transfer situation of the kinetic energy of a flow. It is the ratio of the kinetic energy to the enthalpy (or the dynamic temperature to the temperature) driving force for heat transfer

$$Ec = \frac{U^2}{C_p\left(T_w - T_\infty\right)}$$

where U is an appropriate fluid velocity (e.g., outside the boundary layer or along the centerline of a duct), C_p is the specific heat at constant pressure and $T_w - T_\infty$ is the driving force for heat transfer (e.g., wall temperature minus free stream temperature). For small Eckert number ($Ec \ll 1$) the terms in the energy equation describing the effects of pressure changes, viscous dissipation, and body forces on the energy balance can be neglected and the equation reduces to a balance between conduction and convection.

1.1.3 (f) Peclet number

The combination of the Reynolds and Prandtl number gives another dimensionless group called the Peclet number (Janna, 2009). Peclet number represents the ratio of heat transfer by motion of a fluid to heat transfer by thermal conduction.

$$Pe = \frac{UL}{k} = \mathrm{Re} \cdot \mathrm{Pr}$$

where u is fluid velocity, L is a characteristic dimension, and k is thermal diffusivity of the fluid. Re is Reynolds Number and Pr is Prandtl Number. Heat transported by the fluid per unit area is proportional to $\mu\rho C_p$ where ρ is density and C_p is specific heat capacity, while heat conducted per unit area is proportional to λ/L where λ is thermal conductivity. Hence

$$\frac{\text{heat transported}}{\text{heat conducted}} \approx \frac{\mu\rho C_p}{\lambda/L} = \frac{\mu L}{\lambda/\rho C_p} = \frac{\mu L}{k}$$

1.1.3 (g) Rayleigh number

Rayleigh number, Ra, is a dimensionless term used in the calculation of natural convection

$$\text{Ra} = \frac{g\beta\left(T_w - T_\infty\right)L^3}{\nu\alpha} = \text{Gr}\cdot\text{Pr}$$

where g is acceleration due to gravity, β is coefficient of thermal expansion of the fluid, $T_w - T_\infty$ is temperature difference, x is length, ν is kinematic viscosity and k is thermal diffusivity of the fluid. Gr is the Grashof Number and Pr is the Prandtl Number. The magnitude of the Rayleigh number is also an indication as to whether the natural convection boundary layer is laminar or turbulent (Das et al, 2008).

1.1.4 Boundary Layer Theory

The Navier-Stokes equations are basic equations in fluid mechanics and analytical treatment of the Navier-Stokes equations presents great difficulties. The boundary layer concept, first introduced by Ludwig Prandtl in 1904, provides major simplifications of the Navier-Stokes equation. Ludwig Prandtl showed that the flow past a body can be divided into two regions as depicted in Figure 1.6 (Jiji, 2009):

(1) A very thin layer close to the body where the viscosity is important. This thin layer where friction effects cannot be ignored is called the boundary layer.

(2) the remaining region outside this layer where the viscosity can be neglected. The flow in this region is considered inviscid (Çengel, 2007).

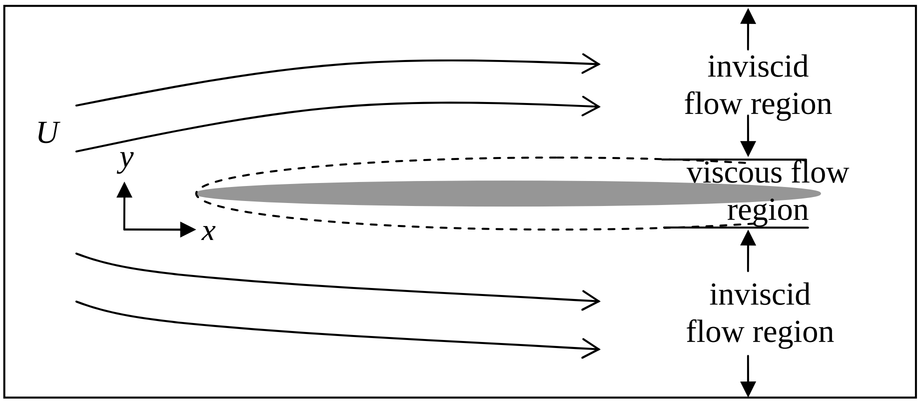

Figure 1.6 The flow of an originally uniform fluid stream over a surface, the regions of viscous flow (next to the surface) and inviscid flow (away from the plate).

According to Prandtl, it might be sufficient in an analysis of a flow field to consider action of viscosity within boundary layer, whereas the flow outside the boundary layer may be considered inviscid. Prandtl then derived the so-called boundary layer equations by simplifying the conservation equation using scale analysis applied to the terms in the conservation equations (Eckert and Drake, 1972). The boundary layer itself can be divided into two types (Cebeci, 2002) as shown in Figure 1.7 for a simple flow configuration over a flat surface:

(1) velocity boundary layer which is also known as hydrodynamic or momentum boundary layer.

- Interaction between the fluid and the surface will produce a region in the fluid where the *x*-component velocity *u* rises from zero at the surface (no slip condition) to an asymptotic value equal to U_∞. This region of large velocity gradient is called the velocity boundary layer. This layer is characterized by the velocity gradient and the shear stress (Cebeci, 2002).

(2) thermal boundary layer or temperature boundary layer.

- Due to temperature difference between the fluid and the surface leads to the formation of a region in the fluid where the temperature also varies rapidly with y near the surface, changing from temperature at the wall T_w to external flow value T_∞. This region with large temperature gradient is called the thermal boundary layer. This thermal boundary layer is characterized by the temperature gradient and the heat transfer (Cebeci, 2002).

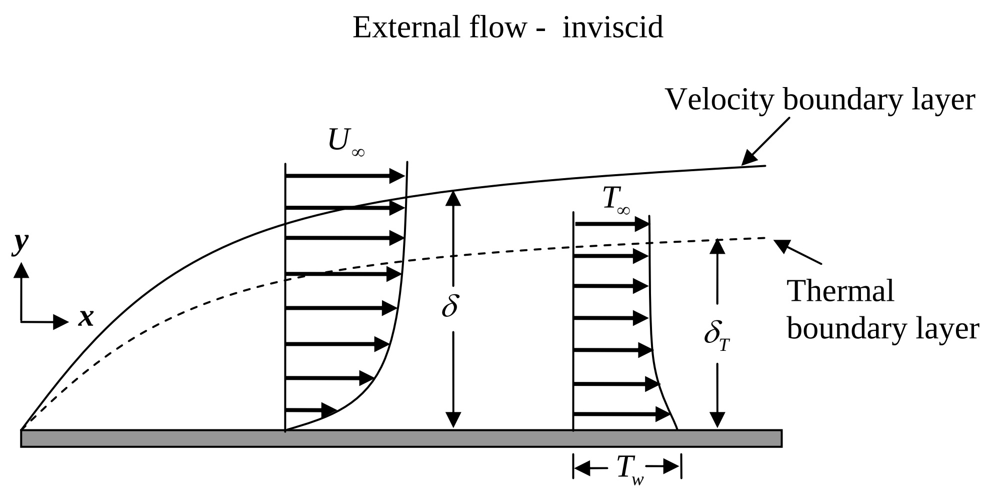

Figure 1.7 Velocity and thermal/temperature boundary layers

In boundary layer concept, under special conditions, certain terms in the governing equations are very small compare to others and therefore can be neglected. The boundary layer equations can be obtained by boundary layer approximations. The intuitive arguments of boundary layer approximation are: velocity component,

$u \gg v$; velocity gradients, $\dfrac{\partial u}{\partial y} \gg \dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ and temperature gradients, $\dfrac{\partial T}{\partial y} \gg \dfrac{\partial T}{\partial x}$ (Çengel, 2007). It is worth to highlight that we assume x and y – axes respectively parallel and perpendicular to the surface. Therefore u and v are velocity components along and perpendicular to x and y axes respectively. In Chapter 2 of this thesis we will further discuss the scale analysis of boundary layer approximations which is a preferred procedure by many authors.

1.1.5 Similarity solutions

The system of equations for boundary layer flow problems is in the form partial differential equations (PDE) and often difficult to be solved (Ali and Hafez, 2012). The term "similarity solution" was introduced in 1908 by Blasius to solve Prandtl's boundary layer equations (Ishak, 2010) and nowadays these solutions have been extensively studied by a number of researchers such as Nazar and Pop (2006), Layek et al. (2007), Fang et al. (2009), Ahmad and Pop (2010), Ishak (2010), Ahmad et al. (2011), Arifin et al. (2011), Lok and Pop (2011), Lok et al. (2011), Ali and Hafez (2012), Hamad and Ferdows (2012), Uddin et al. (2012), Bachok et al. (2013), Mohamed et al. (2013) and many others.

Similarity solutions to PDEs are solutions which depend on certain groupings of the independent variables, rather than on each variable separately. The idea of similarity solution is to simplify the governing equations by reducing the number of independent variables, using a coordinate transformation. The transformation is called similarity transformation and the independent variables x and y in a PDE is combined appropriately as a new independent variable $\eta(x, y)$ which is known as "similarity variable." By employing an appropriate similarity transformation, the partial differential equations are reduced to similarity equations in the form ordinary differential equations and are much easier to be solved. From the physical point of view, the meaning of similarity solutions is that the velocity and temperature profiles of the flow remains geometrically similar in each transversal section of the surface (Nazar and Pop, 2006).

Similarity solutions of the boundary layer equations in fluid mechanics and heat transfer theory proved to be very useful in the interpretation of certain flow and heat transfer characteristics. Although, exact solutions represent highly specialized situations, they do give clues concerning the nature of more realistic behaviour. According to Weidman et al. (2008), a good understanding of the flow physics under consideration usually can be gained from the information of new problems that are using similarity solution. Moreover, when applying similarity results to specific engineering applications, the information can also provide the benchmarks against which numerical codes may be tested (Weidman et al. 2008).

However, not all problems admit similarity solutions, since they depend on various factors, such as the surface geometries, boundary conditions, and the surface heating conditions. According to Wang (2011), similarity solutions exist for flows which show certain symmetries and group properties. In this study, we consider only the problems that admit similarity solutions.

1.2 Research Background

The Navier-Stokes equations consist of a set of nonlinear partial differential equations with very few exact solutions. Similarity transformation renders the Navier-Stokes equations into a set of ordinary differential equations and retains the nonlinearity of the equations. Due to this, similarity solutions might demonstrate nonlinear phenomena such as non-existence and non-uniqueness (Wang, 2011). Dealing with mechanics of nonlinear fluids is a great challenge to physicists, engineers and mathematicians (Layek et al., 2007). According to Mishra and DebRoy (2005), multiple solutions exist for many complex problems in convective heat transfer due to highly non-linear problems. Further, as reported in a book by Schlichting (1979), the nonlinearity of the differential equations, variation of geometric or fluid mechanical parameters can lead to bifurcations in the solution and thus to multiple solutions.

There are significant numbers of studies reporting on multiple solutions in boundary layer flow (Gelfgat and Bar-Yoseph, 2004). In this respect, we mention just the following papers by Ramachandran et al. (1988), Ma and Hui (1990), Aly et al. (2003), Gelfgat and Bar-Yoseph (2004), Liao (2005), Guedda (2006), Lu (2007), Weidman et al. (2008), Fang et al. (2009), Ishak et al. (2009), Yao (2009), Ahmad and Pop (2010), Ishak (2010), Ahmad et al. (2011), Arifin et al. (2011), Lok and Pop (2011), Lok et al. (2011), Turkyilmazoglu (2011), Rohni et al. (2011).

The physical situation described in such situations shows that the studies on multiple similarity solutions are theoretically and practically important. An improved understanding of multiple solutions in boundary layer flows and heat transfer and the applications of the knowledge on these solutions to new design techniques should provide substantial amendments in cost, reliability and performance of many fluid dynamics and heat transfer devices.

1.3 Problem statement

Similarity solutions are often used to take a first look at different problems encountered in Newtonian (viscous), porous media and non-Newtonian flows, just to name a few of direct engineering interest (Wang, 2011). Over the past several decades a number of studies have shown the existence of multiple solutions of boundary layer flows driven by moving surfaces or by buoyancy forces (free and mixed convection boundary layers).

The multiplicity of solutions in fluid dynamics and heat transfer is important to be able to be computed since solutions arising from bifurcations often interact with one another producing otherwise inexplicable phenomena (Cliffe et al., 2000). The transition process provides valuable information of flow evolution and transition to multiple solutions acts as a starting point to turbulence or chaos (Gelfgat and Bar-Yoseph, 2004). In heat transfer engineering, the flow multiplicity may significantly affect the structure and quality of the final product in material processing (Mishra and DebRoy, 2005). An improved understanding of the development of flow states multiplicity can stimulate innovations and lead to enhancement of performance and reliability as well as reducing the costs of various practical flow problems such as rotating machines and crystal growth processes (Gelfgat and Bar-Yoseph, 2004).

Nowadays, it is widely understood that the computational modelling of bifurcation as well as multiplicity of fluid flows are important, but the computation of all possible multiple solutions, still remain a challenge (Gelfgat and Bar-Yoseph, 2004). In real situation, multiple solutions that may exist in boundary layer flow are difficult to visualize. This can cause researchers fail to notice the multiple solutions that might exist within the flows, which is an important aspect of fluid mechanics (Yao, 2009). Therefore, a mathematical computation and analysis are required to determine the existence of multiple solutions. The significance of computational modelling of bifurcation as well as multiplicity of fluid flows as highlighted above leads us to the objective of the present study as given in the following section.

1.4 Objectives and Scope

The objectives of this study are to construct mathematical model, to carry out mathematical formulations and analyses, to obtain the numerical results and to examine the occurrence of multiple similarity solutions in convection boundary layer flows for the following problems:

(1) Steady mixed convection boundary layer flow over an exponentially shrinking vertical sheet with suction.
(2) Steady free convection boundary layer flow over a non-linearly horizontal shrinking sheet with suction and viscous dissipation: Cortell's model.
(3) Unsteady free convection boundary layer flow over a horizontal shrinking sheet with suction in nanofluids.

(4) Unsteady mixed convection boundary layer flow with suction and temperature slip effects near the stagnation-point on a vertical permeable surface embedded in a porous medium.

(5) Steady mixed convection boundary layer flow along a vertical cylinder embedded in a porous medium filled by a nanofluid.

The scope of this study is restricted to two-dimensional, steady or unsteady incompressible laminar boundary layer flows, towards vertical or horizontal shrinking sheets with suction in viscous fluids or nanofluid, vertical plate near stagnation point in porous medium and vertical cylinder in porous medium filled by nanofluid. The governing equations for each problem considered are transformed to similarity equations using similarity transformation. The resulting similarity equations are then solved numerically via shooting technique. All of the problems considered in this thesis are restricted to two-dimensional flows for the purpose of simplicity. However, three-dimensional flows are also can be done in future research as suggested in the last chapter of this thesis.

1.5 Research Methodology

The studies in this thesis undertake the following research methodology:

1.5.1 Problem Formulation

The full boundary layer equations are derived and a mathematical model of every problem mentioned in Section 1.4 is developed.

1.5.2 Mathematical Analysis

An appropriate similarity transformation cited in the literature is employed and the governing equations in the form of partial differential equations are transformed into ordinary differential equations which is easier to be solved.

1.5.3 Numerical Computation

The resulting ordinary differential equations are nonlinear and in a system of boundary value problem. The equations are then solved numerically using a shooting method implemented in Maple program.

1.6 Shooting Method and Maple Implementation

The system of nonlinear ordinary differential equations that governs the boundary layer flows in this thesis is the system of two-point boundary value problem. It is known that a number of methods exist for solving boundary value problem including the shooting, collocation and finite difference methods. Among these, finite difference and the shooting methods are commonly the only competitive methods judging from the viewpoint of efficiency. However, shooting method has many advantages such as easy to programme in a general form, less storage required, suitability for automatic procedures and it also can reveal more detailed flow structures (Yao, 2009).

It has been reported that, shooting method has been successfully used by previous researchers to solve boundary layer problems involving multiple solutions. Examples may be found in Lu (2007), Fang et al. (2009), Bhattacharyya (2011), Bhattacharrya and Vajravelu (2011) and many others. In shooting technique, the systems of boundary value problem (BVP) are first converted to initial value problem (IVP). This well-known technique is an iterative algorithm which attempts to identify appropriate initial conditions for a related IVP that provides the solution to the original BVP. The details of shooting method can be found in the book for example by Na (1979) and Jaluria and Torrance (2003). Nowadays, there are available package of shooting technique with Maple implementation, *shoot* (Meade et al., 1996) which has been successfully used by many researchers, such as Ahmad and Pop (2010), Ali et al. (2011), Remeli et al. (2011), Ariffin et al. (2011), Ahmad et al. (2011), Mat et al. (2012), Mohamed et al. (2013) to solve problems involving multiple similarity solutions.

All problems in this thesis have been solved via the shooting technique with Maple implementation, *shoot*. In this package, they used shooting technique involving fourth order Runge – Kutta method (RK4) to solve initial value problem and Newton Raphson method for correction scheme. The details of shooting method with Maple implementation *shoot*, has been described in paper by Meade et al. (1996). A general formula of RK4 and Newton Raphson method are also provided in Appendix B in this thesis. Basically, the idea of shooting method is to reformulate the BVP to be IVP. This method involves the following steps:

(1) The boundary value problem subject to the related boundary conditions is converted into an equivalent initial value problem.

(2) Using trial and error or some scientific approach, a suitable guess values for the initial values are made so that the calculated values fulfill the actual boundary conditions.

(3) If these boundary conditions are not satisfied to the desired accuracy, the process is repeated with a new set of initial conditions until the desired accuracy is achieved or an iteration limit is reached (Meade et al., 1996).

This procedure for specific problem considered in this thesis is given in details in each chapter of the problem and the Maple program used to solve one of the problems considered is given in Appendix D.

1.7 Thesis Organization

The multiplicity of similarity solutions of boundary layer flows in five different convection situations are presented in this thesis. It comprises nine chapters where Chapter 1 is the introduction chapter which gives a picture and an idea of the whole thesis. The literature reviews related to the problems considered in this thesis are discussed in Chapter 2. Then, Chapter 3 is the derivation of boundary layer equations related to the problems considered in this thesis. Further, Chapter 4 to Chapter 8, respectively, explained in detail each of the five problems considered and finally, this thesis ends with Chapter 9 that is the conclusion part.

In Chapter 1, a basic concept of convection, viscous fluid, nanofluid, porous medium, boundary layer theory, dimensionless numbers and similarity solutions are described in Section 1.1. A brief description about convection itself and the types of convection i.e. free, forced and mixed convection are discussed. Subtopic on viscous fluid, nanofluid, porous medium and dimensionless parameters are also included in Section 1.1. As the present study is on similarity solutions in boundary layer, therefore the boundary layer theory and similarity solutions are also described. Research background related to multiplicity of solutions in boundary layer is highlighted in Section 1.2 and the problem statement is stated in Section 1.3. Realizing the importance of multiplicity of solutions in boundary layer flow and heat transfer leads to the objectives and scope of this study which is listed in Section 1.4. Basically, the main objective of this study is to solve five separate boundary layer problems that admit similarity solutions. The research methodology used to solve the problems is discussed in Section 1.6. All problems considered in this thesis are solved numerically using shooting technique with the aid of Maple software, therefore a specific section on shooting method and Maple implementation, `shoot` is provided in Section 1.7. Finally, thesis outline which gives the overview of thesis is provided in this section 1.8 which is the last section of Chapter 1.

We then discussed the literature reviews of the previous and related studies to the problem considered in this thesis in Chapter 2. This Chapter has been divided into six subsections where the first section is an introduction to the Chapter. In Section 2.2, 2.3, 2.4, 2.5 and 2.6, we discussed literature reviews related to the first, second, third, forth and fifth problem, respectively.

The derivations of boundary layer equations related to the problems considered are given in Chapter 3. The Boussinesq and boundary layer approximations are highlighted in this chapter. Chapter 3 consists of four subsections including introduction to chapter. In this chapter, we derive the general boundary layer equations for two dimensional incompressible viscous flows over vertical plate, two dimensional incompressible flows along vertical plate in porous media and two