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TRANSFER FUNCTIONS IN HIERARCHICAL PRODUCTION PLANNING (HPP)

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Abstract:. In the production planning it is essential to determine simultaneously the production program, the lot sizes as well as sequencing and scheduling the batches: the planning of the production program requires the knowledge of the available manufacturing capacities; these depend, however, on the set-up times determined by lot-sizing, and the idle times of machines induced by sequencing and scheduling. In addition, sequencing and scheduling depend on lot-sizing, as set-up and processing times are influenced by lot-sizes. Lot-sizing and scheduling in turn require the knowledge of the production program. In this paper, the approach of hierarchical production planning and the concept of transfer functions is applied to describe the interactions between the production program and lot-sizing decisions. As a first step, a basic linear model to optimize the production program is formulated. Then transfer functions describing the relation between the quantities to be produced, the optimum lot-sizes and the relevant cost induced by the optimal inventory policy are determined by solving inventory models for all possible quantities of goods to be produced.

Key words: transfer functions, hierarchical production planning, lot-sizing decisions, inventory cost.
Mathematical subject classification number: 90B05, 90B30, 65K10

1. The Basic Idea

Hierarchical models are conceived to avoid the disadvantages of sequential as well as of simultaneous models and to combine the benefits of both approaches. The objective of these models is to reduce the complexity without neglecting the interdependencies between various aspects of a planning problem. In the case of production planning and inventory management, both problems are solved at different hierarchical levels using separate models, which are only interrelated by the targets set by the upper level deciding on the production program, and the feedback reported by the lower level setting the number and sizes of lots. The model of hierarchical production planning to be presented here may be described as follows:

(1) A company produces m goods $j = 1, \ldots, m$.

(2) The maximal quantity of product j which can be sold during period t is $v_{jt}^{\max}$; the demand is considered to be deterministic but varies during the periods $t = 1, \ldots, T$; moreover, there are minimal quantities to be delivered $v_{jt}^{\min}$.

(3) Capacities b_i of machines $i = 1, \ldots, n$ are available; to produce one unit of product j, a_{ij} units of machine capacity i are required.

(4) The selling of one unit of product j provides a net profit of d_j.

(5) The products are produced in lots. Set-up cost for each lot of product j are given by C_j^R; holding cost are equal to c_j^L per unit of quantity and time.

(6) The lot-sizes of different products are determined independently from each other.

At the upper level, the production program is determined by using a linear programming model (LP1). The objective is to maximize net profits:

$$D = \sum_{t=1}^{T} \sum_{j=1}^{m} d_j . x_{jt} \Rightarrow \max! \qquad (1)$$

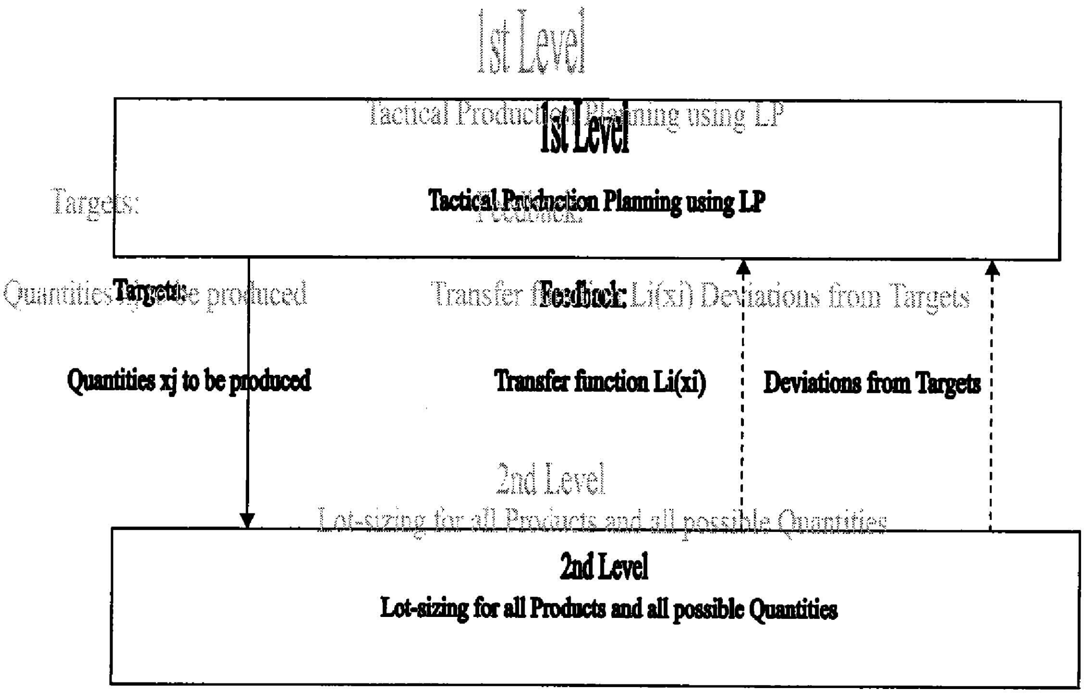
The following restrictions have to be considered:

Capacities: $\sum_{j=1}^{m} a_{ij} . x_{jt} \le b_i$ $\qquad (i = 1, \ldots , n ; t = 1, \ldots ,T)$

Sales: $v_{jt}^{\min} \le x_{jt} \le v_{jt}^{\max}$ $\qquad (j = 1, \ldots ,m ; t = 1, \ldots ,T)$

The costs of inventories are not considered in this type of planning models. To include this type of cost, a transfer function $L_j(x_j)$ has to be established, which describes the functional relation between the quantity x_j of product j to be produced during the next period and the cost of inventory, provided that optimal lot-sizes are determined at the second level (inventory management). The transfer function $L_j(x_j)$ is established for every product j = 1, … ,m using simple inventory models to calculate the optimum lot-size for every quantity x_j. Figure 1 shows the coordination between production planning and inventory management: Targets x_j are set by the upper level and are given to the lower level. As a feedback, the lower level has to report the transfer function, the cost for an optimal inventory handling in relation with the targets, as well as any deviations from the targets.

Figure 1. Coordination in a Hierarchical Production Planning System



2. Determination of a Transfer Function for Inventory Cost

By using inventory models, optimal lot-sizes and costs induced by this choice are determined for all possible quantities of output x_j of products j = 1, … , m. To explain the principle of a feedback by transfer functions, we now consider the Economic-Order-Quantity

(EOQ) model of inventory theory. This implies that sales of products are distributed equally over the planning period of inventory management. If the length of this period is τ, then the rate of demand is given by:

$$\delta_j = \frac{x_j}{\tau} \qquad \text{or, if } \tau = 1 \qquad x_j = \delta_j \qquad j = 1, \ldots, m \tag{2}$$

The relevant cost of inventory induced by a planned production of x_j as a function of lot-sizes q is given by:

$$l_j(x_j, q_j) = \frac{c_j^R}{q_j}.x_j + \frac{1}{2}.q_j.c_j^L \tag{3}$$

An optimal lot-size can be calculated by using the square root formula:

$$q_j^o = \sqrt{\frac{2.c_j^R.x_j}{c_j^L}} \tag{4}$$

Substituting q^o for q in $l_j(x_j, q)$, we get:

$$L_j(x_j) = l_j(x_j, q_j^o) = \sqrt{2.c_j^R.c_j^L.x_j} = K_j.x_j^{\frac{1}{2}} \tag{5}$$

The transfer function has the following characteristics:

(1) It is not defined for $x_j \le 0$, and is continuous and differentiable for $x_j > 0$

(2) It increases monotonously, and is strictly concave:

$$\frac{dL_j(x_j)}{dx_j} = \frac{1}{2}.K_j.x_j^{-\frac{1}{2}} > 0$$

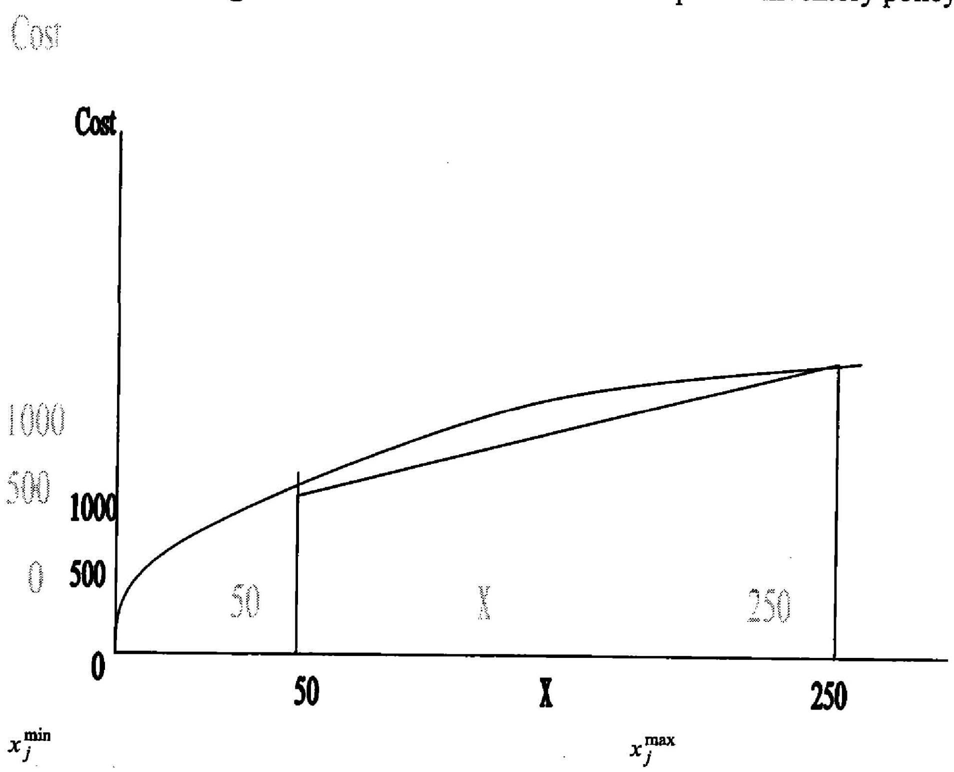
$$\frac{d^2L_j(x_j)}{dx_j^2} = -\frac{1}{4}.K_j.x_j^{-\frac{3}{2}} < 0$$

(3) For sufficiently large quantities x_j, the curvature is very small:

$$\frac{d^3L_j(x_j)}{dx_j^3} = \frac{3}{8}.K_j.x_j^{-\frac{5}{2}}$$

Due to this fact, the transfer function may be approximated in the relevant interval by a linear function or at least by a piecewise linear function. This fact is shown in figure 2 for the coefficients $c_j^R = 1000$, $c_j^L = 10$, $d_j = 20$.

Figure 2. Transfer function of cost of optimal inventory policy



In principle, other inventory models may be applied to establish transfer functions of inventory cost as well.

The transfer function can be used to obtain a minimum lot-size x_{jt}^{min}. This is defined as the break even point, where the gross profit gained by selling x_{jt}^{min} just covers the set-up and holding cost:

$$d_j.x_j = K_j.\sqrt{x_j} \Rightarrow x_j^{min} = \left(\frac{K_j}{d_j}\right)^2 = \frac{2.c_j^R.c_j^L}{d_j^2} \qquad (6)$$

The existence of a single break-even point is guaranteed by the following facts:

(i) For $x_j \Rightarrow 0$, the cost of inventory converge to zero but the gradient goes to ∞

(ii) For $x_j = 0$, the net profit is equal to zero, its increase is constant and equal to d_j

(iii) Net profit minus cost of inventory is negative in the environment of $x_j = 0$

(iv) Due to the strict concavity of the transfer function, the difference of net profit and cost of the inventory is positive for sufficiently large x_j

If the demand x_{jt} of a product *j* in period *t* is less than the break even point x_j^{min}, then this product is considered to be "exotic" and has to be planned separately: That means, the demand of several periods will either be combined in order to obtain a minimum lot-size x_{jt}^{min}; or it will not be produced at all during that period. Hence, the quantity to be produced will then be pre-determined as $x_{jt} = 0$.

3. Production Planning with Transfer Functions

Having established the transfer functions for the inventory cost $L_j(x_j)$ for the minimum quantities to be produced $x_{jt}^{\min}$, we can now determine the production program. In order to take into account cost of inventories, the transfer functions $L_j(x_j)$ have to be included into the objective function of LP2:

$$D = \sum_{t=1}^{T}\sum_{j=1}^{m}\left[d_j.x_{jt} - L_j(x_{jt})\right] \Rightarrow \max! \qquad (7)$$

The capacity restrictions of LP1 remain the same:

$$\sum_{j=1}^{m} a_{ij}.x_{jt} \le b_i \qquad (i = 1, \dots, n\,;\, t = 1, \dots, T)$$

For exotic products, the sales restrictions

$$v_{jt}^{\min} \le x_{jt} \le v_{jt}^{\max} \qquad (j = 1, \dots, m\,;\, t = 1, \dots, T)$$

have to be modified to take into account pre-determined quantities:

$$v_{jt}^{\min} = \begin{cases} 0, \textit{for all exotic products not to be produced} \\ x_j^{\min}, \textit{for all exotic products to be produced with minimal quantity} \\ v_{jt}^{\min}, \textit{for all other products} \end{cases}$$

For all exotic products which are not to be produced, the upper limit has to be set to zero. As $L_j(x_j)$ is concave, the objective function LP2 is convex: as the second derivative of $L_j(x_j)$ is very small, it can, be approximated by a linear function,

$$\tilde{L}_j(x_j) = \alpha_j.x_j + \beta_j \qquad (8)$$

the parameter are given by

$$\alpha_j = \frac{L_j(x_j^{\max}) - L_j(x_j^{\min})}{x_j^{\max} - x_j^{\min}} \qquad \beta_j = L_j(x_j^{\min}) - \alpha_j.x_j^{\min} \qquad (9)$$

The constant term β_j may be neglected, as it does not influence the optimum of solution. Hence, the objective function of LP2 may be approximated by,

$$\tilde{D} = \sum_{t=1}^{T}\sum_{j=1}^{m}\left[d_j.x_{jt} - \alpha_j(x_{jt})\right] \Rightarrow \max! \qquad (10)$$

The optimal production program may be obtained by solving LP2 with the simplex method (or another method of linear programming).

Figure 2 shows that the linear approximation of the transfer function $\tilde{L}_j(x_j)$ underestimates the cost of inventories. The result may be improved by approximating the objective function through a piecewise linear function and by formulating the problem as a separable problem (Hadley, 1964). This approach replaces the objective function by a piecewise linear function. To do, so, net profits

$$d_{jt}^{k} = d_j . x_{jt}^{k} - L_j(x_{jt}^{k}) \quad (11)$$

and calculated for a finite number of x_{jt}^{k} mesh points

$$v_{jt}^{\min} = x_{jt}^{1} < x_{jt}^{2} < ... < x_{jt}^{h_j} = v_{jt}^{\max} \quad (12)$$

The production program may then be determined as an optimal convex combination of mesh points.

The objective function of LP3 is given by,

$$D = \sum_{t=1}^{T} \sum_{j=1}^{m} \sum_{k=1}^{h_j} d_{jt}^{k} . \lambda_{jt}^{k} \Rightarrow \max! \quad (13)$$

The following restrictions have to be considered:

(1) Non-negativity and convexity conditions

$$\lambda_{jt}^{k} \geq 0 \quad (j = 1,...,m; \quad t = 1,...,T; \quad k = 1,...,h_j)$$

$$\sum_{k=1}^{h_j} \lambda_{jt}^{k} = 1 \quad (j = 1,..., m; \quad t = 1,..., T)$$

(2) For each variable, there are at most two neighboring weights λ_{jt}^{k} positive

(3) Capacity restrictions:

$$\sum_{j=1}^{m} a_{ij} . x_{jt} \leq b_i \quad (i = 1,...,n; \quad t = 1,...,T)$$

(4) Sales restrictions:

$$v_{jt}^{\min} \leq x_{jt} \leq v_{jt}^{\max} \quad (j = 1,...,m; \quad t = 1,...,T)$$

(5) The relationship between the weights λ_{jt}^{k} and the variables x_{jt}^{k} is given by,

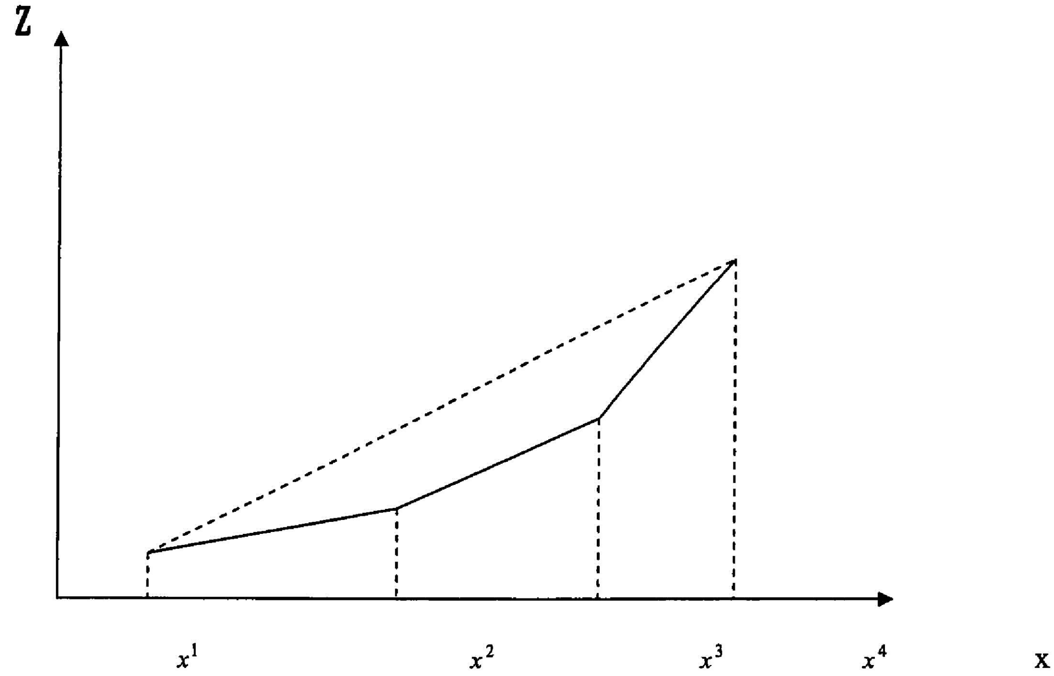
$$x_{jt} = \sum_{k=1}^{h_j} x_{jt}^{k} . \lambda_{jt}^{k} \quad (j = 1,...,m; \quad t = 1,...,T)$$

If a convex function is to be minimized or a concave is to be maximized, then it is always guaranteed that in the optimal solution only adjacent mesh points are considered, as a convex combination of other mesh points come up with a less favorable result. This fact is demonstrated in figure 3: If the function is to be minimized, then the convex combination of the maximal and the minimal mesh point (dashed line) is higher than the combination of the adjacent mesh points, e.g. of points x^1 and x^2 of points x^2 and x^3, and of points x^3 and x^4.

However, in the case to be considered here, a convex function should by maximized. Hence, a convex combination of the minimal and the maximal mesh point will result in higher, but invalid values for the function. Therefore additional restrictions must ensure that only adjacent mesh points are combined. To solve this type of problems, Hadley (1964) suggests a modification of the simplex algorithm of linear programming: The choice of a pivot variable is restricted. λ_{jt}^{k} can only be chosen as a pivot variable if no other λ_{jt}^{p} *with* $p \neq k-1$ and $p \neq k+1$ is basic variable, or if this λ_{jt}^{p} is removed from the basis in the same step.

If an optimal solution of the problem exists, then the algorithm will identify at least a local optimum in a finite number of pivot steps.

Figure 3. Approximation of a convex function by a piecewise linear function



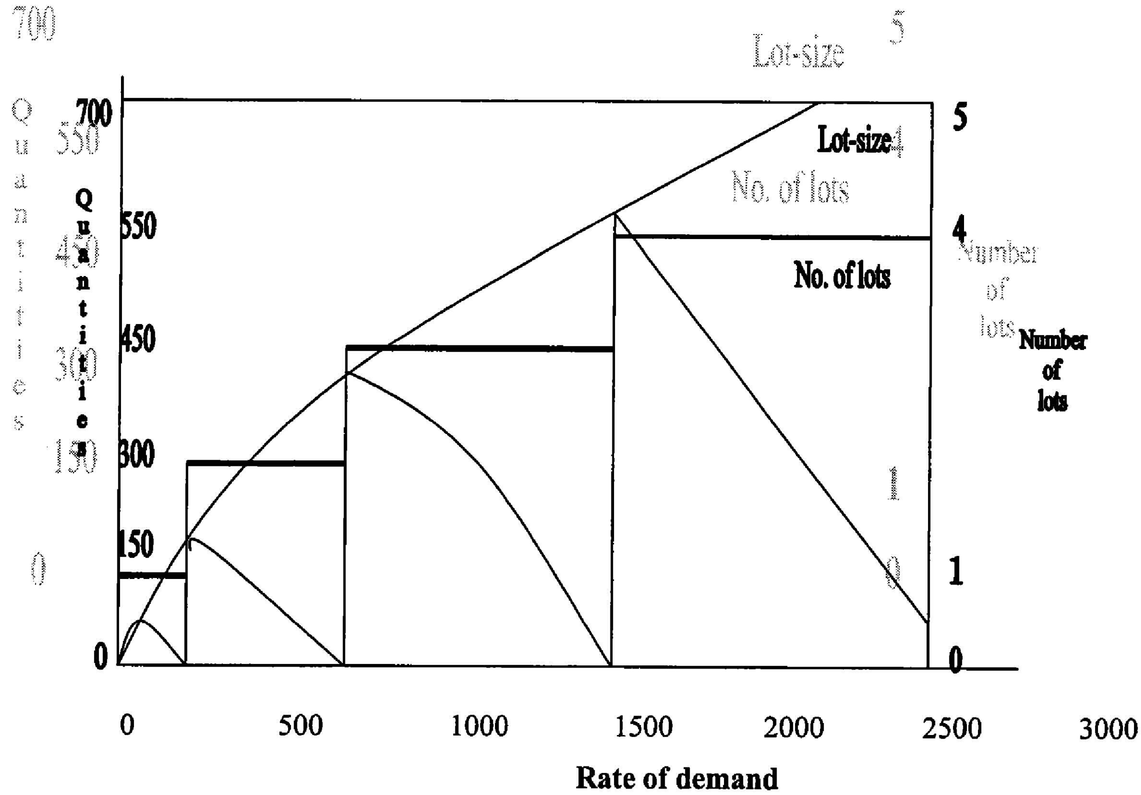
4. Balancing Production Program and Lot-sizing

As indicated above, production planning assigns targets for the production of an entire period. For lot-sizing decisions based on inventory models, this aggregated demand has to be disaggregated. The EOQ model assumes that the sales or the input of goods are distributed equally over the period. Furthermore, inventory models do not concentrate myopically on one single period, but try to take into account the impacts of present lot-sizing decisions on future situation. The EOQ model simply assumes that the present rate of demand will remain constant over time. Consequently, lot-sizes do not exactly match to the targets; they rather have to be adjusted according to the rate of demand. As a matter of fact, the number of lots required to meet the targets may not be integral at all. Since the number of batches to be processes during one period has to be integer, it has to be rounded up, and consequently the quantity actually produced will differ from the target set by production planning. In Figure 4, the optimal lot size, number of batches, and excess production are plotted down as a function of the target.

Obviously, the excess production is zero, if the demand is equal to an integer multiple of the optimal lot-size. As soon as the demand exceeds this critical value, and a further lot is produced the excess production jumps up accordingly.

The question is, how to manage these deviations between production planning and inventory control. This excess production may be accepted, because it anticipates future demands and avoids suboptimal lot-sizes. On the other hand, it may be advisable to control the deviations and to adjust lot-sizes at least partly to quantities allocated by the production planning for the respective period. We consider synchronization approach of how to tune inventory management to targets set by program planning: Lot-sizes are tuned such a way that the targets are completely fulfilled. The number of batches to be produced is either rounded up or down to the next integer value and the lot-sizes are adapted accordingly.

Figure 4. Lot-sizes without restrictions on excess production, coefficients $c_j^R = 1000,\ c_j^L = 10,\ d_j = 20$



Lot-sizes are adapted exactly to the targets set by production planning. If the number of batches, which corresponds to the optimal lot-size q_j^o, is not integer, it is either rounded down and the lot-size is increased, or the number of batches is rounded up and the lot-size is reduced accordingly. To optimize adaptation, the alternative with the minimum of cost is chosen.

Let $z_j^- = \left[\frac{x_j}{q_j^o}\right] \quad z_j^+ = \left[\frac{x_j}{q_j^o}\right] + 1$ with [y] = greatest integer strictly less than y (14)

Then the adapted lot-size is defined by the solution of

$$l_j(x_j, q_j^*) = \min\left\{ c_j^R.z_j^+ + \frac{1}{2}.c_j^L.\frac{x_j}{z_j^+} ; c_j^R.z_j^- + \frac{1}{2}.c_j^L.\frac{x_j}{z_j^-} \right\} \quad (15)$$

Figure 5 shows that the resulting transfer function has similar characteristics to the case of unrestricted lot-size q_j^o: The transfer function is continuous, concave and piecewise linear. This is due to the optimization of adjustment: At the break even points, the lot-sizes switches from z_j^- *to* z_j^+. At these points, the cost of inventory management are equal, hence the transfer function is continuous. Between two critical points, where the number of batches changes, the cost function is linear.

Figure 6 shows the differences between the synchronized inventory policy and the EOQ model: In principle, relative deviations of lot-sizes as well as of cost of synchronization decrease with the rate of demand. The functions are, however, not monotonous but have declining peaks at the critical points.

Figure 5. Lot-size and transfer function in case of synchronization, coefficients $c_j^R = 1000,\ \ c_j^L = 10,\ \ d_j = 20$

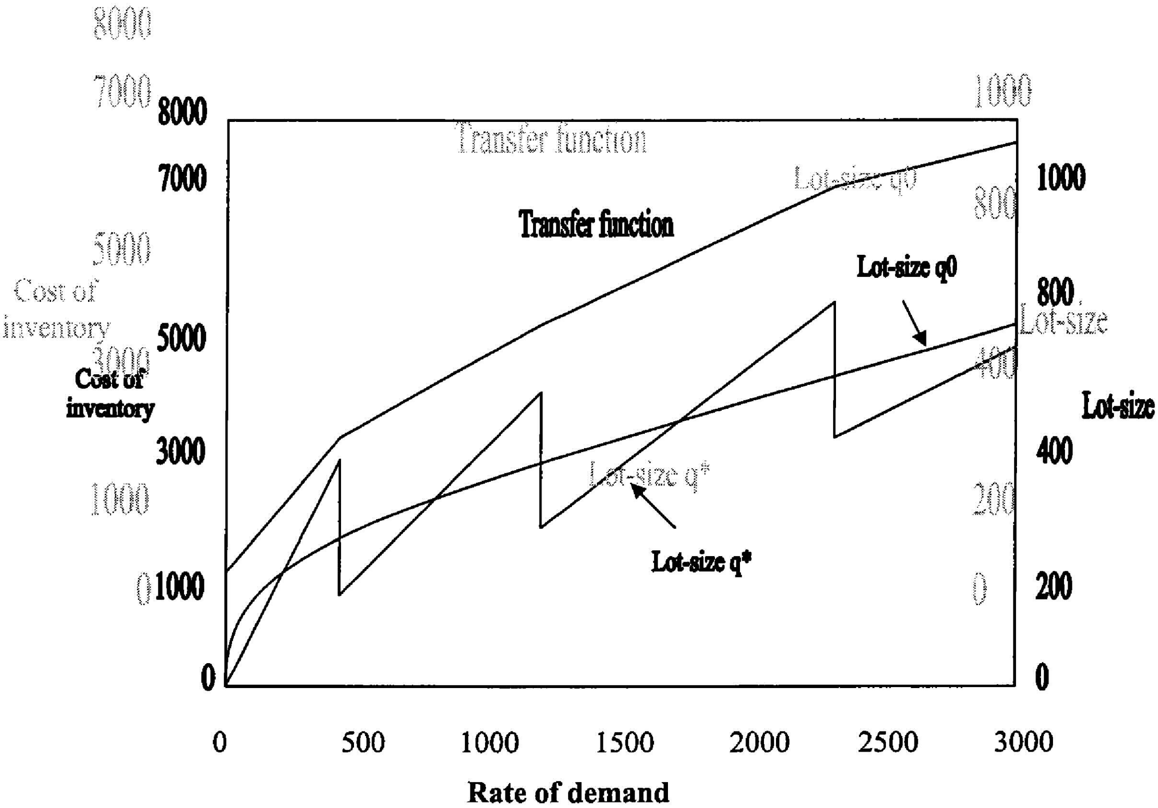
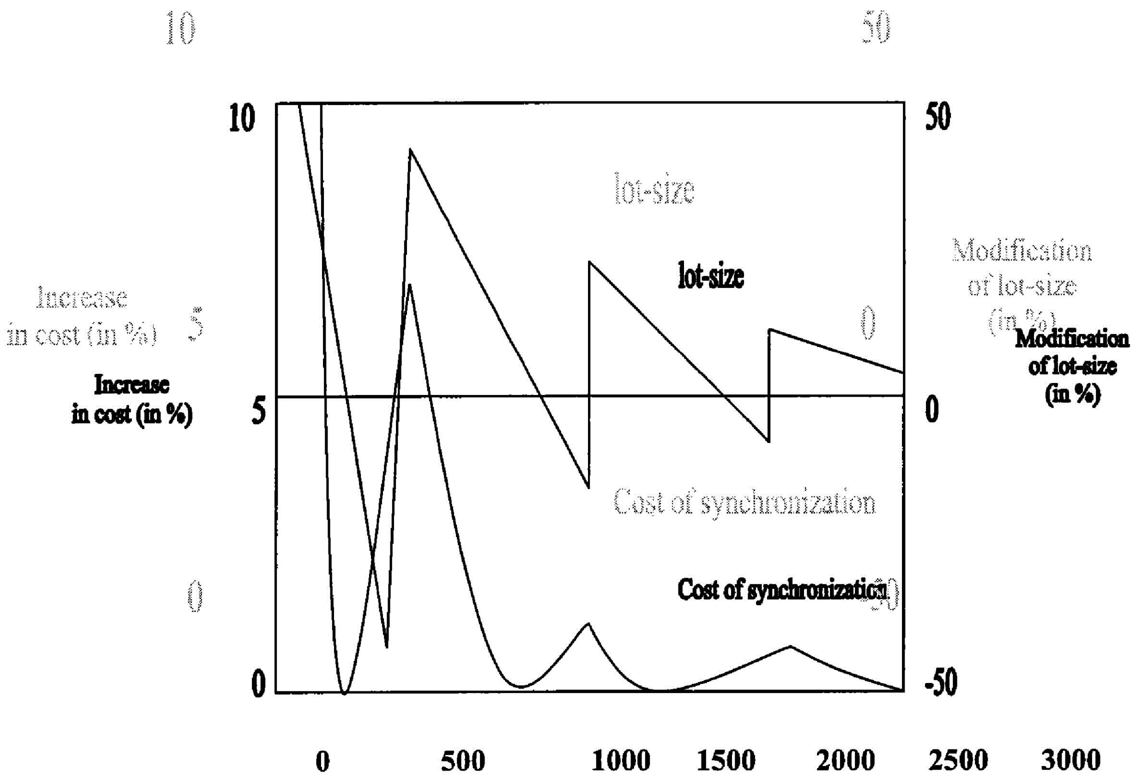


Figure 6. Relative changes of lot-sizes and inventory cost in case of synchronization, coefficients

$c_j^R = 1000, \quad c_j^L = 10, \quad d_j = 20$



5. Integration into Hierarchical Production Planning

As the transfer function is continuous and convex in the case mention above, and as its curvature is – apart from finite number of vertices – quite small, the approach proposed for a transfer function derived from unrestricted EOQ model may be applied to the transfer functions. In particular, vertices of the transfer functions may be used as mesh points for separable programming. In order to approximate strictly convex transfer functions, additional mesh points may be introduced.

6. Conclusions

In this paper, we present a concept of transfer functions to integrate targets set by a superior level of decision making with decisions on a subordinate level in the framework of a hierarchical production system. Using synchronization methods to adjust lot-sizes set by production planning, transfer functions of cost of inventories are derived. These functions can be applied to coordinate production planning based on the traditional mathematical programming approach and lot-sizing based on the EOQ model.

In the case considered, the transfer functions for inventory cost have the structure:

- Increase monotonously and slightly concave.
- The second derivative is small in the significant region.

- There is a finite number of vertices; the transfer function is piecewise linear.
- The increase in cost implied by the adjustment of lot-sizes is relatively small in the relevant region; it oscillates lower with damped amplitudes with the quantity to be produced

Transfer functions facilitate to consider cost of inventories in the objective function of production planning. The set-up cost can be charged against quantities produced according to the principle of causality: If a certain quantity has to be produced and a fixed rule to control inventories is applied, then a given amount of set-up and holding cost have to be accepted. This approach makes it possible to solve the fixed charge problem without introducing binary variables. For exotic products with small demand, however, the decision whether a lot with a run out time of several periods should be produced or not must be taken before starting the programming model.

A production planning model, which considers the cost of inventory control, can be formulated as a separable program and be solved using a modified simplex algorithm.

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