CHROMATICITY OF CERTAIN K_4-HOMEOMORPHS

By

SABINA CATADA GHIMIRE
B.S.Math., Cert.Prof.Educ., M.S.Math.

Thesis Submitted to
Universiti Sains Malaysia in Fulfilment of the Requirements
for the Degree of Doctor of Philosophy

2009

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation for the support, assistance and encouragement extended to me by my main supervisor, Dr. Roslan bin Hasni @ Abdullah, throughout the course of my study .

I thank my co-supervisor, Dr. Andrew Rajah A/L Balasingam Gnanaraj, for his moral support, advice and concern.

My profound gratitude is due to my field supervisor from Universiti Putra Malaysia, Professor Peng Yee Hock, for his invaluable guidance, expert advice, encouragement and constructive comments which are relevant for the improvement and completion of this thesis.

I would like to express my sincere thanks to the panel of examiners; Professor Teoh Siang Guan (Chairman), Professor Chia Gek Ling (External Examiner from the University of Malaya), Dr. Hailiza Binti Kamrulhaili @ Harun (Internal Examiner) and Dr. Ang Miin Huey (Internal Examiner) for their helpful and valuable comments.

Special thanks for the assistance of the Dean, Associate Professor Ahmad Izani Md. Ismail, and the staff of the School of Mathematical Sciences, Universiti Sains Malaysia.

I also thank Associate Professor Low Heng Chin for her help and assistance in various situations during the period of my study.

Special thanks also goes to the Dean, Associate Dean and staff of the Institute of Graduate Studies, Universiti Sains Malaysia for their assistance.

I am grateful to my parents, brothers, sisters and colleagues for their love which serves as an inspiration for me to finish this thesis.

I share the success of this study with my beloved husband, Ashok B. Ghimire

and my two sons, Prem Amiel and Ashish Hans. Their love, encouragement, understanding and support are the sources of my motivation to accomplish this thesis. I dedicate this piece of work to them.

All glory and honour are due to the Almighty God who continuously sustains me with knowledge, strength and determination. Praise and thanksgiving to Him.

TABLE OF CONTENTS

NOTATIONS

$P(G, \lambda)$ or $P(G)$: the chromatic polynomial of G.

χ-*unique* : chromatic unique.

χ-*equivalent* : chromatic equivalent.

$\chi(G)$: the chromatic number of G.

$g(G) =$ the length of the shortest cycle of G; i.e., the girth of G.

$G \sim H$: G and H are $\chi-$equivalent; i.e., $P(G, \lambda) = P(H, \lambda)$ or $P(G) = P(H)$.

$G \cong H$: G and H are isomorphic.

$|S| =$ the number of elements in the finite set S.

$\binom{n}{r} =$ the number of r-element subsets of an n-element set $= \frac{n!}{r!(n-r)!}$.

$V(G)$: the vertex set of G.

$E(G)$: the edge set of G.

$V(H)$: the vertex set of H.

$E(H)$: the edge set of H.

$d_G(v)$: the degree of v in G, where $v \in V(G)$.

uv : an edge of a graph whose endpoints are vertices u and v.

$v(G)$ or $|V(G)|$: the number of vertices of G or the order of G.

$e(G)$ or $|E(G)|$: the number of edges of G or the size of G.

$v(H)$ or $|V(H)|$: the number of vertices of H or the order of H.

$e(G)$ or $|E(G)|$: the number of edges of H or the size of H.

K_n : the complete graph with n vertices; i.e., the complete graph of order n.

O_n : the empty graph of order n.

P_n : the path of order n.

$K_4(a, b, c, d, e, f)$: the graph derived from K_4, the edges of which are replaced by six paths of length a, b, c, d, e, f.

$G - v$: the subgraph of G obtained by removing v and all edges incident with v from G, where $v \in V(G)$.

$G - e$: the subgraph of G obtained by removing e from G, where $e \in E(G)$.

$G \cdot xy$: the graph obtained from G by contracting x and y and removing any loop and all but one of the multiple edges, if they arise, where $x,\ y \in V(G)$.

$G + xy$: the graph obtained by adding a new edge xy to G, where $x,\ y \in V(G)$ and $xy \notin E(G)$.

$[G] = \{H : H \sim G\}$.

LIST OF TABLES

LIST OF FIGURES

LIST OF PUBLICATIONS AND PRESENTATIONS

1. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic equivalence of a family of K_4-homeomorphs. *Far East Journal of Mathematical Sciences.* **30(2)** (2008), 389–400.

2. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic equivalence of a pair of K_4-homeomorphs II. *Far East Journal of Mathematical Sciences.* **30(3)** (2008), 499–512.

3. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic equivalence of K_4-homeomorphs II. *Advances and Applications in Discrete Mathematics.* **3(1)** (2009), 53–66.

4. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic uniqueness of a family of K_4-homeomorphs. *Bull. Malays. Math. Sci. Soc. (2)* **32(3)** (2009), 327–333.

5. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic equivalence pairs of a family of K_4-homeomorphs. *Global Journal of Pure and Applied Mathematics, in press.*

6. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic equivalence of K_4-homeomorphs. *The Southeast Asian Bulletin of Mathematics* **33** (2009), 749–757.

7. S.Catada-Ghimire, H.Roslan, Y.H. Peng, Chromatically equivalent pairs of K_4-homeomorphic graphs. *Acta Mathematicae Application Sinica (English Series), in press.*

8. S.Catada-Ghimire, H.Roslan, Y.H. Peng, On chromatic uniqueness and equivalence of a family of K_4-homeomorphs. *Proceedings of the Third International Conference on Research and Education in Mathematics 2007:*

Strengthening Human Capital Through Research and Education in Mathematics (The Legend Hotel, Kuala Lumpur), 85–91. Universiti Putra Malaysia Press, Kuala Lumpur, Malaysia.

9. S.Catada-Ghimire, H.Roslan, Y.H. Peng, A lemma to study the chromaticity of K_4-homeomorphs. *Proceedings of the Fifth Asian Mathematical Conference 2009* (Putra World Centre, Kuala Lumpur), *in press*.

KEKROMATIKAN GRAF K_4-HOMEOMORPH TERTENTU

ABSTRAK

Kekromatikan graf adalah suatu terma merujuk kepada persoalan kekromatikan setara dan kekromatikan unik bagi suatu graf. Sejak perhatian ditimbulkan terhadap kekromatikan setara graf dan kekromatikan unik graf, pelbagai konsep dan keputusan dalam bidang kajian tersebut ditemui dan banyak famili graf berkenaan diperoleh. Tujuan tesis ini ialah untuk menyumbang keputusan baru tentang kekromatikan setara dan kekromatikan unik graf, khususnya, graf K_4-homeomorfik.

Graf K_4-homeomorfik ialah suatu graf diterbitkan daripada graf lengkap K_4. Graf homeomorfik ini ditandakan sebagai $K_4(a, b, c, d, e, f)$ dimana enam sisi digantikan dengan enam lintasan yang masing-masing panjangnya a, b, c, d, e dan f.

Keputusan dalam tesis ini merangkumi dua topik. Topik pertama melibatkan kekromatikan unik bagi empat famili graf K_4-homeomorfik. Tiga daripada famili ini adalah famili graf dengan tepatnya dua lintasan yang masing-masing panjangnya dua. Kajian ini dimotivasikan daripada masalah diberikan oleh Dong et al. in [10]. Maka, dalam Bab tiga hingga enam kami menentukan syarat cukup dan perlu untuk graf-graf berikut supaya bersifat kromatik unik: $K_4(a, b, 2, d, 2, f)$, dimana $a \geq 3,\ b \geq 3,\ d \geq 3,\ f \geq 3$; $K_4(1, b, 2, d, 2, f)$, dimana $b \geq 3,\ d \geq 3,\ f \geq 3$; $K_4(a, 2, 2, d, 1, f)$, di mana $a \geq 3,\ d \geq 3,\ f \geq 3$; dan $K_4(3, 3, 4, d, e, f)$, di mana $d + e \geq 6,\ f + d \geq 6,\ f + e \geq 6$. Pengukuhan kekromatikan unik bagi graf pertama dan kedua melengkapkan kajian kekromatikan unik bagi suatu famili graf K_4-homeomorfik dengan tepatnya dua lintasan tak bersebela-

han yang panjangnya dua. Dari keputusan diperoleh dalam kajian kekromatikan unik $K_4(3,3,4,d,e,f)$ dengan kitar yang mempunyai panjang terpendek sepuluh, suatu teorem umum tentang kekromatikan unik $K_4(a,a,a+1,d,e,f)$, dimana $a \geq 3$, $d \geq 3$, $e \geq 3$, $f \geq 3$ diformulasikan dan dibuktikan.

Topik kedua mengandungi kekromatikan setara sepuluh pasangan graf K_4-homeomorfik. Daripada kesudahan kajian tiga pasangan pertama graf berkenaan dalam Bab tujuh, suatu keputusan umum tentang pasangan kekromatikan setara dalam bentuk $K_4(1,2,c,d,e,f) \sim K_4(1,2,c',d',e',f')$ diperoleh. Bab lapan membincangkan kekromatikan setara bagi tujuh pasangan berikutnya.

Dalam Bab sembilan, keputusan diperoleh oleh penulis yang mengkaji kekromatikan graf K_4-homeomorfik dengan kitar yang mempunyai panjang terpendek g, dimana $3 \leq g \leq 9$ diringkaskan. 24 jenis graf K_4-homeomorfik dengan kitar yang mempunyai panjang terpendek sepuluh juga diberikan dalam tajuk yang sama. Keputusan dari Bab dua hingga Bab enam bersama dengan keputusan oleh penulis lain diaplikasikan untuk mengukuhkan syarat cukup dan perlu untuk 14 daripada 24 jenis graf K_4-homeomorfik supaya bersifat kromatik unik. Kekromatikan untuk 10 jenis selebihnya ditinggalkan sebagai kajian lanjutan.

CHROMATICITY OF CERTAIN K_4-HOMEOMORPH GRAPHS

ABSTRACT

The chromaticity of graphs is the term used referring to the question of chromatic equivalence and chromatic uniqueness of graphs. Since the arousal of the interest on the chromatically equivalent and chromatically unique graphs, various concepts and results under the said areas of research have been discovered and many families of such graphs have been obtained. The purpose of this thesis is to contribute new results on the chromatic equivalence and chromatic uniqueness of graphs, specifically, K_4-homeomorphs.

A K_4-homeomorph is a graph derived from a complete graph K_4. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ where the six edges are replaced by the six paths of length a, b, c, d, e and f.

The results in this thesis cover two main topics. The first topic involves the chromatic uniqueness of four families of K_4-homeomorphs. Three of such families are with exactly two paths of length two. Such study is motivated by the problems posted by Dong et al. in [10]. Thus, in Chapters 3, 4, 5 and 6 we establish sufficient and necessary condition for the following respective graphs to be chromatically unique: $K_4(a, b, 2, d, 2, f)$, where $a \geq 3$, $b \geq 3$, $d \geq 3$, $f \geq 3$; $K_4(1, b, 2, d, 2, f)$, where $b \geq 3$, $d \geq 3$, $f \geq 3$; $K_4(a, 2, 2, d, 1, f)$, where $a \geq 3$, $d \geq 3$, $f \geq 3$; and $K_4(3, 3, 4, d, e, f)$, where $d + e \geq 6$, $f + d \geq 6$, $f + e \geq 6$. Establishing the chromatic uniqueness of the first and second graphs completes the study on the chromatic uniqueness of a family of K_4-homeomorphs with exactly two non-adjacent paths of length two. From the result obtained in the study of

the chromatic uniqueness of $K_4(3, 3, 4, d, e, f)$ with girth ten, a more general theorem involving the chromatic uniqueness of $K_4(a, a, a + 1, d, e, f)$, where $a \geq 3$, $d \geq 3$, $e \geq 3$, $f \geq 3$ is formulated and proved.

The second topic includes the chromatic equivalence of ten pairs of K_4-homeomorphs. From the outcome on the study of the first three pairs of such graphs in Chapter 7, a more general result on the chromatic equivalence pair of the form $K_4(1, 2, c, d, e, f) \sim K_4(1, 2, c', d', e', f')$ is obtained. Chapter 8 discusses the chromatic equivalence of the remaining seven pairs.

In Chapter 9, the results obtained by authors who studied the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 9$ are summarised. The 24 types of K_4-homeomorphs with girth ten are also given in the same chapter. The results from Chapters Two to Six together with results by other authors are applied to establish the sufficient and necessary condition for the 14 of the said 24 types of K_4-homeomorphs to be chromatically unique. The chromaticity of the remaining 10 types are left for further studies.

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Euler (1707-1782) has been regarded as the father of graph theory for solving the famous Königsberg Bridge Problem. However, it was the Four-Colour Map Problem or the well known Four-Colour Conjecture which has spawned the development of graph theory. The problem asks whether four colours are sufficient to colour any geographical or planar graph so that no neighbouring countries have the same colour. It was first conjectured by Francis Guthrie, a graduate student of University College, London. But its first recognition in mathematical world took place in De Morgan's classes, and it was formally presented to the London Mathematical Society by Cayley in 1878. Since then, the most determined well-known mathematicians who attempted to prove this deceptive problem were not able to produce a solution. In 1912, Birkhoff proposed a way of dealing with this problem by introducing a function $P(M, \lambda)$, defined for all positive integer λ, to be the number of proper λ-colourings of a map M. It was proven that $P(M, \lambda)$ is a polynomial in λ and termed as the *chromatic polynomial of M*. To prove that $P(M, 4) > 0$ for all maps M would solve the four-colour problem.

Whitney [38], in 1932, established many fundamental results by generalising the notion of a chromatic polynomial to that of an arbitrary graph. In 1946, Birkhoff and Lewis investigated the distribution of real roots of chromatic polynomials of planar graphs. They conjectured that these polynomials have no real roots greater than or equal to four. The conjecture remains open. They also proved that $P(G, \lambda)$ is a polynomial for any graph G. The minimum integer λ such that $P(G, \lambda)$ is nonzero is called the *chromatic number* of G, denoted by $\chi(G)$. More information on the development of chromatic polynomials can be found in [6], [7], [27], [28], [29] and [33].

In 1978, Chao and Whitehead Jr. [2] defined a graph with no other graphs sharing its chromatic polynomial as *chromatically unique*. They found several families of such graphs. Since then, various results on chromatic equivalence have been obtained successively (see [8], [9], [15] and [16]). The problem of chromatic equivalence and uniqueness is termed as the *chromaticity of graphs*.

Birkhoff's hope to solve the Four-Colour Conjecture using the chromatic polynomial has not come to reality. But it gave birth to the interest of many who wanted to explore more on the topic he introduced, especially with respect to the roots of chromatic polynomial. In 1993, Hutchinson [14] used the notion of solving the roots of chromatic polynomial to prove her conjecture that an earth/moon non-planar graph is 8, 9 or 12 colourable. A graph is $\chi-$colourable if its vertices can be coloured with χ colours such that no two adjacent vertices have the same colour. Hutchinson applied her results to the testing of printed circuit boards for quality control on the production of such electronic chips. In an expository paper of Catada-Ghimire [1], the author of this thesis, Hutchinson's work was examined and explicitly discussed. According to Dong et.al [10], *"More recently, Thomassen discovered a relationship between the roots of the chromatic polynomial and hamiltonian paths. There has also been an influx of new ideas from statistical mechanics due to the recent discovery of a connection to the Potts Model in Physics".*

One of the most popular families of graphs being studied with regard to chromaticity of graphs is the family of K_4-homeomorphs (see [5] and [19] for other examples of families of graphs). A K_4-homemorph is a graph derived from a complete graph with four vertices by subdividing its edges. Such homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of K_4 are replaced by the six paths of length a, b, c, d, e, f, respectively, as shown in Figure 1.1. In 2005, Dong et al. [10] gave two tasks to tackle after summarising the works done in this particular area of research. The first task is to study the chromaticity of K_4-homeomorphs with exactly two equal paths of length greater than or equal to two. The second task is to study the chromaticity of K_4-homeomorphs with exactly one path of length one. Motivated by these problems, this thesis aims to tackle a specific area of such tasks, i.e., the chromaticity of certain types of K_4-homeomorphs with exactly two paths of length two.

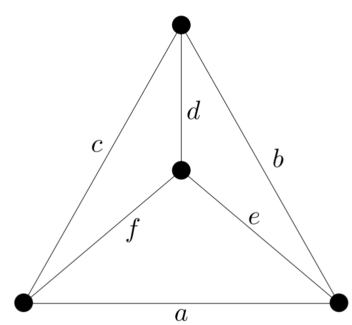

Figure 1.1: $K_4(a, b, c, d, e, f)$

The study of the chromaticity of K_4-homeomorphs with exactly two paths of length two involves two cases: Case 1, when the two paths of length two are adjacent and Case 2, when the two paths of length two are non-adjacent. For each case the following conditions on the four paths not of length two need to be considered.

(i) all four paths are of length one;

(ii) three paths are of length one and one path is of length greater than two;

(iii) two paths are of length one and two paths are of length greater than two;

(iv) one path is of length one and three paths are of length greater than two; and

(v) all four paths are of length greater than two.

The problem of the chromaticity of K_4-homeomorphs with at least two paths of length one has been completely solved (see [32], [17], [12], [36], [41], [25]). Thus, only conditions (iv) and (v) must be considered to study the chromaticity of K_4-homeomorphs with exactly two paths of length two in both cases.

As we discuss the chromaticity of K_4-homeomorphs, we shall discover that the chromaticity of such graphs with girth g, where $3 \leq g \leq 9$ has been studied by many authors. In this thesis, we shall continue to discover the chromaticity of certain families of K_4-homeomorphs with girth ten. We shall also investigate certain chromatically equivalent pairs of K_4-homeomorphs.

1.2 Objective of the Study

The first part of this thesis shall focus on the chromaticity of the following four types of K_4-homeomorphs:

(1) with exactly two non-adjacent paths of length two with no path of length one (Case 2, condition (v)),

(2) with exactly two non-adjacent paths of length two and with exactly one path of length one (Case 2, condition (iv)),

(3) with two adjacent paths of length two and with exactly one path of length one (one specific type of such K_4-homeomorph only) (Case 1, condition (iv)), and

(4) K_4-homeomorph $K_4(3, 3, 4, d, e, f)$ with girth ten.

These results shall then be applied to establish the sufficient and necessary condition for four types of K_4-homeomorphs with girth ten to be chromatically unique.

The chromaticity of a more general K_4-homeomorph of the form $K_4(a, a, a + 1, d, e, f)$ with girth $3a + 1$, where $a \geq 3$ shall be investigated based on the outcome on the study of the chromaticity of K_4-homeomorph $K_4(3, 3, 4, d, e, f)$ with girth ten.

The chromaticity of K_4-homeomorphs with exactly two paths of length two has a relatively wide range. This study is limited to the chromaticity of only three specific types of such graphs.

Obtaining the chromatic equivalence classes of ten pairs of K_4-homeomorphs is the topic of the second part of this research. Motivated by the outcome of the study on the first three pairs of such graphs, a more general result shall be established. These results are important tools in the study of the general K_4-homeomorphic graph $K_4(a, b, c, d, e, f)$.

The list of 24 types of K_4-homeomorphs with girth ten shall be presented. However, application of the results shall be limited only to 14 of these types and the remaining types shall be for further study.

Graph theory is an indispensable source of the basic concepts used to formulate and prove the theorems considered in this thesis. However, from this vast field, only the fundamental notions related to the discussion shall be presented. The main theorems shall be proven in detail. However, the proofs of related theorems shall not be included. References, where detailed and substantive analysis of the proof of such theorems, shall be given. Figures, tables and definition of important terms shall be provided to make the concepts comprehensible.

The purpose of this study is to provide new results significant to the present chain of research on the chromaticity of graphs, specifically, K_4-homeomorphs. Moreover, this thesis aims to contribute in the development of graph theory since its facts and techniques are used in this study.

1.3 Organisation of Thesis

Chapter 1 contains the introduction of the study. It includes the literature review on the chromaticity of graphs, objective of the study and the organisation of the thesis.

In Chapter 2, we provide definitions of terms, preliminary concepts and some known results needed for the comprehension of the succeeding chapters.

In Chapter 3, we examine the chromatic uniqueness of $K_4(a, b, 2, d, 2, f)$, where $\min\{a, b, d, f\} \geq 3$. We study the chromatic uniqueness of $K_4(1, b, 2, d, 2, f)$, where $\min\{b, d, f\} \geq 3$ in Chapter 4. The chromatic uniqueness of $K_4(a, 2, 2, d, 1, f)$, where $\min\{a, d, f\} \geq 3$ is discussed in Chapter 5.

In Chapter 6, we investigate the chromatic uniqueness of K_4-homeomorphs with girth $3a + 1$, where $a \geq 3$. This result is a consequence of the study on the chromaticity of one type of such K_4-homeomorphs with girth 10, i.e., $K_4(3, 3, 4, d, e, f)$.

Chapter 7 provides results that can be used to study the chromatic equivalence (or simply $\chi-$equivalence) and chromatic uniqueness (or simply $\chi-$uniqueness) of a general K_4-homeomorph of the form $K_4(1, 2, c, d, e, f)$, where $c \geq 7$.

In Chapter 8, we present the chromatic equivalence pairs of seven types of K_4-homeomorphs.

In the final chapter, we summarise the work done by authors who studied the chromaticity of K_4-homeomorphs with girth g, $3 \leq g \leq 9$. We also give the complete list of 24 types of K_4-homeomorphs with girth ten and investigate the chromatic uniqueness of 14 of these types applying the theorems formulated by several authors together with the main results obtained in this thesis. The conclusion and some open problems are also given for further study in this area of research.

CHAPTER 2

CHROMATICITY OF GRAPHS

2.1 Introduction

In this chapter, we shall introduce basic terminology from graph theory which will be assumed throughout the whole study. For further explanation of these terms and detailed proofs of stated results, the reader may refer to [13] and [34]. Definitions and results not included here will be presented later on as they are needed, or may be found in the references given above. In Section 2.2, we give the formal definition and properties of the chromatic polynomial of a graph. We shall also state the fundamental results on such field of study. In Section 2.3, we shall define isomorphic, chromatic equivalent and chromatic unique graphs. The conditions for two graphs to be chromatically equivalent and typical examples of chromatically unique graphs are also presented in the same section. We shall define homeomorphic graphs and a K_4-homeomorph in Section 2.4. We shall also cite some results on the study of the chromaticity of K_4-homeomorphs in the last section of this chapter.

Definition 2.1 *A graph $G = (V(G), E(G))$ is a non-empty finite set consisting of a vertex set $V(G)$ together with an (possibly empty) edge set $E(G)$ of unordered* *pairs of distinct elements of* $V(G)$*. An edge* $\{u, v\}$ *in* $E(G)$*, where* $u, v \in V(G)$*, is often denoted by* uv *or* vu*. In this case, the vertices* u *and* v *are said to be adjacent. Furthermore,* u *is also called a neighbour of* v*, and vice versa. If* $e = uv$ *is an edge of* G*, then* e *joins* u *and* v*, or the vertices* u *and* v *are the two endpoints of* e*. Moreover,* e *is incident with* u *and* v*. The degree of* v *in* G*, denoted by* $d_G(v)$*, is the number of edges of* G *incident with* v*. The symbols* $v(G)$ *or* $|V(G)|$ *refer to the number of vertices in* G*.* $e(G)$ *or* $|E(G)|$ *are used to represent the number of edges in* G*. A graph* G *is said to be of order* n *if* $v(G) = n$ *and of size* m *if* $e(G) = m$*. An* (n, m)*-graph is a graph of order* n *and size* m*. A graph* G *is said to be trivial if* $v(G) = 1$*, and non-trivial otherwise. The repeated edges and edges with the same endpoints are called multiple edges and loops, respectively. A graph is simple if it has no loops and no multiple edges. A directed graph is a finite nonempty set* $V(G)$ *with a set* $E(G)$ *of ordered pairs of distinct elements of* $V(G)$*, where set* $E(G)$ *is disjoint from* $V(G)$*. All graphs considered here are finite, undirected, simple and loopless.*

Definition 2.2 *A graph in which every two vertices are adjacent is called a complete graph; the complete graph with* n *vertices and* $(\frac{1}{2})n(n-1)$ *edges is denoted by* K_n*.*

Definition 2.3 *A graph* H *is said to be a subgraph of a graph* G *if* $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$*. A simple subgraph is a subgraph without loops and multiple edges.*

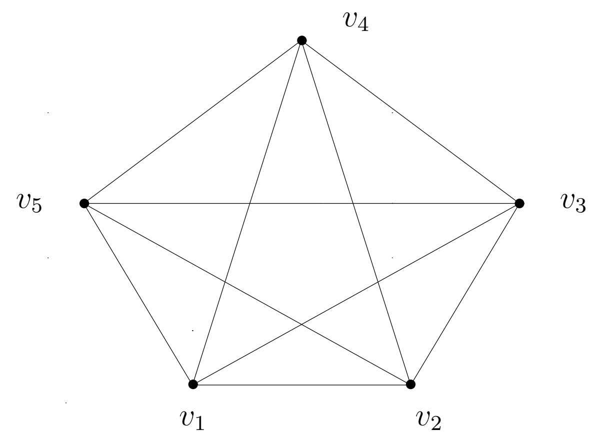

Figure 2.1: A complete graph with five vertices, K_5.

Definition 2.4 *A walk in G is a finite sequence $v_0v_1v_2\cdots v_k$ of vertices such that v_iv_{i+1} is an edge for each $i = 0, 1, \cdots, k-1$. It is a path if all the points (and thus necessarily all the lines) are distinct. A path is said to be hamiltonian (or spanning if it contains all the vertices in G. A cycle is a closed walk $v_1v_2\cdots v_kv_1$ in which the v_i's are distinct. Such a cycle C of order k in G is also called a k-cycle C in G. A $3-$ cycle is often called a triangle. A chord in cycle C is an edge joining two non-consecutive vertices along C. The girth of G, denoted by $g(G)$, is the length of the shortest cycle in G.*

Consider the graph K_5 in Figure 2.1: $v_4v_3v_5v_4v_1v_2$ is a walk of length 5; $v_3v_2v_1v_4$ is a path of length 3; $v_5v_1v_3v_4v_3v_1v_5$ is a closed walk of length 6; $v_1v_4v_3v_2v_1$ is a cycle of length 4 and v_1v_3 is an example of a chord in this cycle.

Definition 2.5 *A graph G is connected if every two vertices u, $v \in G$ are joined by a path. It is disconnected otherwise. A component of G is a connected subgraph of G which is not a proper subgraph of any connected subgraph of G. The graph*

K_5 in Figure 2.1 is a connected graph. The graph in Figure 2.2 is a disconnected graph.

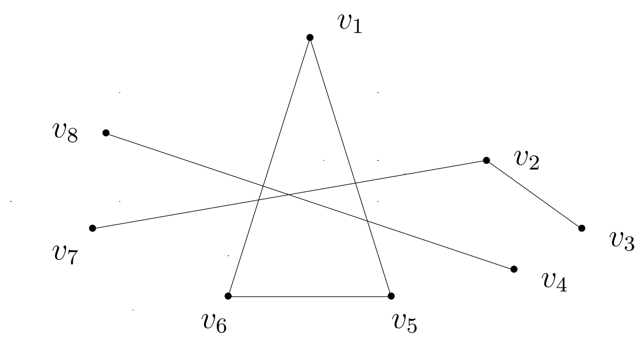

Figure 2.2: A disconnected graph of eight vertices.

Definition 2.6 *A forest is a graph containing no cycles; such a graph is also said to be acyclic. A tree is a connected forest. Thus, every component of a forest is a tree.*

2.2 The Fundamental Results on Chromatic Polynomial

Definition 2.7 *An assignment of at most λ colours to the vertices of a graph G is a $\lambda-$colouring of G. Such a colouring of G is proper if two adjacent vertices are assigned two distinct colours. More precisely, a proper $\lambda-$colouring of G is a mapping*

$$f : V(G) = \{ v_1, v_2, \ldots, v_n \} \rightarrow \{ 1, 2, \ldots, \lambda \}$$

such that $f(v_i) \neq f(v_j)$ whenever $v_iv_j \in E(G)$. Two proper $\lambda-$colourings f and g of G are considered different if $f(v_i) \neq g(v_i)$ for some vertex v_i in G. Let $P(G, \lambda)$ (or simply $P(G)$) denote the number of different proper $\lambda-$colourings of G.

Definition 2.8 *An empty graph is a graph with no edges while a complete graph is a graph in which each pair of distinct vertices is joined by an edge. Thus, for instance, if* O_n *is the empty graph of order* n*, then* $P(O_n, \lambda) = \lambda^n$*; and if* K_n *is the complete graph of order* n*, then* $P(K_n, \lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)$*. Observe that* $P(O_n, \lambda)$ *and* $P(K_n, \lambda)$ *are polynomials in* λ*. It turns out (see Theorem 2.5) that for any graph* G*,* $P(G, \lambda)$ *is in fact a polynomial in* λ*, called the chromatic polynomial of* G*.*

The following result is very useful in determining $P(G, \lambda)$ or in showing that certain graphs are chromatically unique.

Theorem 2.1 (Fundamental Reduction Theorem)(Whitney [39]) *Let* G *be a graph, and* e *an edge in* G*. Then*

$$P(G) = P(G - e) - P(G \cdot e)$$

where $G - e$ *is the graph obtained from* G *by deleting* e*, and* $G \cdot e$ *is the graph obtained from* G *by contracting the two vertices incident with* e *and removing all but one of the multiple edges, if they arise.*

By means of Theorem 2.1, the chromatic polynomial of a graph can be expressed in terms of the chromatic polynomials of a graph with an edge less, and another with one fewer vertices. When applying this theorem repeatedly, we can express the chromatic polynomials as a sum of the chromatic polynomials of empty graphs.

The Fundamental Reduction Theorem can also be used in another way. Let $v_i, v_j \in V(G)$ such that $v_i v_j \notin E(G)$. Then

$$P(G, \lambda) = P(G + v_i v_j, \lambda) + P(G \cdot v_i v_j, \lambda)$$

where $G+v_iv_j$ is the graph obtained from G by adding the edge v_iv_j and $G\cdot v_iv_j$ is the graph obtained from G by identifying the vertices v_i and v_j. In this way, one can express $P(G,\lambda)$ as a sum of the chromatic polynomials of complete graphs.

Suppose G_1 and G_2 are the graphs each containing a complete subgraph K_r, where $r \geq 1$. Let G be the graph obtained from the union of G_1 and G_2 by identifying the two subgraphs K_r in arbitrary way, then G is called a K_r-*gluing* of G_1 and G_2. Note that K_1-gluing and K_2-gluing are called a *vertex-gluing* and *edge-gluing* of G_1 and G_2, respectively. The following two lemmas will provide a shortcut for computing $P(G,\lambda)$.

Lemma 2.1 (Zykov [42]) *Let G be a K_r-gluing of graphs G_1 and G_2. Then*

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)} = \frac{P(G_1)P(G_2)}{\lambda(\lambda-1)\cdots(\lambda-r+1)}.$$

Lemma 2.2 (Read [27]) *If a graph G has connected components $G_1, G_2, \ldots, G_k$, then*

$$P(G) = P(G_1)P(G_2)\ldots P(G_k).$$

The following are some properties of the chromatic polynomial $P(G,\lambda)$ of a graph G.

Theorem 2.2 (Read [27]) *Let G be a graph of order n and size m. Then $P(G,\lambda)$ is a polynomial in λ such that*

(i) deg$(P(G,\lambda)) = n$;

(ii) all the coefficients are integers;

(iii) the leading term is λ^n;

(iv) the constant term is zero;

(v) the coefficients alternate in sign;

(vi) the absolute value of the coefficient of λ^{n-1} is the number of edges of G;

(vii) either $P(G,\lambda) = \lambda^n$ or the sum of the coefficients in $P(G,\lambda)$ is zero.

The following two results for determining $P(G,\lambda)$ are due to Whitney, which can be proven using the Principle of Inclusion and Exclusion.

Theorem 2.3 (Whitney [39]) *Let G be a graph of order n and size m. Then*

$$P(G,\lambda) = \sum_{k=1}^{n} \Big(\sum_{r=0}^{m} (-1)^r N(k,r) \Big) \lambda^k$$

where $N(k,r)$ denotes the number of spanning subgraphs of G having exactly k components and r edges.

Suppose G is a graph with an arbitrary bijection $\beta : E(G) \to \{ 1, 2, \ldots, m \}$. Let C be any cycle in G and e be an edge in C such that $\beta(e) \geq \beta(x)$ for each edge x in C. Then the path $C - e$ is called a *broken cycle* in G induced by β. Then we have the following theorem.

Theorem 2.4 (Broken-Cycle Theorem)(Whitney [39]) *Let G be a graph of order n and size m, and let $\beta : E(G) \to \{ 1, 2, \ldots, m \}$ be a bijection. Then*

$$P(G,\lambda) = \sum_{i=0}^{n-1} (-1)^i h_i \lambda^{n-i}$$

where h_i is the number of spanning subgraphs of G that have exactly i edges and that contain no broken cycles induced by β.

Let G be a graph of order n. By using Theorems 2.3 and 2.4, we then can derive the coefficient of λ^i, where $n-3 \leq i \leq n$, expressed in terms of the numbers of certain simple subgraphs of G.

Theorem 2.5 (Farrell [11]) *Let G be a graph of order n and size m. Then in the polynomial $P(G,\lambda)$, the coefficient of*

(i) λ^n *is 1;*

(ii) λ^{n-1} *is* $-m$*;*

(iii) λ^{n-2} *is* $\binom{m}{2} - t_1(G)$*;*

(iv) λ^{n-3} *is* $-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G)$.

where

$t_1(G)$ = *the number of triangles* K_3 *(i.e., complete graphs with three vertices) in* G*,*

$t_2(G)$ = *the number of cycles of order 4 without chords in* G*,*

$t_3(G)$ = *the number of* K_4 *(i.e., complete graphs with four vertices) in* G.

2.3 Chromatically Unique Graphs and Chromatically Equivalent Graphs

Definition 2.9 *Two graphs G and H are said to be isomorphic, in notation: $G \cong H$, if there exists a bijection φ: $V(G) \to V(H)$ which preserves adjacency; i.e., $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. Such a bijection φ is called an isomorphism of G onto H.*

Definition 2.10 *Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent or simply $\chi-$equivalent, symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$.*

It can be proved by using Lemma 2.1 that for any tree T of order n, $P(T,\lambda) = \lambda(\lambda-1)^{n-1}$. Thus, there exists non-isomorphic graphs which have the same chromatic polynomial. These observations lead to the following definitions.

Definition 2.11 *A graph G is chromatically unique or simply $\chi-$unique if $G \cong H$ for any graph H such that $H \sim G$. Trivially, the relation $\sim$ is an equivalence relation on the class of graphs. We shall denote by $[G]$ the chromatic equivalence class determined by G under $\sim$, indeed $[G]$ is the set of all graphs having the same chromatic polynomial $P(G,\lambda)$. Clearly, G is $\chi-$unique if and only if $[G] = \{G\}$, that is each graph in $[G]$ is isomorphic to G.*

The following result is obvious.

Lemma 2.3 *Let G be a graph of size m. Then $m \geq 1$ if and only if $\lambda(\lambda-1)|P(G,\lambda)$.*

The following are some typical examples of $\chi-$unique graphs.

(a) The empty graph O_n of order n is $\chi-$unique and $P(O_n,\lambda) = \lambda^n$.

(b) The complete graph K_n of order n is $\chi-$unique and $P(K_n,\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$.

(c) Let C_n be the cycle of order n, $n \geq 3$. Then $P(C_n,\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$. Chao and Whitehead [2] proved that every cycle is $\chi-$unique.

(d) A $\theta-$graph denoted by $\theta(p,q)$, consists of a cycle C_p and C_q with a an edge in common. Then

$$P(\theta(p,q),\lambda) = \frac{P(C_p,\lambda)P(C_q,\lambda)}{\lambda(\lambda-1)}.$$

Chao and Whitehead [2] showed that $\theta(p,q)$ is $\chi-$unique.

2.4 Chromaticity of K_4-homeomorphs

Definition 2.12 *Harary [13] defines two graphs to be homeomorphic if both can be obtained from the same graph by a sequence of subdivisions of lines, i.e., divisions of edges into segments. A K_4-homeomorph is a graph homeomorphic to the complete graph with four vertices, K_4. In simple terms, a K_4-homeomorph is a subdivision of the complete graph K_4. Figures 2.3(b) and (c) show the two ways of representing K_4-homeomorphic graphs. For simplicity, we shall use Figure 2.3(c) (see also Figure 1.1 in Chapter 1) to refer to a general K_4-homeomorph, where the six edges of K_4 are subdivided into segments (similarly saying, are replaced by the six paths of length) a, b, c, d, e and f.*

Definition 2.13 *A path $v_0v_1\cdots v_k$ is called a chain if $d_G(v_i) = 2$ for each $i = 1, 2, \cdots, k-1$; and is a maximal chain if, in addition, $d_G(v_0) \geq 3$ and $d_G(v_k) \geq 3$.*

The six paths of the homeomorph are its *maximal chains*. The length of a chain P is denoted by $l(P)$.

The chromaticity of K_4-homeomorphs was first studied by S. Kahn while writing his doctoral dissertation entitled *Chromatic Equivalence and Chromatic Uniqueness* in George Washington University (1980)(see [3]). Chao and Zhao (see [3])

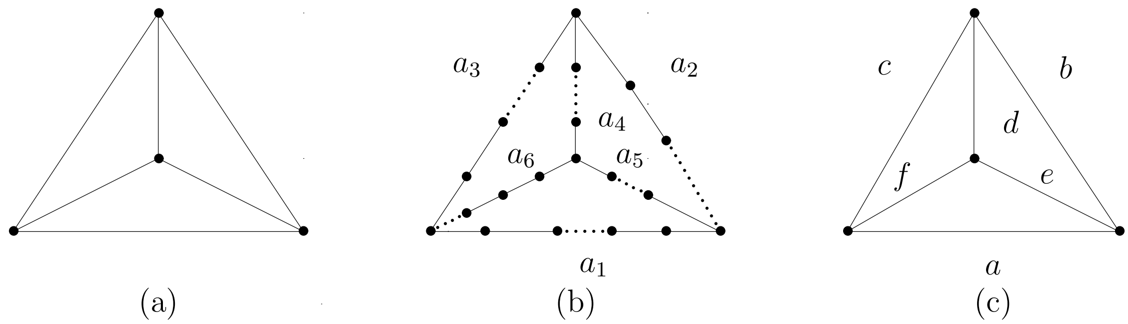

Figure 2.3: (a)K_4, (b)$K_4(a_1, a_2, a_3, a_4, a_5, a_6)$, (c)$K_4(a, b, c, d, e, f)$.

computed all the chromatic polynomials of all connected $(n, n+2)$-graphs to prove the following lemma.

Lemma 2.4 *Any graph that is $\chi-$equivalent to K_4-homeomorph must itself be a K_4-homeomorph .*

The chromatic polynomial of K_4-homeomorph can be found easily by using the Fundamental Reduction Theorem, i.e., Theorem 2.1, as shown by Dong et al. in [10]. Ren (see [30]) combined the results of Li [17], Whitehead and Zhao [36] to express the chromatic polynomial of a K_4-homeomorph as follows.

Lemma 2.5 *Let $G = K_4(a, b, c, d, e, f)$. Then*

$$P(G, \lambda) = \tfrac{1}{\lambda^2}(-1)^m w[w^{m-1} + Q(G, w) - (w+1)(w+2)],$$

where $w = 1 - \lambda$, $m = \mid E(G) \mid$ and

$$Q(G, w) = -(w^{a+f+c} + w^{a+b+e} + w^{b+c+d} + w^{d+e+f} + w^{a+d} + w^{b+f} + w^{c+e})$$

$$+(1+w)(w^a + w^b + w^c + w^d + w^e + w^f).$$

$Q(G, w)$ or simply $Q(G)$ is called the essential polynomial of G.

Li in [18] proved the following.

Lemma 2.6 *Two K_4-homeomorphs with the same order are chromatically equivalent if and only if they have the same essential polynomials.*

$K_4(4, 1, 3, 1, 3, 1)$ and $K_4(2, 3, 1, 1, 5, 1)$ are examples of $\chi-$equivalent but non-isomorphic graphs. This shows that not all K_4-homeomorphs are $\chi-$unique. The question on the proportionality of $\chi-$unique K_4-homeomorphs on the family of K_4-homeomorphs was answered by W.M. Li in (1987)(see [17]). He first established the following result.

Theorem 2.6 *If $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets and*

$$K_4(a, b, c, d, e, f) \sim K_4(a', b', c', d', e', f'),$$

then the two graphs are isomorphic.

From this, W.M. Li derived the following result.

Corollary 2.1 *The graph $G = K_4(a, b, c, d, e, f)$ is $\chi-$unique if each of the following conditions is satisfied:*

(1) $min\{a, b, c, d, e, f\} > 1$;

(2) for any maximal chains P, Q, R in G such that P and Q are disjoint,

$$l(P) + l(Q) \neq \lceil \tfrac{1}{2}(l(P) + l(Q) + l(R)) \rceil;$$

(3) for any maximal chains P, Q, R, S in G such that P, Q and R have a vertex in common,

$$l(P) + l(Q) + l(R) \neq \lceil \tfrac{1}{2}(l(P) + l(Q) + l(R) + l(S)) \rceil.$$

Dong et al in [10] summarise the result of W.M. Li in [17] as follows:

Let S_0 be the family of all the K_4-homeomorphs of order n with the four vertices of degree three labeled. It is known that the number of ways of distributing k identical objects into r distinct boxes is given by $\binom{k+r-1}{r-1}$. Since each member of S_0 can be constructed by inserting $n-4$ vertices into the six labeled edges of a given K_4,

$$|S_0| = \binom{n-4+(6-1)}{6-1} = \binom{n+1}{5}.$$

Let S_j, where $j \in \{1, 2, 3\}$ be the family of those members of S_0 which violate condition (j) in the above corollary. Then it is not hard to see that $|S_1| \leq 4\binom{n}{4}$. Li further showed that each of $|S_2|$ and $|S_3|$ is bounded above by a polynomial in n of degree four. Thus,

$$lim_{n\to\infty} \frac{|S_1| + |S_2| + |S_3|}{|S_0|} = 0.$$

This means that almost every K_4-homeomorph is $\chi-$unique.

Definition 2.14 *Let $\chi(G)$ denote the chromatic number of G. Then $\chi(G)$ is the smallest integer λ such that $P(G, \lambda) > 0$.*

The following lemma can be derived from Theorem 2.6.

Lemma 2.7 *Let G and H be graphs such that $G \sim H$. Then*

(i) G and H have the same order;

(ii) G and H have the same size;

(iii) $t_1(G) = t_1(H)$;

(iv) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$;

(v) $\chi(G) = \chi(H)$;

(vi) G is connected if and only if H is connected,

where

$t_1(G)$ = the number of triangles K_3 (i.e., complete graphs with three vertices) in G,

$t_1(H)$ = the number of triangles K_3 (i.e., complete graphs with three vertices) in H,

$t_2(G)$ = the number of cycles of order 4 without chords in G,

$t_2(H)$ = the number of cycles of order 4 without chords in H,

$t_3(G)$ = the number of K_4 (i.e., complete graphs with four vertices)in G,

$t_3(H)$ = the number of K_4 (i.e., complete graphs with four vertices) in H,

$\chi(G)$ = the chromatic number of G, and

$\chi(H)$ = the chromatic number of H.

There are no general methods for constructing families of $\chi-$unique graphs, thus, it is very important to know as many as possible necessary conditions for two graphs to be $\chi-$equivalent. The above lemma is just the necessary conditions for two graphs G and H to be $\chi-$equivalent.

Theorem 2.7 *For any $n \geq 4$, let $f(n)$ (resp., $g(n)$) denote the number of K_4-homeomorphs (resp., $\chi-$unique K_4-homeomorphs) of order n. Then*

$$lim_{n\to\infty}\frac{g(n)}{f(n)} = 1.$$

The following result is obtained by Whitehead Jr. and Zhao in [36].

Theorem 2.8 *Suppose $K_4(a,b,c,d,e,f) \sim K_4(a',b',c',d',e',f')$. Then $min\{a,b,c,d,e,f\} = min\{a',b',c',d',e',f'\}$ and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that this minimum occurs in the list $\{a',b',c',d',e',f'\}$.*

W.M. Li proved the following useful result.

Lemma 2.8 *Let G be a K_4-homeomorph. If at most a pair of terms in the essential polynomial $Q(G)$ can be cancelled, then G is $\chi-$unique.*

The reader may refer to [10] for the summary of some known results on the chromaticity of K_4-homeomorphs. We shall restate such results as we use them to prove our theorems in the succeeding chapters and in our discussion in Chapter 9.

CHAPTER 3

CHROMATIC UNIQUENESS OF $K_4(a, b, 2, d, 2, f)$

3.1 Introduction

Recall that the study of the chromaticity of K_4-homeomorphs with exactly two paths of length two involves two cases: Case 1, when the two paths of length two are adjacent and Case 2, when the two paths of length two are non-adjacent. In this chapter, we shall investigate the chromaticity of one family of such graphs under Case 2.

3.2 Chromaticity of $K_4(a, b, 2, d, 2, f)$

As what we have mentioned in Chapter 1, to complete the study of the chromaticity of K_4-homeomorphs with exactly two non-adjacent paths of length two (as shown in Figure 3.1(a)) we need to consider two families of graphs, namely, $K_4(a, b, 2, d, 2, f)$, where $\min\{a, b, d, f\} \geq 3$ and $K_4(a, b, 2, d, 2, f)$, where exactly one of a, b, d, f is of length one and the remaining three paths are of length greater than two. If we refer to Figure 3.1(a) then by symmetry, we can assume

$min\{a, b, d, f\} = a$. Therefore, $K_4(a, b, 2, d, 2, f)$, where $\min\{a, b, d, f\} = a$, $a \geq 3$ and $K_4(1, b, 2, d, 2, f)$, where $\min\{b, d, f\} \geq 3$ represent the two families of such K_4-homeomorphs, respectively.

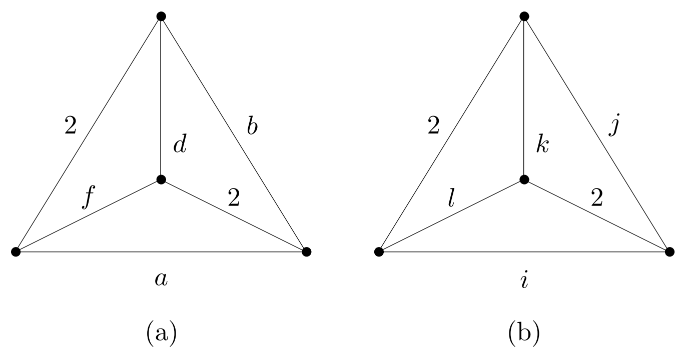

Figure 3.1: (a) $K_4(a, b, 2, d, 2, f)$ and (b) $K_4(i, j, 2, k, 2, l)$

In this chapter, we shall study the chromaticity of K_4-homeomorph $K_4(a, b, 2, d, 2, f)$, where $\min\{a, b, d, f\} \geq 3$. We shall investigate the other family in Chapter 4.

In Section 3.3, we give some known results and notations used to obtain the main result. We show the detailed proof of such result in Section 3.4.

3.3 Preliminary Results and Notations

Lemma 3.1 *Assume that G and H are $\chi-$equivalent. Then the following statements are proven to be true.*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, *i.e., G and H have equal number of vertices (order) and equal number of edges (size) (see [15]);*