MONOPOLE-ANTIMONOPOLE PAIR, VORTEX DYONS OF THE SU(2) YANG-MILLS-HIGGS FIELD THEORY

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by

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Abstrak

Monokutub magnet dan dyon merupakan penyelesaian topologi soliton dalam ruang tiga dimensi, ia muncul dalam teori tolok Yang-Mills-Higgs di mana kumpulan tolok non-Abelian SU(2) dipecah secara spontan oleh medan Higgs kepada baki kumpulan simetri U(1). Walaupun cas magnet adalah dikuantumkan dari segi topologi, tetapi cas elektrik tidak.

Dalam tesis ini, monokutub magnet dan dyon dikaji dalam konteks teori medan SU(2) Yang-Mills-Higgs yang juga dikenali sebagai model SU(2) Georgi-Glashow. Matlamat kajian ini adalah untuk mendapatkan informasi tentang kewujudan dan ciri-ciri soliton topologi tersebut, struktur dan kelakuan ia dengan mengaji persamaan medan klasik.

Secara umumnya, ansatz yang bersesuaian amat penting dalam menyelesaikan persamaan-persamaan pergerakan tertib kedua. Langkah yang seterusnya adalah samada menyelesaikan persamaan tersebut secara analitik atau secara berangka. Ulasan tulisan yang awal mengenai monokutub magnet dan dyon telah diterangkan dalam tesis ini.

Dengan menggunakan transformasi tolok yang sesuai, kami mendapati bahawa cas magnet monokutub boleh dipindahkan dari medan Higgs ke medan tolok dan sebaliknya. Keputusan menunjukkan bahawa singulariti tali dari tolok Abelian boleh dialihkan dengan mengubah parameter dari sudut kutub terkawal (Boulware et al., 1976) selepas transformasi tolok.

Kami juga mengkaji penyelesaian monokutub yang bercas topologi satu per dua. Penyelesaian tersebut tidak semestinya mematuhi persamaan Bogomol'nyi tertib pertama dan ia mempunyai ketumpatan tenaga yang tak terhingga pada asalan. Keputusan kami menunjukkan bahawa monokutub separuh ini sebenarnya adalah monokutub separuh jenis Wu-Yang dan ia boleh memiliki cas elektrik dan menjadi dyon sparuh.

Penyelesaian monokutub yang baru bersimetri paksian telah dikaji secara berangka dengan mengeneralisikan penyelesaian jarak jauh asimptotik dari monokutub 't Hooft-Polyakov kepada fungsi eliptik Jacobi. Keputusan menunjukkan bahawa sesetengah daripada monokutub yang bersimetri paksian ini adalah terherot jika dibandingkan dengan monokutub 't Hooft-Polyakov.

Dengan mengkaji secara berangka, kami mendapati dyon pasangan monokutub-antimonokutub dan dyon vorteks cincin boleh membawa cas elektrik dan memiliki cas magnet yang lenyap. Dalam kes di mana keupayaan Higgs adalah lenyap, kesemua sifat-sifat dyon seperti jumlah tenaga, jarak antara monokutub dan antimonokutub, diameter dyon vorteks cincin meningkat secara eksponen ke ketakhinggaan semasa jumlah cas elektrik menghampiri nilai tak terhingga. Untuk kes keupayaan Higgs yang tak lenyap, jumlah cas elektrik akan akhirnya mencapai nilai genting yang terhingga dan kesemua sifat-sifat tersebut juga akan menghampiri nilai gentingnya.

Akhirnya, kaedah berangka yang digunakan dalam kajian kami telah diterangkan. Masalah komputasi berangka yang berkaitan diterangkan dengan teliti dan langkah-langkah balas untuk meningkatkan prestasi komputeran diperkenalkan.

Monopole-antimonopole Pair, Vortex Dyons of The SU(2) Yang-Mills-Higgs Field Theory

Abstract

Magnetic monopoles and dyons are topological soliton solutions in three space dimensions, which arise in Yang-Mills-Higgs gauge theory where the non-Abelian gauge group SU(2) is spontaneously broken by the Higgs field to a residual symmetry group U(1). While the magnetic charge is quantized due to topological arguments, the electric charge is not.

In this thesis, the magnetic monopoles and dyons are studied in the context of the SU(2) Yang-Mills-Higgs field theory which is also known as the SU(2) Georgi-Glashow model. The aim is to gain information on the existence and properties of these topological solitons, their structure and behaviour by studying the classical field equations.

Generically, an appropriate ansatz is pivotal in solving the second order equations of motion which are a set of nonlinear partial differential equations. Then the next step is either solving them analytically or numerically. The early literature review on magnetic monopoles and dyons are described in this thesis.

By applying a proper gauge transformation, we found that the magnetic charge of the monopole can be transferred from the Higgs field to the gauge field and vice versa. The results show that the string singularity from the Abelian gauge can be removed by varying the parameter from the regulated polar angle (Boulware et al., 1976) after gauge transformation.

We also study the one half topological charge monopole solutions. These solutions do not necessarily satisfy the first order Bogomol'nyi equations and they possess infinite energy density at the origin. Our results show that these half-monopoles are actually a half Wu-Yang type monopole and they can possess electric charge and become half-dyons.

New axially symmetric monopole solutions are studied numerically by generalizing the large distance asymptotic solution of the 't Hooft-Polyakov monopole to the Jacobi elliptic functions. The results show that some of these axially symmetric monopoles are distorted if compared with the 't Hooft-Polyakov monopole.

By study numerically, we found that the monopole-antimonopole pair dyon and vortex ring dyon can carry electric charges and possess vanishing magnetic charge. In the case when Higgs potential is vanishing, all the properties of the dyons such as total energy, separation between monopole and antimonopole, diameter of the vortex ring dyon increase exponentially to infinity when the net electric charge approach infinity. For non-vanishing Higgs potential case, the net electric charge will eventually reach its finite critical value and all the aforementioned properties also approach their critical values as well.

Finally, the numerical methods that are used in our aforementioned works are described. The related numerical computational problems are discussed in detail and the countermeasures to improve the computing performance are explained.

Chapter 1

Introduction

1.1 Symmetry in Physics

The beauty of physics often reveals itself as a symmetry or duality in our physical theories. The principle of symmetry has been permeating into theoretical physics and playing a main role in the description of all fundamental forces. Even Einstein himself was quite convinced that beauty was a guiding principle in the search for important results in theoretical physics. To put simply, symmetry is an operation that doesn't change how thing behaves relative to the outside world. So from a physicist's point of view, symmetry can be referring to an operation in space, like rotation, that doesn't change the result of an experiment.

There exist four fundamental forces which can describe all the interactions among matter constituents, namely the gravitational force, electromagnetic force, strong nuclear force and weak nuclear force. Albeit we understand quite well about the electromagnetic and the nuclear (strong and weak) forces, however the gravitational force still remains as a puzzle up to now. All of these forces are governed by gauge principle which is best understood in terms of symmetry group. To put in a nutshell, the symmetries of the SU(3) group describe the strong force, the SU(2) group describes the weak interaction, the U(1) group describes the electromagnetic force and the symmetry of Lorentz group describes the gravitational force (Carmeli, 1982). As expressed by C. N. Yang, the theoretical physicist, the role of symmetry is central to the entire modern physics in describing the properties of quantum particles (Yang, 2003).

All the physical phenomenon that happen either in macroscopic world or in microscopic world can be described in the language of mathematics. Physicists used to regard those most powerful physical theories which have a compact form of mathematical expression as the beauty or elegance of these theories. Paul

Adrien Maurice Dirac famously expressed this as *"It is more important to have beauty in one's equations than to have them fit experiment"*. Although it sounds a bit exaggerated but it reflects well how important is the role of mathematics in physical theories. However, many physicists would disagree with the statement above if science is solely following the aesthetic theory without referring to the experiment. A theory can only be considered true if it could be possibly tested by any feasible experiment. Without experimental guidance, one would be easily lost.

1.2 Magnetic Monopoles and Dyons

In physics, a magnetic monopole is a hypothetical particle which possesses only magnetic charge. On the other hand, a dyon is a magnetic monopole, that also carry electric charge (Julia and Zee, 1975).

Unlike electric charges which can be isolated, magnetic materials always have two 'poles', namely north pole and south pole. If one tries to split a magnetic bar into two pieces, it always ends up with two smaller magnetic bars with both north and south poles. Evidently it is impossible to isolate a single magnetic pole and only the combination of north and south poles seem to exist.

Electricity and magnetism are quite well-known to people for centuries. Supposedly, the field equations of electromagnetism are symmetrical between electric and magnetic field in vacuum. However, the symmetry between electricity and magnetism is ruined by the fact that a single electrical charge particle such as electron is ubiquitous, while a single magnetic pole has not been observed yet. From what has been discussed above, Maxwell's equations are no longer symmetric under the duality transformation. Hence, the absence of magnetic monopoles leads to the broken symmetry in electrodynamics. Paul A. M. Dirac, one of the founders of quantum mechanics and quantum electrodynamics, is the one who first introduced the quantum theory of magnetic charge (Dirac, 1931). He found that quantum mechanics literally allow the existence of magnetic monopoles.

This idea was consistent with Maxwell's equations and as a result provided an explanation for the observed quantization of electric charge.

Quantum field theory (QFT) was born of the necessity of dealing with the marriage of special relativity and quantum mechanics (Zee, 2003). It is the best and most complete theoretical framework to describe the quantum behaviour of elementary particles at high speed. QFT comprises the modern particle theory's ideas such as non-Abelian gauge theories, spontaneous symmetry breaking, soliton concept and so on. Once we consider in the framework of non-Abelian gauge theories, the new versions of Maxwell's equations are no longer linear and hence it may advocate soliton solutions. Solitons are non-linear wave solutions with finite nonzero energy that can describe particles with structure when the theory respects the principle of relativity.

In the context of non-Abelian gauge theories, the soliton also requires a Higgs field, that is a scalar field which provides a means of ascribing mass to other particles and to itself. This Higgs mechanism 'spontaneously break' the gauge symmetry to a subgroup and consequently the soliton that arises is actually the magnetic monopole. In next chapter we will find that how this spontaneous symmetry breaking is intimately pertaining to the existence of topological monopole solutions. The cardinal difference between topological monopole and Dirac monopole is that the former appear as regular, soliton-like solutions and they are natural and ineluctable. Moreover, the conservation of magnetic charge arises as the outcome of topological defect and not due to some symmetry argument.

1.3 Why Magnetic Monopoles?

Readers may wonder why we need a magnetic monopole and what is the outcome that it brings to our physical world? In principle, a magnetic monopole can solve many open questions in physics. For instance, with the existence of magnetic monopole we could preserve the symmetrization of electromagnetism in term of Maxwell's equations. If one allows for the possibility of "magnetic charge" analogous to electric charges, Maxwell's equations become completely symmetric under the interchange of electric and magnetic fields, i.e. duality transformation.

The existence of magnetic monopoles could explain the quantization of electric charge (Dirac, 1931, 1948). In nature, all electric charges that are found on particles seem to carry an integer multiples of the electron's charge. Let say, we denote the electron's charge as e, then all electric charges that are found in nature can be written as ne, for some integer n. This peculiar characteristic of electric charge is known as the quantization of electric charge. No one could explain this phenomenon until Dirac introduced the idea of magnetic monopole, whereby he proposed the Dirac quantization condition which says that in the presence of magnetic monopole, the product of electric and magnetic charges must be equal to an integer multiple of 1/2. The detailed description of quantization condition will be discussed in Chapter 2.

Magnetic monopoles and dyons are very common predictions of some Grand Unified Theory (GUT) models. GUT is a theory that seeks to unify the three fundamental forces i.e. electromagnetic, weak and strong force into a single fundamental interaction described by a larger gauge symmetry such as SU(5) or SO(10), which is larger than the standard model SU(3) $\times$ SU(2) $\times$ U(1). Some properties of monopoles such as their mass are model dependent. Monopoles seem to appear as a price that one has to pay in any theories that are intentionally unifying electromagnetism with other fundamental forces such as GUT, supersymmetry, extra dimension, string theory and so forth (Bais, 2005).

One of the significant predictions of GUT (Georgi and Glashow, 1974) and yet has not been observed experimentally is the proton decay. Proton decay is an analogical form of radioactive decay in which the proton decays into lighter subatomic particles, such as a neutral pion and a positron. Rubakov (1981, 1982) and Callan (1982a,b) proposed that the grand unified monopoles could catalyze proton decay. With the presence of GUT monopoles, the inert baryon, proton can decay with dramatic rates at low energy. After all, proton decay is a feasible indirect observation that could prove the validity of GUT.

On the other hand, the existence of monopole could provide an explanation for the quark confinement which is a phenomenon that quarks cannot be isolated. It was widely believed that due to the condensation of color-magnetic monopoles, the quarks are confined in flux lines through dual Meissner effect (Nambu, 1974; Mandelstam, 1976; 't Hooft, 1981; Polyakov, 1977).

1.4 The Existence of Monopoles

Given that the exotic monopoles should exist, but the question is why they are not seen up to now. Certain grand unified theories have them because magnetic monopoles are an inextricable prediction of grand unification in the early universe. According to the aforementioned section, it can be seen that the existence of monopoles are able to solve several open questions in physics. Obviously, magnetic monopoles play an important role in our universe which could not be ignored.

It has been generally believed that the standard Weiberg-Salam model, which is in accord with nature, does not have monopole solutions. However, in 1997 Cho and Maison (1997) have succeeded in obtaining model equations for a new type of monopole or dyon solution in the electroweak Weinberg-Salam model. This one-monopole is a non-trivial hybrid between the Abelian Dirac monopole and the non-Abelian 't Hooft-Polyakov monopole. However, this Cho-Maison monopole solution is still a theoretical result and the existence of such monopole is still waiting for empirical evidence. Whatever it is, the crucial challenge before searching for the magnetic monopole is to find the Higgs particle in the first place. This is because without the Higgs particle, the whole mathematical structure of the theory which leads to monopole solution will be in doubt.

To date, many attempts have been made to detect magnetic monopoles but none have been found. One of the tantalizing recorded event is by Blas Cabrera (Cabrera, 1982) on the night of February 14, 1982 (sometimes referred to as the " Valentine's Day Monopole"), had the perfect signature hypothesized for a magnetic monopole. Their monopole detectors are using the technique called "superconducting quantum interference device", or SQUID. A superconducting ring is used to detect the moving magnetic monopole based on the long range electromagnetic interaction between the magnetic particle and the macroscopic quantum state of the ring. However, when Cabrera's laboratory later on built an improved detector and other research groups tried to repeat the experiment, no similar reproducible evidence was found.

Another interesting cosmic ray experiment in 1975 which was carried out by the team of Price et al. (1975) engendered the announcement of the detection of a moving magnetic monopole. But that result was soon withdrawn in 1978 after serious errors were found by further analysis by Price et al. (1978) group. Recently, the searches of magnetic monopole have been carried out at high energy accelerators. Researchers try to detect the magnetic monopole immediately after their production in high-energy collisions such as e^+-e^-, e-p, p-p, and p-$\bar{p}$ interactions at various high energy colliders. For instance in 1990, a search at the Fermilab Tevatron collider using plastic track detectors seems to rule out magnetic monopole with masses up to 850 GeV (Bertani et al., 1990). Experiments at the Large Electron-Positron Collider 2 (LEP2) excluded masses below 102 GeV (OPAL Collaboration, 2008).

Recently in 2008, attention has turned to condensed matter system because a monopole-like quasiparticle could be observed in a kind of crystalline magnetic material know as spin ice (Castelnovo et al., 2008). It can show emergent phenomena that resemble magnetic monopoles in some respect. A year later, two team of researchers from France and Germany (Fennell et al., 2009; Morris et al., 2009) successfully reported the observation of certain states of spin ice that resemble magnetic monopoles configurations by using neutron scattering experiments. The materials used are holmium titanate $Ho_2Ti_2O_7$ and dysprosium titanate $Dy_2Ti_2O_7$. However, these 'quasi-monopoles' should not be confused with the actual monopole particles because they are not elementary particles.

Paul Adrien Maurice Dirac's quantum theory of magnetic monopole had inspired a large number of subsequent developments. He is often quoted on the importance of mathematical elegance in one's equations. Ironically towards the end of Dirac's career, he became less certain about the existence of monopoles due to the complete lack of experimental evidence. In 1981, Dirac was invited to attend a symposium at Abdul Salam International Center for Theoretical Physics in Trieste to commemorate the 50th anniversary of his monopole paper. He wrote a letter in response (Dirac, 1981):

> "I am inclined now to believe that monopoles do not exist. So many years have gone by without any encouragement from the experimental side."

Seemingly Dirac has abandoned his earlier dictum: *"It is more important to have beauty in one's equations than to have them fit experiment"*. In fact, beauty is very difficult to define therefore it is not supposed to follow any aesthetic theory alone. All the physical theory has to be justified by feasible experiment, otherwise it will not be considered as science regardless of how 'elegant' or 'beautiful' the theory is. Ultimately, magnetic monopole has to pass the rigorous experimental test or else it will still remain as a hypothetical particle and not a physical one.

To sum up, the lack of observational evidence however does not preclude the possibility that magnetic monopoles do exist. The question of whether the magnetic monopoles really exist or not still remains as an open question to the whole world. Even though their existence remains a mystery, we can at least be certain that if they do exist, they are a very rare phenomenon in our world.

1.5 SU(2) Yang-Mills-Higgs Theory

Throughout this thesis, the field theory model that will be used in which monopole solutions arise is the SU(2) Yang-Mills-Higgs (YMH) theory. This theory also known as the SU(2) Georgi-Glashow model, was once considered as an alternative to the Standard Model of electroweak interactions (Georgi and Glashow, 1972*a*). It is a SU(2) gauge group model with triplet of real Higgs fields. The SU(2) YMH Lagrangian in 3+1 dimensions with non vanishing Higgs potential is:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - V, \\ V &= \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2, \end{aligned} \tag{1.1}$$

where μ is the Higgs field mass and λ is the strength of the Higgs potential which are constants. The vacuum expectation value of the Higgs field is $\upsilon = \mu/\sqrt{\lambda}$ and the metric used is $g_{\mu\nu} = (- + ++)$. The SU(2) internal group indices a, b, c run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2,$ and 3 in Minkowski space. Here we use the *Einstein summation convention* to represent sums: any index that is repeated twice is summed over.

The Lagrangian (1.1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$\begin{aligned} D_\mu\Phi^a &= \partial_\mu\Phi^a + e\epsilon^{abc}A^b_\mu\Phi^c, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu. \end{aligned} \tag{1.2}$$

Since the gauge field coupling constant e can be scaled away, we can set e to one without any loss of generality.

The equations of motion that follow from the Lagrangian (1.1) are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + e\epsilon^{abc} A^{b\mu} F^c_{\mu\nu} = e\epsilon^{abc}\Phi^b D_\nu \Phi^c, \tag{1.3}$$

$$D^\mu D_\mu \Phi^a = \lambda \Phi^a (\Phi^b \Phi^b - \frac{\mu^2}{\lambda}). \tag{1.4}$$

The energy-momentum tensor $T_{\mu\nu}$ which follows from the Lagrangian (1.1) and the equations of motion (1.3) and (1.4) is (Prasad and Sommerfield, 1975)

$$T_{\mu\nu} = F^a_{\mu\rho} F^{a\rho}_{\nu} + D_\mu \Phi^a D_\nu \Phi^a + g_{\mu\nu}\mathcal{L} \tag{1.5}$$

and the Hamiltonian density is given explicitly by

$$\begin{aligned} T_{00} &= F^a_{0\rho} F^{a\rho}_0 + D_0\Phi^a D_0 \Phi^a + g_{00}\mathcal{L} \\ &= F^a_{0\rho} F^{a\rho}_0 + D_0\Phi^a D_0 \Phi^a + g_{00}\left(-\frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta} - \frac{1}{2}D_\alpha\Phi^a D^\alpha \Phi^a - V\right) \\ &= E^a_i E^a_i + D_0\Phi^a D_0\Phi^a - \frac{1}{2}(E^a_i E^a_i - B^a_i B^a_i) \\ &\quad -\frac{1}{2}D_0\Phi^a D_0\Phi^a + \frac{1}{2}D_i\Phi^a D_i \Phi^a + V \\ &= \frac{1}{2}\left(E^a_i E^a_i + B^a_i B^a_i + D_0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a\right) + V, \end{aligned} \tag{1.6}$$

where

$$E^a_i = F^a_{i0} \quad \text{and} \quad B^a_i = -\frac{1}{2}\epsilon_{ijk}F^a_{jk}. \tag{1.7}$$

It is conserved by virtue of the field equations:

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.8}$$

Therefore, the static Hamiltonian, also known as the total energy of the system, is

$$E = \int d^3x\, T_{00} = \int d^3x \left(E^a_i E^a_i + B^a_i B^a_i + D_0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a\right) + V, \tag{1.9}$$

where the potential of the scalar fields is given in the second equation of (1.1).

1.5.1 The SU(2) Gauge Group

Consider now the SU(2) group, the gauge potential and the field strengths can be written as (Rubakov, 2002),

$$A_\mu = e\frac{\sigma^a}{2i}A_\mu^a, \quad F_{\mu\nu} = e\frac{\sigma^a}{2i}F_{\mu\nu}^a, \quad \Phi = \frac{\sigma^a}{2i}\Phi^a, \tag{1.10}$$

and σ^a are the Pauli matrices with commutation relations $[\sigma^a/2, \sigma^b/2] = \epsilon_{abc}\sigma^a/2$ and it have the following properties,

$$\sigma_a\sigma_b = i\epsilon_{abc}\sigma_c + \delta_{ab}, \quad \mathrm{Tr}(\sigma^a) = 0, \quad \mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}. \tag{1.11}$$

Here all Roman indices a, b, c take the values $1, 2, 3$.

Under SU(2) gauge transformation ω, the gauge potentials and Higgs field transform as

$$A_\mu \;\rightarrow\; A'_\mu = \omega A_\mu \omega^{-1} - (\partial_\mu\omega)\omega^{-1}, \tag{1.12}$$

$$\Phi \;\rightarrow\; \Phi' = \omega\Phi\omega^{-1}. \tag{1.13}$$

The SU(2) gauge transformation $\omega(x)$ can also depend on space-time point x_μ as,

$$\begin{aligned}\omega(x) &= \exp(-\theta_a(x)T_a) \\ &= \cos\left(\frac{1}{2}\theta(x)\right) I \,+\, i\hat{n}_a(x)\sigma_a \sin\left(\frac{1}{2}\theta(x)\right),\end{aligned} \tag{1.14}$$

where I is identity matrix, $T_a = \frac{\sigma_a}{2i}$ are the generators of the SU(2) gauge transformation and $\hat{n}_a(x)$ is the unit vector defined by

$$\theta_a(x) \;=\; \hat{n}_a(x)\theta(x). \tag{1.15}$$

Here $\theta_a(x)$ is real for a real SU(2) gauge transformation and complex for a complex

SU(2) gauge transformation. The pure gauge term in Eq.(1.12) then becomes

$$e\frac{\sigma^a}{2i}A^a_\mu\,(\text{pure}) \;=\; -(\partial_\mu\omega)\omega^{-1}, \tag{1.16}$$

which can be written as

$$\begin{aligned} eA^a_\mu\,(\text{pure}) &= -i\mathrm{Tr}\{\sigma_a(\partial_\mu\omega)\omega^{-1}\} \\ &= \frac{1}{2}\mathrm{Tr}\Big\{\sigma_a\sigma_b\big(\hat{n}_b\partial_\mu\theta + (\partial_\mu\hat{n}_b)\sin\theta\big) + 2\sigma_c\sigma_d\epsilon_{abd}(\partial_\mu\hat{n}_b)\hat{n}_c\sin^2\frac{\theta}{2}\Big\} \\ &= \hat{n}_a\partial_\mu\theta + (\partial_\mu\hat{n}_a)\sin\theta + 2\epsilon_{abc}(\partial_\mu\hat{n}_b)\hat{n}_c\sin^2\frac{\theta}{2}. \end{aligned} \tag{1.17}$$

By using the formula (Actor, 1979),

$$\omega\sigma_a\omega^{-1} \;=\; \sigma_a\cos\theta + \epsilon_{abc}\hat{n}_b\sigma_c\sin\theta + 2\hat{n}_a(\hat{n}_b\sigma_b)\sin^2\frac{\theta}{2}, \tag{1.18}$$

the right hand side of gauge potential Eq.(1.12) can be written explicitly as

$$\begin{aligned} A'^a_\mu = \cos\theta A^a_\mu \;+\; \sin\theta\epsilon_{abc}A^b_\mu\hat{n}_c \;+ 2\sin^2\frac{\theta}{2}\hat{n}_a(\hat{n}_bA^b_\mu) \\ +\,\frac{1}{e}\big\{\hat{n}_a\partial_\mu\theta \;+\; \sin\theta\,\partial_\mu\hat{n}_a \;+\; 2\sin^2\frac{\theta}{2}\epsilon_{abc}(\partial_\mu\hat{n}_b)\hat{n}_c\big\}, \end{aligned} \tag{1.19}$$

whereas the Higgs field from Eq.(1.13) can be written as

$$\Phi'^a = \cos\theta\Phi^a + \sin\theta\epsilon_{abc}\Phi^b\hat{n}_c + 2\sin^2\frac{\theta}{2}\hat{n}_a(\hat{n}_b\Phi^b). \tag{1.20}$$

1.5.2 Spontaneous Symmetry Breaking of SU(2) group

By using spontaneous symmetry breaking, mass can be introduced into the Yang-Mills-Higgs theory (Rubakov, 2002). The coupling of Higgs field in Yang-Mills theory will force the gauge bosons to acquire mass. Let us consider again the Lagrangian density that is given in Eq.(1.1). The SU(2) gauge symmetry can be spontaneously broken to U(1). The minimum of the potential energy is realized at $|\Phi^a_{\text{vac}}| = \mu/\sqrt{\lambda} = \upsilon$. If we take the ground state or vacuum field configuration of Φ_0^a to be

$$\Phi_0^1 = \Phi_0^2 = 0, \quad \Phi_0^3 = \upsilon, \tag{1.21}$$

the gauge transformation from Eq.(1.13) gives the same minimum, so we have a symmetry. In order to break this symmetry and find the particle spectrum, we have to perturb around this minimum. Let us consider the perturbations $\eta(x)$ about the ground state υ, it become

$$\Phi_0^1 = \Phi_0^2 = 0, \quad \Phi_0^3 = \upsilon + \eta(x). \tag{1.22}$$

By substituting the above equations into the Lagrangian (1.1), considering only to quadratic order and neglecting the higher order in η, the potential become

$$\begin{aligned} V &= \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2 \\ &= \left(\frac{\sqrt{\lambda}}{2}(\upsilon + \eta(x)) - \frac{\mu^2}{2\sqrt{\lambda}}\right)^2 \\ &= \mu^2\eta^2. \end{aligned} \tag{1.23}$$

Assuming that the field A_μ^a is small and to linear order, we have

$$F_{\mu\nu}^a \approxeq \mathcal{F}_{\mu\nu}^a \quad \text{where} \quad \mathcal{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \tag{1.24}$$

After substituting equations (1.22) to Lagrangian (1.1), it becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2 \\ &= -\frac{1}{4}\mathcal{F}_{\mu\nu}^a\mathcal{F}^{a\mu\nu} + \frac{1}{2}\left(\partial_\mu\eta\right)^2 + \frac{1}{2}e^2\upsilon^2 A_\mu^2 A_\mu^2 + \frac{1}{2}e^2\upsilon^2 A_\mu^1 A_\mu^1 - \mu^2\eta^2. \end{aligned} \tag{1.25}$$

By doing so, we can recognize the mass term in the Lagrangian, which means that the mass term is written in analogy to the mass term in scalar field Lagrangian (Zee, 2003). Overall, the particle spectrum of the theory then consists of a massless photon, two massive vector bosons and a massive scalar field. Two components of massive vector field A^1_μ, A^2_μ, which also known as intermediate vector boson will acquire the same mass $M_W = ev$. The massive scalar field is $\eta(x)$ and its mass is equal to $M_H = \sqrt{2\lambda}v$. Instead of two real fields, it is convenient to consider a single complex vector field:

$$W^{\pm}_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \pm iA^2_\mu\right). \tag{1.26}$$

This complex vector fields $W^{\pm}_\mu$ will have electric charge $\pm e$ (This is the reason for calling the gauge coupling e). We would like to interpret A^3_μ as the Abelian gauge potential which corresponding to unbroken U(1) electromagnetic subgroup. It is representing the massless photon, so that electromagnetism is embedded into the theory. In conclusion, the nonzero vacuum expectation value of the Higgs field breaks the non-Abelian SU(2) gauge symmetry to the Abelian U(1) symmetry.

The total number of degrees of freedom is conserved. According to the Lagrangian density Eq.(1.1), there are 3 massive scalar fields (Φ^1, Φ^2, Φ^3) and three massless vector fields $(A^1_\mu, A^2_\mu, A^3_\mu)$. So that the initial number of degrees of freedom is

$$3 + 2 \times 3 = 9. \tag{1.27}$$

Each scalar field contributes one degree of freedom whereas each vector field component contribute two degrees of freedom. This is because electromagnetic wave has two polarization of **E** and **B**. After the symmetry breaking there are one massive scalar η and two massive vector bosons (A'^1_μ, A'^2_μ) and one massless boson A'^3_μ so that

$$1 + 2 \times 3 + 2 = 9, \tag{1.28}$$

where each massive vector boson contributes 3 degrees of freedom.

1.6 Natural Units and Dimension Analysis

In physics, natural units are physical units of measurement based only on universal physical constants. According to the convention of particle physics, we use natural units in which the speed of light c and the Planck constant divided by 2π, $\hbar$ are both set equal to 1. In other words, we take c and $\hbar$ to be dimensionless. Therefore, the only non-trivial dimension is the dimension of mass. In natural units, length and time have the same dimension which is the inverse of the dimension of mass (and of energy and momentum). Particle physicists tend to count dimension in term of mass as they are used to thinking of energy scales. This is in contrast to condensed matter physicists, who usually count dimension in term of length scales. For a detailed description of natural units and dimension analysis, readers are advised to refer the book by Rubakov (2002) and Zee (2003). Whereas for the detailed of conversion factors in natural units, please refer to the book by Dominguez-Tenreiro and Quiros (1988).

Chapter 2

Literature Review on Monopoles and Dyons

2.1 Electromagnetic Duality

The idea of magnetic monopoles is closely related to the idea of electromagnetic duality in classical electrodynamics. The four well-known Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho_e, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2.3}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_e. \tag{2.4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ_e is the electric charge density, and $\mathbf{J}_e$ is the electric current density. Let us consider the Maxwell's equations in vacuum first; when $\rho_e = \mathbf{J}_e = 0$, all the right hand side of the above equations appear to be zero and they become symmetric and invariant under the transformation

$$\mathbf{E} \to \mathbf{B} \quad \text{and} \quad \mathbf{B} \to -\mathbf{E}. \tag{2.5}$$

This symmetry is known as electromagnetic duality. We can generalize the duality transformation (2.5) to duality rotations which are parameterized by an arbitrary angle θ (Harvey, 1996)

$$\begin{aligned} \mathbf{E} &\to \cos\theta \mathbf{E} + \sin\theta \mathbf{B}, \\ \mathbf{B} &\to -\sin\theta \mathbf{E} + \cos\theta \mathbf{B}. \end{aligned} \tag{2.6}$$

Unfortunately, this symmetry seems to be ruined by the fact that we have not yet observed any magnetic charges but only electric charges. To understand the argument, let there be no magnetic charge but only electric charge density ρ_e and current density $\mathbf{J}_e$. The Maxwell equations become exactly the same as the previous equations (2.1)-(2.4) where the ρ_e and $\mathbf{J}_e$ are nonzero.

The equations (2.2) and (2.3) seem to be missing something on their right hand sides. This is because the above Maxwell's equations assume there is no magnetic charge, therefore there is no magnetic current density $\mathbf{J}_m$. Consequently, the absence of magnetic charge ruins the symmetrization of electromagnetic duality. In physics jargon the absence of magnetic charge breaks the symmetry.

In order to retain the electromagnetic duality again, we assume that there exist magnetic charge ρ_m and magnetic current density $\mathbf{J}_m$ (Jackson, 1999). The Maxwell equations after modification would then be

$$\begin{aligned} &\nabla \cdot \mathbf{E} = \rho_e, && \nabla \cdot \mathbf{B} = \rho_m, \\ &\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}_m, && \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_e. \end{aligned} \tag{2.7}$$

Now the above Maxwell's equations with both electric and magnetic source terms look more symmetric. Besides that, they are left unchanged under the following duality transformations (Song, 1996)

$$\begin{aligned} &\mathbf{E} \to \mathbf{B}; && \mathbf{B} \to -\mathbf{E} \\ &\rho_e, \mathbf{J}_e \to \rho_m, \mathbf{J}_m; && \rho_m, \mathbf{J}_m \to -\rho_e, -\mathbf{J}_e. \end{aligned} \tag{2.8}$$

Apparently, the invariance of the equations of electrodynamics under duality transformations conjecture the existence of isolated magnetic poles which would be the counterparts of electric charges. In other words, the existence of magnetic monopole in the universe is essential in order to keep the symmetrization of Maxwell's equations in electromagnetism.

2.2 Dirac Monopole and Charge Quantization

The idea of magnetic monopoles began with Dirac's paper on *Quantized Singularities in the Electromagnetic Field* which was published in the Proceedings of the Royal Society of London A on 29 May 1931 (Dirac, 1931). The Dirac monopole is based upon a straightforward generalization of the electric monopole. By analogy with the electric field **E** of a point electric charge, the magnetic field **B** of a point magnetic monopole can be written as,

$$\mathbf{E} = e\frac{\mathbf{r}}{r^3} \quad \rightarrow \quad \mathbf{B} = g\frac{\mathbf{r}}{r^3}. \tag{2.9}$$

These fields can be expressed in terms of potential, $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$. But we will have a contradiction here because we know that in vector calculus the divergence of a curl is equal to zero, i.e. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

In order to evade this problem, we introduce the Dirac delta function

$$\delta(r) = \left\{ \begin{array}{cc} 0, & \text{if } r \neq 0 \\ \infty, & \text{if } r = 0 \end{array} \right\} \tag{2.10}$$

and Maxwell's equations are generalized to

$$\nabla \cdot \mathbf{E} = 4\pi e \delta^3(r) \quad \rightarrow \quad \nabla \cdot \mathbf{B} = 4\pi g \delta^3(r). \tag{2.11}$$

So there is a delta function singularity in the **A** field. Take a sphere surrounding the point monopole. At the top of sphere, there is a small circle that is centered around the north pole.

The flux of the magnetic field through the circle is given by,

$$\int \mathbf{B} \cdot d\mathbf{S} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l}. \tag{2.12}$$

If the circle is infinitesimally small, including only the north pole, then the line integral around this small circle is zero. But if the circle is made successively larger, until it includes the entire sphere, then the surface integral over the **B** is $4\pi g$. However the line integral over the **A** field must be zero because the loop has become an infinitesimally small loop surrounding the south pole. It seems that we have encounter a new contradiction here.

To avoid this, the **A** field must be singular along the negative z-axis. There must be an unphysical singularity that extends from the origin down to the south pole and beyond. This singularity is called the Dirac String. The vector potential cannot be defined on the Dirac string, but it is defined everywhere else.

The wave function in the presence of monopole must be single-valued when we go around the Dirac string. The wave function for free particle is

$$\Psi \approx \exp\left(\frac{i}{\hbar}\right)(\mathbf{p}\cdot\mathbf{r} - Et). \tag{2.13}$$

In the presence of electromagnetic field, we make the standard substitution $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\,\mathbf{A}$. With this substitution, the wave function picks up new phase factor Λ given by,

$$\Lambda = \exp\frac{-ie}{c\hbar}\left(\mathbf{A}\cdot\mathbf{r}\right). \tag{2.14}$$

In order for the wave function to be single-valued when we go around a loop, this phase factor α must be equal to one. The line integral around the Dirac string must therefore be $2\pi n$, where n is an integer. Then we have

$$2\pi n = \frac{e}{c\hbar}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{e}{c\hbar}\int \mathbf{B}\cdot d\mathbf{S} = \frac{e}{c\hbar}4\pi g. \tag{2.15}$$

Hence, the final Dirac quantization condition is

$$e = n\frac{\hbar c}{2g}. \tag{2.16}$$

The above charge quantization condition implies that the existence of a single monopole would explain the observed quantization of all electric charges.

2.3 Wu-Yang Monopole

In 1975, T.T. Wu and C.N. Yang found a method to describe magnetic monopole without the need for Dirac string singularities (Wu and Yang, 1975). To avoid introducing singularities in the coordinate system, one divides the sphere into more than one overlapping region and defines a singularity-free coordinate system in each region.

The space outside of a magnetic monopole will be divided into two overlapping regions, R_a and R_b (Wu and Yang, 1976). A vector potential $(A_\mu)_a$ in R_a and a vector potential $(A_\mu)_b$ in R_b will be defined as below:

$$\begin{aligned} R_a: &\quad 0 \leq \theta < \tfrac{1}{2}\pi + \delta, \quad r > 0, \quad 0 \leq \phi < 2\pi \\ R_b: &\quad \tfrac{1}{2}\pi - \delta < \theta \leq \pi, \quad r > 0, \quad 0 \leq \phi < 2\pi \\ R_{ab}: &\quad \tfrac{1}{2}\pi - \delta < \theta < \tfrac{1}{2}\pi + \delta, \quad r > 0, \quad 0 \leq \phi < 2\pi \end{aligned} \tag{2.17}$$

where we assume δ such that $0 < \delta \leq \frac{1}{2}\pi$. The above mathematical expressions tell us that the region R_a covers a bit more than the upper hemisphere, whereas the region R_b covers a bit more than the lower hemisphere as shown in Fig.2.1. Take the vector potentials to be

$$\begin{aligned} (A_r)_a = (A_\theta)_a = 0, \quad (A_\phi)_a = \tfrac{g}{r\sin\theta}(1-\cos\theta), \\ (A_r)_b = (A_\theta)_b = 0, \quad (A_\phi)_b = \tfrac{-g}{r\sin\theta}(1+\cos\theta). \end{aligned} \tag{2.18}$$

After that, we will "glue" the two vector potentials along the equator. The final gluing process between these two different field configurations is accomplished by making a gauge transformation between the two configurations along the equator. In the overlap region the $\Delta\mathbf{A}$ is given by,

$$\Delta\mathbf{A} = \mathbf{A}_a - \mathbf{A}_b = (A_a - A_b)\hat{e}_\phi = \frac{g}{r}\frac{2}{\sin\theta}\hat{e}_\phi. \tag{2.19}$$

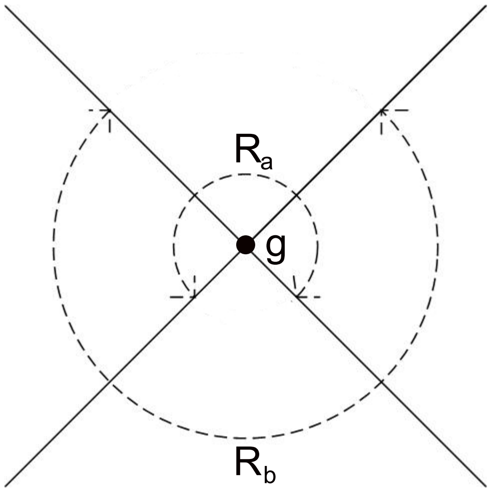

Figure 2.1: Division of space outside of monopole g into overlapping regions of R_a and R_b

After that, take a loop at the polar angle θ,

$$\oint \Delta \mathbf{A} \cdot d\mathbf{r} = \int \frac{g}{r} \frac{2}{\sin\theta} r \sin\theta d\varphi = 4\pi g. \tag{2.20}$$

But according to the Bohm-Aharonov experiment (Aharonov and Bohm, 1959), the relevant phase factor is $\exp\left(ie/c\hbar \oint \mathbf{A} \cdot d\mathbf{r}\right)$. Therefore we need or allow

$$\begin{aligned} &\exp\left(i\frac{e}{c\hbar}\oint \mathbf{A} \cdot d\mathbf{r}\right) = 1 \\ &\frac{e}{c\hbar}\oint \mathbf{A} \cdot d\mathbf{r} = 2n\pi \\ &eg = \frac{n\hbar c}{2}. \end{aligned} \tag{2.21}$$

Hence, the magnetic monopoles can exist but it must be quantized in units of $g_0 = \frac{\hbar c}{2e}$ (in unit $\hbar = c = 1$). Once again we obtain the Dirac quantization,

$$eg = \frac{1}{2}n. \tag{2.22}$$

2.4 't Hooft Polyakov Monopole

The solution of finite energy stringless magnetic monopoles in non-Abelian gauge theories with spontaneously broken symmetry was first discovered by 't Hooft

(1974) and Polyakov (1974). They found that such objects were actually three-dimensional topological solitons. A soliton is a stable localized solution to the classical field equations which has finite nonzero energy. These regular magnetic monopoles are not put in by hand (as in the original work of Dirac) but are an inevitable outcome of the non-Abelian SU(2) gauge group which upon spontaneous symmetry breaking to U(1) yields solitonic solutions that carry magnetic charge.

Lets consider the SU(2) Georgi-Glashow model with gauge group SU(2) symmetry group which has been mentioned in Chapter 1 (1.1), containing the gauge field strength $F^a_{\mu\nu}$ (a is the internal group indices run from 1 to 3) and triplet of real Higgs fields Φ^a. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2. \tag{2.23}$$

The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by equations (1.2), whereas the equations of motion that follow from the Lagrangian (2.23) are given by equations (1.3) and (1.4). The 't Hooft-Polyakov ansatz is:

$$\Phi^a = \frac{r^a}{er^2}H(r), \quad A^a_n = \epsilon_{amn}\frac{r^m}{er^2}[1 - K(r)], \quad A^a_0 = 0. \tag{2.24}$$

These ansatz will reduce the equations of motion (1.3) and (1.4) to

$$\begin{aligned} r^2\frac{d^2K}{dr^2} &= KH^2 + K(K^2 - 1), \\ r^2\frac{d^2H}{dr^2} &= 2K^2H + \frac{\lambda}{e^2}H(H^2 - r^2). \end{aligned} \tag{2.25}$$

A solution of these equations must satisfy the following boundary conditions

$$\begin{aligned} K(r) \to 1, \quad H(r) \to 0, \quad \text{as} \quad r \to 0, \\ K(r) \to 0, \quad H(r) \to r, \quad \text{as} \quad r \to \infty. \end{aligned} \tag{2.26}$$

Evidently, these ordinary differential equations (ODE) cannot be solved analytically for all general values of λ, unless it is considered in the BPS limit where $\lambda = 0$ which will be discussed later. However, a numerical solution is still possible and it was reported in Bais and Primarck (1976) and Kirkman and Zachos (1981).

2.4.1 Electromagnetic Field and Charge Quantization

Nevertheless we still can prove the existence of monopole without solving the equations of motion (2.25) directly. Here, we are interested in static solutions in which the gauge potential have the non-trivial form

$$A_i^a = -\epsilon_{iab}\frac{r^b}{er^2} \quad (r \to \infty); \qquad A_0^a = 0 \tag{2.27}$$

$$\Phi^a = \upsilon\frac{r^a}{r} \quad (r \to \infty), \tag{2.28}$$

where the vacuum expectation value, $\upsilon = \frac{\mu}{\sqrt{\lambda}}$. Equation (2.28) implies $|\Phi| = \upsilon$, therefore the right hand side of the equation (1.4), vanishes at infinity. One can show that Φ^a is covariantly constant at infinity, namely $D_\mu\Phi^a = 0$. At spatial infinity the field Φ is pointing radially outward, so this configuration is known picturesquely as a hedgehog. 't Hooft found a gauge invariant definition of the Abelian electromagnetic field tensor,

$$F_{\mu\nu} = \frac{1}{|\Phi|}\Phi^a F_{\mu\nu}^a - \frac{1}{e|\Phi|^3}\epsilon^{abc}(D_\mu\Phi^b)(D_\nu\Phi^c). \tag{2.29}$$

For the special case $\Phi = (0, 0, 1)$, one gets $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and it reduces the electromagnetic field to the usual one. Now, by defining

$$A_\mu = \frac{1}{|\Phi|}\Phi^a A_\mu^a, \tag{2.30}$$

after a straightforward calculation it gives,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e}\epsilon_{abc}\hat{\Phi}^a(\partial_\mu\hat{\Phi}^b)(\partial_\nu\hat{\Phi}^c), \quad \hat{\Phi}^a = \Phi^a/|\Phi|. \tag{2.31}$$

With this definition, we can now calculate the magnetic and electric charge of the monopole. We find that $A_\mu = 0$ and that:

$$F_{0i} = 0, \quad F_{ij} = -\frac{1}{er^3}\epsilon_{ijk}r^k, \quad B_k = \frac{r^k}{er^3}. \tag{2.32}$$

With this value of the magnetic field, then we can show that the total flux through a sphere surrounding the monopole is given by $\frac{4\pi}{e}$. But the total flux of a monopole is given by $4\pi g$. Therefore, the monopole magnetic charge obeys the constraint

$$eg = 1, \tag{2.33}$$

which is twice the Dirac case.

2.4.2 Topological Charge

The topological magnetic current (Arafune et al., 1975) is defined to be

$$k_\mu = \frac{1}{8\pi}\ \epsilon_{\mu\nu\rho\sigma}\ \epsilon_{abc}\ \partial^\nu\hat{\Phi}^a\ \partial^\rho\hat{\Phi}^b\ \partial^\sigma\hat{\Phi}^c, \tag{2.34}$$

which is also the topological current density of the system and the corresponding conserved topological magnetic charge is

$$\begin{aligned} M &= \int d^3x\ k_0 = \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon^{abc}\partial_i\left(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right)d^3x \\ &= \frac{1}{8\pi}\oint d^2\sigma_i\left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) \\ &= \frac{1}{4\pi}\oint d^2\sigma_i\ B_i. \end{aligned} \tag{2.35}$$

For such configuration, the static energy for the system is

$$E = \int d^3x\left(\frac{1}{2}B_i^aB_i^a + \frac{1}{2}D_i\Phi^aD_i\Phi^a + \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2\right). \tag{2.36}$$

A necessary condition for finiteness of the energy is the requirement

$$\Phi^a \Phi^a = \upsilon^2, \qquad r \to \infty. \tag{2.37}$$

The boundary condition (2.27) and the above condition suggest that the direction of the fields Φ^a in internal space may depend on the direction in physical three-dimensional space.

$$\Phi^a|_{r\to\infty} = \Phi^a(\mathbf{n}) \quad \text{where} \quad \mathbf{n} = \frac{\mathbf{r}}{r}.$$

This particular boundary condition defines a map of the sphere at spatial infinity, S^2_∞, onto a sphere S^2 in the internal space SU(2) manifold. Such mappings fall into a denumerable infinity of homotopy classes which form the group $\pi_2(S^2) = Z$, where π_2 is the second homotopy group and the elements of Z are integers, i.e., the winding number n.

2.4.3 The Mass of the Monopole

The mass of the monopole (its static energy) can be estimated by rewriting Eq.(2.36) in term of the dimensionless quantities by operate the rescaling (Weinberg and Yi, 2007)

$$\Phi \to \Phi/\upsilon, \quad A_\mu \to A_\mu/\upsilon, \quad r \to e\upsilon r. \tag{2.38}$$

This isolates the dependence on e and υ, and leads to the following mass formula:

$$M_{\text{soliton}} = \frac{M_W}{\alpha} f(\lambda/e^2), \quad \alpha = e^2/4\pi, \quad M_W = e\upsilon, \tag{2.39}$$

where M_W is the mass of the charged intermediate vector boson ($M_W \approx 50 - 60$GeV) and $f(\lambda/e^2)$ is a monotonically increasing function which satisfies $f(0) = 1$. Therefore this soliton is expected to have a very large mass $\geq 10^4$GeV (Marciano, 1978; Georgi and Glashow, 1972*b*).

2.5 Julia-Zee Dyon

So far we have only considered solutions with zero electric charge. In this section we will discuss solutions which predict particles with non-zero electric and magnetic charge. Julia and Zee (1975) have shown that the same SU(2) model that is used in 't Hooft-Polyakov monopole also yields a dyon solution with non-zero electric and magnetic charge. The Julia-Zee ansatz is

$$\Phi^a = \frac{r^a}{er^2}H(r), \quad A_n^a = \epsilon_{amn}\frac{r^m}{er^2}[1-K(r)], \quad A_0^a = \frac{r^a}{er^2}J(r). \tag{2.40}$$

Substituting the ansatz back into equation (1.3) and (1.4) yields

$$\begin{aligned} r^2\frac{d^2K}{dr^2} &= K(H^2-J^2)+K(K^2-1), \\ r^2\frac{d^2H}{dr^2} &= 2K^2H+\frac{\lambda}{e^2}H(H^2-r^2), \\ r^2\frac{d^2J}{dr^2} &= 2K^2J. \end{aligned} \tag{2.41}$$

One cannot solve the equations (2.41) in closed form, but numerical solutions can be obtained when it satisfies the following boundary conditions (Julia and Zee, 1975; Actor, 1979):

As $r \to 0$, $K(r) \to 1, H(r) \to 0, J(r) \to 0$.

As $r \to \infty$,

$$\begin{aligned} K(r) &\to Ar\exp\left(-r\sqrt{\beta^2-M^2}\right), \\ J(r) &\to Mr+C+O(1/r), \\ H(r) &\to \left(\frac{e\mu}{\sqrt{\lambda}}\right)r+\ldots, \end{aligned} \tag{2.42}$$

where A is constant, M is a new parameter with the dimension of mass, C is the unknown constant which has to be found numerically and $\beta = \left(\frac{e\mu}{\sqrt{\lambda}}\right)^{1/2}$. The