THE REVISED *M*-OF-*K* RUNS RULES BASED ON MEDIAN RUN LENGTH

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by

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CHAPTER 1
INTRODUCTION

1.1 Statistical Quality Control (SQC)

Statistical Quality Control (SQC) is a branch of Total Quality Management. SQC deals with the collection, analysis and interpretation of data, for use in quality control activities. Two major parts of SQC are Statistical Process Control (SPC) and acceptance sampling. All planned or systematic actions necessary to provide adequate confidence that a product or service will satisfy given requirements for quality are called quality assurance (Shirland, 1993). Quality assurance is usually associated with some form of measurement and inspection activity and it has been an important aspect of production operations throughout history.

Prior to the late 1920s, quality control consisted mainly of 100% inspection. Most often, subjective decisions were made regarding the conformance or nonconformance of a product. It is generally accepted that Walter A. Shewhart of the Bell Telephone Laboratories was the originator of modern statistical quality control (Montgomery, 2009). In 1924, he developed a statistical chart for the control of product variables. In the later part of the 1920s, H. F. Dodge and H. G. Romig, both of the Bell Telephone Laboratories, developed the area of acceptance sampling as a substitute for 100% inspection (Montgomery, 2009).

Recognition of the usefulness of statistical quality control became apparent during World War II, where products had to be produced quickly and with high quality. In 1946, the American Society for Quality was formed. This organization has been instrumental in standardizing the symbols and terminology used in quality control. It promotes the use of quality for all types of productions and services through numerous technical publications, conferences and training sessions.

In 1950, W. Edwards Deming and Joseph M. Juran were hired to give a series of lectures on statistical methods to Japanese engineers and on quality responsibilities to the chief executive officers (CEOs) of large organizations in Japan. In the decades that followed, Japanese industries were instrumental in developing quality as a management philosophy and this sets the quality standards for the rest of the world to follow (Shirland, 1993).

By the late 1970s and early 1980s, a quality renaissance began to occur in U.S. products and services and by the middle of 1980s, the concepts of Total Quality Management (TQM) were being publicized. In the late 1980s, industries and the U.S. Department of Defense began to emphasize on SPC. Japan's Genichi Taguchi introduced his concepts of parameter and tolerance design and brought about a resurgence of design of experiments (DOE) as a valuable quality improvement tool (Montgomery, 2009).

In 1990s, quality concepts continued to be emphasized in the auto industry. In addition, the introduction of ISO 9000 became the worldwide model for a quality system. ISO 9000 has been modified by the automotive industry to place greater emphasis on customer satisfaction, continuous improvement and manufacturing capabilities.

In the past ten years, the quality focus has been shifting towards information technology within an organization and externally via the Internet.

1.2 Control Charting Techniques for $\bar{X} - R$ Charts

It is helpful to follow a set of procedure in order to set up a pair of control charts, each for the average, $\bar{X}$ and range, R. The steps in this procedure are as follows (Besterfield, 2009):

Step 1. Determine the quality characteristics.

Step 2. Select the rational subgroup size.

Step 3. Collect the data.

Step 4. Determine the trial center line and control limits.

Step 5. Establish the revised center line and control limits.

The primary step in setting up the $\bar{X}$ and R charts is to determine the characteristics to be measured. A candidate for measurement should be something that is causing problems or has the potential to cause problems, such as length, overall weight, luminous intensity or other related characteristics.

Next, the rational subgroup size is selected. A rational subgroup is one in which the variation within the group is due only to chance causes. This within-subgroup variation is used to determine the control limits whereas the variation between subgroups is used to evaluate long-term stability. Since the purpose of control charts is to determine when the process has gone out-of-control, there will be a greater probability of detecting a change if the units sampled are as nearly alike as possible. Typically, an ideal subgroup size is about four or five measurements (Shirland, 1993).

The actual process mean and standard deviation are usually not known in practice; therefore, estimates are usually used to determine the control limits of a control chart. For subgroups with greater than 10 measurements, the subgroup standard deviation provides a good estimation of the process standard deviation. As the subgroup size increases, the control limits become closer to the center value, which makes the control chart more sensitive to small variations in the process mean. On the other hand, for subgroups with less than 10 measurements, either the subgroup range or the subgroup standard deviation provides a good estimation of the process standard deviation. In practice, the $\bar{X}$ and R charts are used with subgroups having sizes of four or five each, for ease of data collection and simple calculation of the range.

After the subgroup size is selected, the next step is to collect data and calculate the trial control limits. It is necessary to collect a minimum of 25 subgroups of data. Fewer subgroups would not provide a sufficient amount of data for accurate computation of the center line and control limits, while more subgroups would delay the introduction of the control chart. Tradition dictates the use of the three standard deviation width for calculating the upper and lower control limits of control charts. When three standard deviation is used, there is only a 0.0027 probability that a point will fall either above the upper control limit or below the lower control limit if the process is in-control. The use of the three standard deviation width in setting the control limits will help quality engineers in determining whether a process is out-of-control so that corrective actions can be taken immediately.

Next, the mean and range of each subgroup are plotted on their respective $\bar{X}$ and R charts after the upper and lower trial control limits have been determined. A good process can be briefly described as one, where no out-of-control point or unusual pattern of variation on a control chart is present. Any out-of-control condition will trigger a search for an assignable cause. If an assignable cause for the point of out-of-control exists, the data point can be thrown out and the new centerline and control limits are computed using the remaining data points. The remaining data points are then plotted on the control chart using the new centerline and control limits. This process is repeated until all the data points are plotted within the control limits which indicate that the process is stable so that the trial revised limits can be used for the monitoring of a future process.

1.3 Objectives of the Study

The main objective of this thesis is to propose the design of the various revised *m*-of-*k* runs rules based on Median Run Length (MRL), as a criterion to be minimized. The second objective is to study the standard deviation of the run length (SDRL) distribution of the revised *m*-of-*k* rules. An example of application to show how the revised *m*-of-*k* rules, designed based on MRL is put to work in a real situation is also given. In the revised *m*-of-*k* runs rules, the general form of the transition probability matrix for the transient states does not exist because its dimension changes with the values of *m* and *k*. The values of *m* and *k*, where m = 2, 3 and 4 and k = 2, 3, 4 and 5 are considered in this thesis due to practical reasons. Note that small values of *m* and *k* are considered because the dimension of the transition probability matrix for the transient states will be extremely large when *m* and *k* are large. A large dimension of the transition probability matrix will lead to complex transient states.

1.4 Organization of the Thesis

This thesis is organized in the following manner: Chapter 1 gives some descriptions on Statistical Quality Control and control charting techniques for the Shewhart $\bar{X}-R$ charts, besides explaining the objectives of the study. In Chapter 2, the importance and applications of control charts are discussed. The Shewhart $\bar{X}$ chart, various types of runs rules and performance measures of control charts are also discussed in Chapter 2. In Chapter 3, the Markov chain approaches for computing the median run length (MRL) and the standard deviation of the run length (SDRL) distribution of the revised runs rules are presented. Chapter 4 gives a detailed explanation on the proposed designs of the revised runs rules based on MRL. The descriptions and operations of the computer programs are also described in this chapter. Performance evaluations of the revised runs rules, in terms of MRL and SDRL, and an illustrative example to show how some of these rules are implemented in a real situation are also given in Chapter 4. Finally, conclusions and suggestions for further research are presented in Chapter 5.

CHAPTER 2
SOME PRELIMINARIES AND REVIEW ON RUNS RULES AND CONTROL CHARTS

2.1 Introduction

In this chapter, the importance and examples of applications of control charts are given. In addition, the Shewhart $\bar{X}$ control charts for variables data, which include the $\bar{X}_R$ and $\bar{X}_S$ charts that are widely used to monitor the mean of a process under the normality assumption will be briefly discussed.

One of the ways to enhance the $\bar{X}$ chart to increase its sensitivity in detecting shifts in the process mean is to incorporate the chart with runs rules. This chapter will also discuss the various types of runs rules.

Furthermore, several performance measures of control charts, like the ARL, MRL and SDRL will also be explained in this chapter.

2.2 Importance of Control Charts

Control charts are among the most important management control tools for analyzing data and have had a long history of use in U.S. industries and in many offshore industries (Montogomery, 2009).

Control charts are effective in defect prevention. They can be used to determine whether a process is operating in a state of statistical control. This is because control charts show the degree and nature of variation over time. Control charts help to keep the process in-control, which is consistent with the "do it right the first time" philosophy.

Control charts are also used to estimate process parameters, like the mean and variance. They help us to recognize, understand and identify the variability and changes in process performance. A control chart can distinguish between background noise and abnormal variation. If process operators adjust a process based on periodic tests unrelated to a control charting program, they will often overreact to the background noise and make the unnecessary adjustments. These unnecessary adjustments can actually result in a deterioration of process performance. In other words, a control chart is consistent with the "if it isn't broken, don't fix it" philosophy (Montgomery, 2009).

Control charts can be used to improve a process. Once a process is in a state of statistical control, efforts to reduce process variability can begin. By reducing the variability of a process, the overall quality of the final product increases, which reduces scrap and rework and thus increases profitability.

In addition, control charts provide diagnostic information. Frequently, the pattern of points on a control chart contains useful diagnostic information to an experienced operator or engineer. This information allows the implementation of a change in the process that improves its performance.

2.3 Applications of Control Charts

2.3.1 Applications in Industries

Control charts have been widely used in industries since the 1920s. Control charts can be used in practically any type of industry. Some examples of industrial applications are given as follows:

(i) Improving a supplier's process in the aerospace industry (Montgomery, 2009):

A large aerospace manufacturer purchased aircraft components from two suppliers. These components frequently exhibited excessive variability on a key dimension that made it impossible to assemble them into the final product. This problem always resulted in expensive rework costs and occasionally caused delays in finishing the assembly of an airplane. The materials-receiving group performed a 100% inspection of these parts in an effort to improve the situation. They maintained the $\bar{X}$ and R charts on the dimension of interest for both suppliers and find out the main problems faced by the two suppliers. Consequently, corresponding actions were taken, where the use of control charts had successfully helped the company to increase its profit.

(ii) Reducing the nonconforming production of electric insulators (Gitlow et al., 1989):

Consider the case of a small manufacturer of low tension electric insulators. The insulators are sold to wholesalers who subsequently sell them to electrical contractors. Each day during a one-month period the manufacturer inspects the production of a given shift; the number inspected varies somewhat. Based on carefully laid out operational definitions, some of the production which is deemed nonconforming is downgraded. The p chart is used to help to reduce the fraction of nonconforming production.

(iii) Monitoring the stability of a plastic film process (Gitlow et al., 1989):

In a converting operation, a plastic film is combined with paper coming off a spooled reel. As the two come together they form a moving sheet that passes as a web over a series of rollers. The operation runs in a continuous feed, and the thickness of the plastic coating is an important product characteristic. Coating thickness is monitored by a highly automated piece of equipment that uses ten heads to take ten measurements across the web at half-hour intervals. The data are then analyzed using the $\bar{X}$ and S charts to monitor the stability of the process.

(iv) Identifying the variation in the diameters of ceramic insulators (Gitlow et al., 1989):

In the manufacturing of low tension ceramic insulators, a mixture of various clays and water is crushed, milled, de-aired, pre-shaped and then turned on a wheel to achieve the proper final shape. The manufacturer's clients specify that the diameter of the center hole is a critical dimension for proper ultimate use of the insulator. Consistency in the diameter of the center holes has been identified as an important characteristic. The firm's clients and hence the manufacturer, would like this dimension to be consistently on the nominal value with respect to time. To accomplish this goal the manufacturer must identify and remove any special causes of variation. The plant is located in an economically depressed area. The work force is poorly educated but is willing to learn and improve their processes. The median and range control charts are effective control charting tools that can be introduced to the work force, for process monitoring, as these charts involve the median and range statistics that are easily comprehensible by this level of the work force.

2.3.2 Applications in Healthcare

Control charts also have direct applications in healthcare. In fact, many of the Performance Improvement Standards from *Joint Commission on Accreditation of Healthcare Organizations (*JCAHO) can be met by using control charts. In general, control charts have the following uses in healthcare (Carey, 2003):

(i) Understanding

Control charts are very useful to help understand a process and its capabilities, such as in studying the waiting time of patients in an outpatient center. Control charts provide an understanding of the performance and restraints of the system. For example, knowing that the average waiting time in an outpatient center is 28 minutes, management would not readily sign a service provider agreement that promises an average waiting time of 15 minutes unless the system is revamped. Control charts are able to show that the system will simply not perform well enough to meet the 15-minutes average unless changes are made in the procedures.

(ii) Monitoring

Control charts help to monitor things over time. Healthcare organizations may choose to use control charts to display health or clinical data that they are required to gather for regulatory compliance, insurance purposes or for their own needs. Healthcare organizations can also use control charts, particularly in high-risk areas like in an emergency ward, to provide ongoing, current information on the performances of the system, facilities and equipments, instead of discovering problems long after they occur.

(iii) Improving

Control charts can be used to improve the processes or systems in healthcare organizations. When a process is monitored, problems may be identified and improvement priorities may surface. For example, the systolic blood pressure (mmhg) readings for a patient who is diagnosed with high blood pressure can be recorded over a period of time deemed necessary by a medical doctor (Mohammed et al., 2008). The individuals and moving range charts can be used to observe the blood pressure of the patients so that actions are taken to improve the blood pressure of the patient.

(iv) Verifying

Control charts are very helpful to verify if changes made to a system result in an improved system performance. For example, a new form may be used to decrease patients waiting time in an outpatient center. A control chart will show visually if the form is effective or not in reducing the waiting time.

2.3.3 Applications in Accounting

The potential applications of control charts in accounting are numerous. Control charts can measure efficiency, such as days it takes to process an invoice from a shipping document or days it takes to complete a monthly close. They can also be used to chart weekly payroll in an account's department of an organization so as to alarm management whenever an error occurs. The use of control charts in accounting helps to improve performance and efficiency, which in turn reduces cost and increases profitability. Table 2.1 shows some of the applications of control charts in accounting and the measurements plotted on the chart (Walter et al, 1990).

Table 2.1. Some applications of control charts in accounting

Applications	Measurements plotted on the control chart
Payroll	Number of audit exceptions in samples of employee pay records
Accounts receivable billing	Average billing time
Tax preparation	Proportion of unusable returns due to error
Management travel and entertainment	Number of improperly authorized or documented expense vouchers
Accounts payable	Number of invoices processed
General accounting	Time required for monthly closing and statement preparation
Accounts receivable and cash management	Age of accounts receivable
Purchasing	Number of purchase discounts lost
Sales personnel	Sales returns per salesperson when commissions are based on gross sales

2.3.4 Applications in Environment

Control charts are also being used for environmental purposes. For example, the Time Variant (TV) control chart is used to assess the quality and reasonableness of hourly and daily air temperature, wind speed and vapour pressure data collected at the California Irrigation Management Information System (CIMIS) weather stations (Eching and Snyder, 2003). The TV control chart helps to provide an efficient means of inspecting the data and identify potential data quality problems easily.

2.4 Shewhart $\bar{X}$ Chart

The Shewhart $\bar{X}$ chart was proposed by Walter A. Shewhart while working for the Bell Labs in the 1920s (Montgomery, 2009). It is used to determine whether or not a manufacturing or business process is in a state of statistical control. The Shewhart $\bar{X}$ chart is based on the assumption that the distribution of the quality characteristic is normal or approximately normal. Generally, the Shewhart $\bar{X}$ chart, which signals an out-of-control condition when a single point falls beyond the three sigma limits, has been the standard control chart for variables data since the first quarter of this century. Variables data involve numerical measurements such as length, volume or weight which are measured on a continuous scale.

The Shewhart $\bar{X}$ chart consists of three lines, namely, (i) upper control limit (UCL), (ii) center line (CL), and (iii) lower control limit (LCL). The UCL and LCL are chosen to help identify nonrandom patterns on the $\bar{X}$ chart, which represent occurrences that are sufficiently unusual and warrant special attention.

Assume that a quality characteristic is normally distributed with mean μ, and standard deviation σ, where both μ and σ are known. For $X_1, X_2, ..., X_n$ as a sample of size n, the average of this sample is

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n} \tag{2.1}$$

where $\bar{X}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ (Montgomery, 2009). Given the probability is $1-\alpha$ that an arbitrary sample mean will fall between

$$\mu \pm Z_{\alpha/2}\sigma_{\bar{X}} = \mu \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \tag{2.2}$$

the control limits on the $\bar{X}$ chart can be obtained from Equation (2.2) if μ and σ are known (Montgomery, 2009).

The values of μ and σ are usually unknown in real life situations and are estimated from an in-control historical data set consisting of *m* samples, each of size, *n*. Let $\bar{X}_1$, $\bar{X}_2$, ..., $\bar{X}_m$ be the averages of the *m* in-control samples in a Phase–I process, then μ is estimated as follows (Shirland, 1993):

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_m}{m}, \tag{2.3}$$

where $\bar{\bar{X}}$ is the center line of the $\bar{X}$ chart.

The standard deviation σ can be estimated from either the ranges or standard deviations of the *m* samples. The range, *R* of a sample can be calculated as the difference between the largest and smallest observations in the sample, i.e.,

$$R = X_{\max} - X_{\min}. \tag{2.4}$$

Let R_1, R_2, ..., R_m be the ranges of the *m* samples. Then, the average range is

$$\bar{R} = \frac{R_1 + R_2 + \ldots + R_m}{m}. \tag{2.5}$$

Therefore, the limits of the $\bar{X}$ chart when parameters are unknown are given as follows (Montgomery, 2009):

$$\text{UCL} = \bar{\bar{X}} + A_2\bar{R}, \tag{2.6a}$$

$$\text{CL} = \bar{\bar{X}} \tag{2.6b}$$

and

$$\text{LCL} = \bar{\bar{X}} - A_2\bar{R}. \tag{2.6c}$$

The value of the constant, A_2 which depends on the sample size, *n*, is given in most statistical quality control textbooks (see Table A1 in Appendix A).

If the sample standard deviation, *S* is used to estimate σ, where

$$S = \sqrt{\frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}}, \tag{2.7}$$

then the average sample standard deviation estimated from *m* preliminary samples in Phase–I is

$$\bar{S} = \frac{S_1 + S_2 + ... + S_m}{m}. \tag{2.8}$$

It follows that the limits of the $\bar{X}$ chart when parameters are estimated are (Montogomery, 2009)

$$\text{UCL} = \bar{\bar{X}} + A_3\bar{S}, \tag{2.9a}$$

$$\text{CL} = \bar{\bar{X}} \tag{2.9b}$$

and

$$\text{LCL} = \bar{\bar{X}} - A_3\bar{S}, \tag{2.9c}$$

where A_3 is the control limit constant whose value is given in most quality control reference books (see Table A1 in Appendix A).

2.5 Runs Rules

2.5.1 Classical Rules

Western Electric (1956) and Nelson (1984) provided excellent discussions on numerous runs rules schemes. They described an out-of-control condition as depicted by k of r successive points falling beyond the one, two or three sigma limits, where $2 \leq k \leq r$. The effect of using runs rules on the $\bar{X}$ chart has been studied by Champ and Woodall (1987) who found that the chart incorporating runs rules achieve its goal but its false out-of-control signal rate increases significantly.

Derman and Ross (1997) considered two additional schemes, each of which used specially designated (smaller than three sigma) control limits. In their first scheme, given two successive points, an out of control signal is obtained if either point is above an upper control limit and the other is below a lower control limit, or if both points are beyond one limit. In their second scheme, an out-of-control signal is obtained if any two of three successive points are beyond any of the control limits, i.e., among the two points contributing to the out-of-control signal, either a point is beyond the UCL while the other is beyond the LCL; or two points are beyond the same limit. In short, they showed that both schemes provided increased sensitivity to moderate process average shifts over that of a Shewhart $\bar{X}$ chart. Figures 2.1 and 2.2 show the graphical displays explaining how the two schemes work. Both examples generate out-of-control signals at the last sample point.

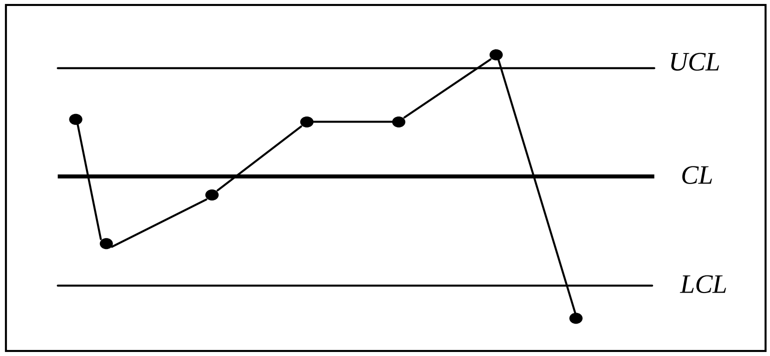

Figure 2.1. An illustration of the first scheme

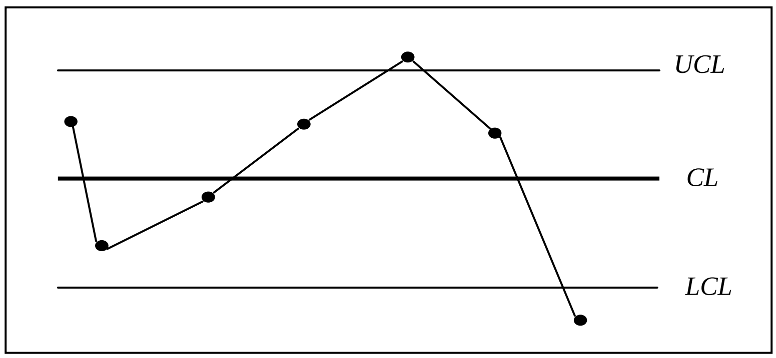

Figure 2.2. An illustration of the second scheme

Motivated by Derman and Ross (1997), Klein (2000) suggested two types of rules for the $\bar{X}$ chart, namely the two of two (2/2) and the two of three (2/3) rules, having symmetric upper and lower control limits. The designs of both the rules are based on the Markov chain approach. For the 2/2 rule, either two successive points plotted above an upper control limit (UCL) or below a lower control limit (LCL) are required to signal an out-of-control. For the 2/3 rule, an out-of-control signal is produced if either two of three successive points are plotted above the UCL or below the LCL. The $\bar{X}$ chart incorporating any of these rules demonstrated better average run length (ARL) performances than the standard $\bar{X}$ chart, for a process mean shift of up to 2.6 standard deviations (Klein, 2000). Figures 2.3 and 2.4 show the graphical display explaining how the 2/2 and 2/3 rules generate out-of-control signals at the last sample point.

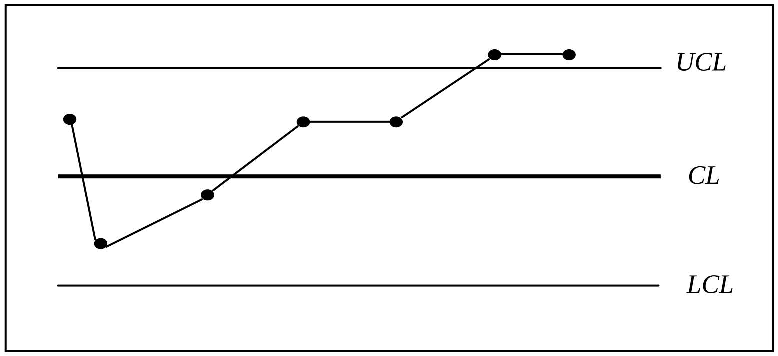

Figure 2.3. An illustration of the 2/2 rule

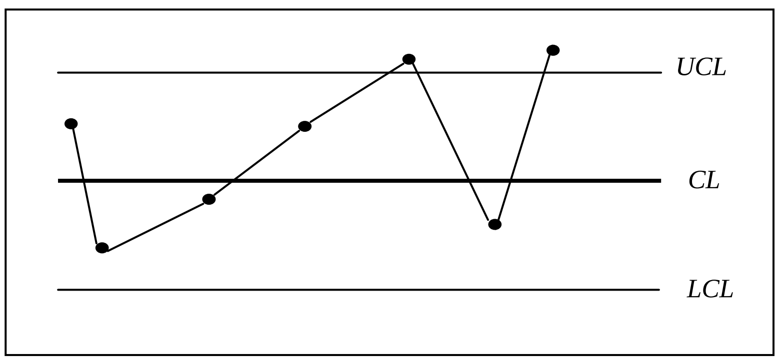

Figure 2.4. An illustration of the 2/3 rule

2.5.2 Improved Runs Rules

Khoo (2003) extended the work of Klein (2000) by suggesting the 2/4, 3/3 and 3/4 rules. Khoo and Ariffin (2006) proposed two new rules, by combining the classical 1/1 rule with each of the 2/2 and 2/3 rules. These combined rules are referred to as the improved two-of-two and improved two-of-three rules (denoted by $I - m/k$, where m = 2 and k = 2 or 3). The I – 2/2 rule signals an out-of-control if either a point plots beyond the outer limits $\left(\mathrm{LCL}_2,\ \mathrm{UCL}_2\right)$ or two consecutive points plot between UCL_1 and UCL_2 (or between LCL_1 and LCL_2). On the contrary, an out-of-control signal is issued by the *I* – 2/3 rule if either a point plots beyond $\mathrm{LCL}_2 / \mathrm{UCL}_2$ or two of three consecutive points plot between UCL_1 and UCL_2 (or between LCL_1 and LCL_2). The two *I* – *m*/*k* rules give better ARL performances than the corresponding rules of Klein (2000), in detecting small and moderate mean shifts while maintaining the same sensitivity in the detection of large shifts. Figures 2.5 and 2.6 show the graphical displays explaining how the *I* – 2/2 and *I* – 2/3 rules work, where out-of-control signals are detected at the last sample point.

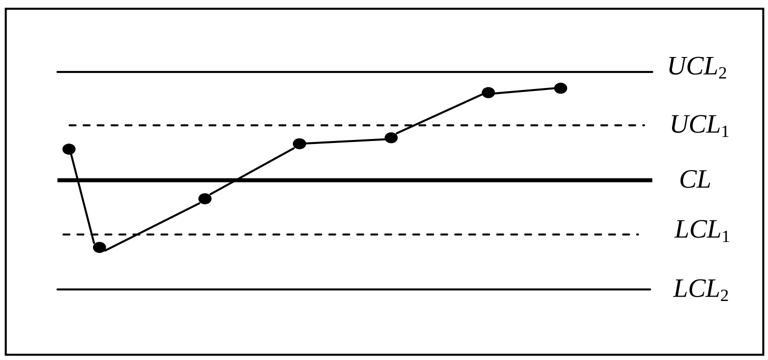

Figure 2.5. An illustration of the *I* – 2/2 rule

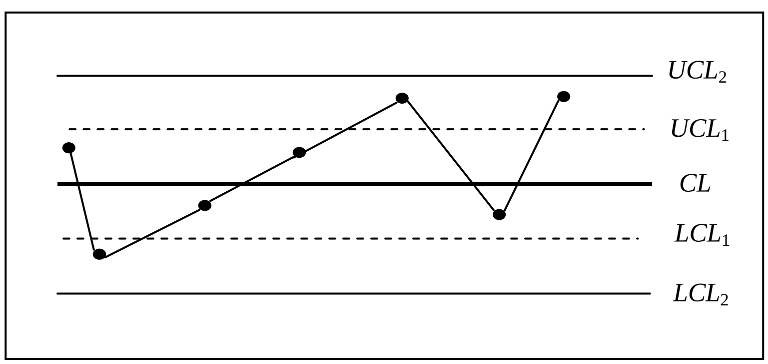

Figure 2.6. An illustration of the *I* – 2/3 rule

2.5.3 Modified Runs Rules

Recently, Antzoulakos and Rakitzis (2008a) suggested a modified *r* out of *m* rule, which is denoted by $M - r/m$. This modified rule gives an out-of-control signal if among *m* consecutive points, either *r* points are all plotted above an upper control limit, while at most $(m - r)$ points fall between the lower control limit and the upper control limit or *r* points are all plotted below a lower control limit, while at most $(m - r)$ points fall between the upper control limit and the lower control limit. Antzoulakos and Rakitzis (2008a) recommended the use of the $M - 4/5$ rule for the detection of small process average shifts and $M - 3/5$ and $M - 2/5$ rules for moderate shifts. Figures 2.7 and 2.8 are examples showing the detection of out-of-control signals by the $M - r/m$ rule at the last sample point. In Figure 2.8, there is reasonable doubt about the shift of the process average to a higher level since between the r (= 2) points falling above the UCL, there are $m - r$ (= 2) points that plot far away from the former two points. Thus, it seems useful to take into account the location of the $(m - r)$ points relative to the location of the *r* points, in order to make a decision about a process shift, following an out-of-control signal. This setback of the modified rules has led to the revised *m*-of-*k* rules, proposed by Antzoulakos and Rakitzis (2008b), discussed in the next section.

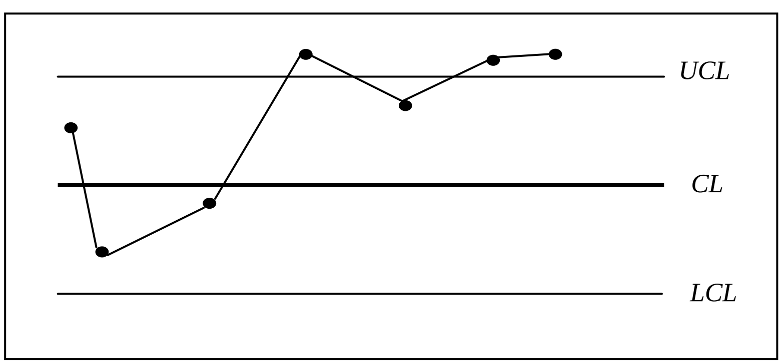

Figure 2.7. An illustration of the $M - 3/4$ rule

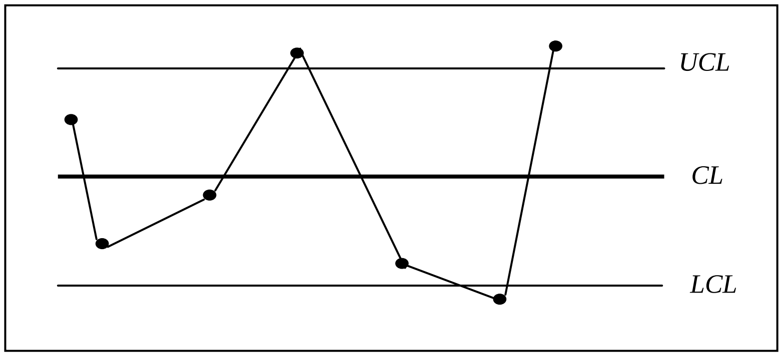

Figure 2.8. An illustration of the $M - 2/4$ rule

2.5.4 Revised Runs Rules

Antzoulakos and Rakitzis (2008b) suggested the revised *m*-of-*k* rule, denoted as $R - m/k$. In using the $R - m/k$ rule, consider the $\bar{X}$ chart having a center line (CL) and two sets of limits, the outer limits $(\text{LCL}_2, \text{UCL}_2)$ and inner limits $(\text{LCL}_1, \text{UCL}_1)$, where $\text{LCL}_2 < \text{LCL}_1 < \text{CL} < \text{UCL}_1 < \text{UCL}_2$. For $k \geq 2$ and $2 \leq m \leq k$, the $R - m/k$ rule signals an out-of-control if (Antzoulakos and Rakitzis, 2008b).

(i) a sample point is plotted beyond $\text{LCL}_2/\text{UCL}_2$ or

(ii) *m* out of *k* successive sample points fall between UCL_1 (LCL_1) and UCL_2 (LCL_2), and the cluster of points taking part in the out-of-control signal lies between CL and $\text{UCL}_2\,(\text{LCL}_2)$.

For every shift in the mean, Antzoulakos and Rakitzis (2008b) showed that the $R - 2/3$ and $R - 4/5$ rules exhibit better ARL performances than the corresponding $I - 2/3$ and $I - 4/5$ rules. Note that the latter rules were suggested by Khoo and Ariffin (2006). The evaluation of the ARL performances of the $R - m/k$ rules were based on the Markov chain technique, developed by Fu and Koutras (1994). The $R - m/k$ rule allows the user to specify the desired ARL_0 value. In comparison to most of the runs rules in the literature having this desirable property, the $R - m/k$ rule provides the quickest speed in the detection of an out-of-control signal. Figures 2.9 and 2.10 provide the graphical displays showing how the R – 2/3 and R – 4/5 rules detect out-of-control signals at the last sample point.

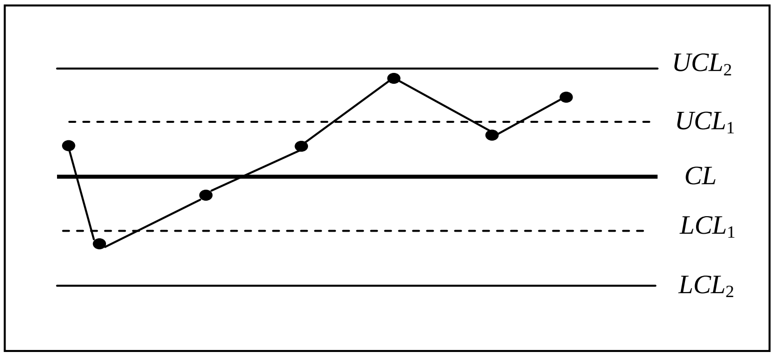

Figure 2.9. An illustration of the R – 2/3 rule

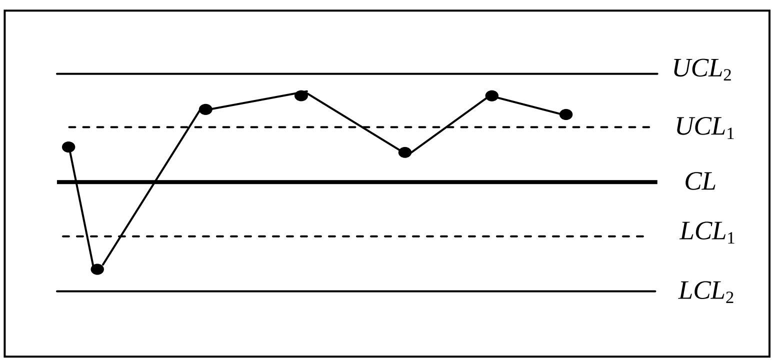

Figure 2.10. An illustration of the R – 4/5 rule

2.6 Performance Measures of a Control Chart

2.6.1 Average Run Length (ARL)

The average run length (ARL) is often used to evaluate the performances of control charts. The ARL is defined as the average number of sample points plotted on a chart before the first out-of-control signal is obtained. A sequence of ARL values corresponding to the various sizes of process shifts is called an ARL profile. ARL profiles are useful when alternative quality control schemes are evaluated and compared (Klein, 1997). If we use an $\bar{X}$ chart with limits set at plus/minus three standard deviations, the probability that a sample mean will fall outside the chart's limits is 0.0027 when the process is in-control. This means that if a process is in statistical control, we would expect to observe a sample mean falling outside the chart's limits about once in every 370 subgroups. The in-control ARL is calculated as follows:

$$\text{ARL}_0 = \frac{1}{0.0027} = 370\,. \tag{2.10}$$

In general, the ARL of a Shewhart chart is computed as

$$\text{ARL} = \frac{1}{p}\,, \tag{2.11}$$

where p is the probability of a sample mean falling outside the chart's limits.

2.6.2 Median Run Length (MRL)

Barnard (1959), Bissell (1969), Woodall (1983), Waldmann (1986a and 1986b) and Gan (1993), to name a few, have all criticized the sole dependence on ARL as a measure of a chart's performance. They suggested the use of median run length (MRL) as an additional performance measure. The MRL is the 50th percentage point of the probability distribution of the run length.

The main setback of the ARL is its difficulty of interpretation. The difference between the ARL and MRL decreases as the magnitude of a shift increases.

Interpretation based on ARL is complicated and could be misleading to quality control practitioners, as the shape of the run length distribution changes with the magnitude of the shift. On the contrary, the MRL does not have the interpretation problem faced by ARL. For instance, an MRL of 20 indicates that 50% of all the run lengths are less than 20, or to a layman, it simply means that an out-of-control will occur by the 20^{th} sample, in 50% of the time.

2.6.3 Standard Deviation of the Run Length (SDRL)

The standard deviation of the run length (SDRL) is the standard deviation of the probability distribution of the run length. Besides ARL and MRL, the SDRL is sometimes used to evaluate a chart's performance. Smaller SDRL values are more desirable than larger ones.

CHAPTER 3
MARKOV CHAIN APPROACH FOR THE REVISED RUNS RULES

3.1 Introduction

This chapter discusses the Markov chain approach to design the $R-m/k$ rule, based on MRL. It also explains the computation of the SDRL, by means of the Markov chain technique.

The computations of the MRL and SDRL using the Markov chain technique give more accurate values than the simulation method. This is because the Markov chain method uses the exact formulae in calculating the MRL and SDRL while the simulation method only provides approximated values.

3.2 The Markov Chain` Approach for Computing the Median Run Length (MRL) of the Revised Runs Rules

For the $R-m/k$ scheme, the $\bar{X}$ control chart is assumed to have the inner and outer limits shown in Figure 2.9. Here, the $\bar{X}$ chart is divided into five regions and the probability of a sample point falling in each region is defined as follows (Antzoulakos and Rakitzis, 2008b):

(i) p_1 denotes the probability that a point falls between UCL_1 and UCL_2 (region 1),

(ii) p_2 denotes the probability that a point falls between CL and UCL_1 (region 2),

(iii) p_3 represents the probability that a point plots between LCL_1 and CL (region 3),

(iv) p_4 represents the probability that a point plots between LCL_2 and LCL_1 (region 4), and

(v) p_5 denotes the probability of a point plotting beyond $\text{UCL}_2/\text{LCL}_2$ (region 5).