THE STUDY OF THE SOLITON SOLUTIONS IN (2+1) YANG-MILLS-HIGGS FIELD THEORY

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THE STUDY OF THE SOLITON SOLUTIONS IN (2+1) YANG-MILLS-HIGGS FIELD THEORY

by

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Table of Contents

List of Figures

KAJIAN PENYELESAIAN SOLITON DALAM TEORI MEDAN (2+1) YANG-MILLS-HIGGS

Abstrak

Tujuan tesis ini ialah mengkaji penyelesaian soliton teori medan Yang-Mills-Higgs dalam dimensi (2+1). Soliton ditakrifkan sebagai penyelesaian gelombang soliton, gelombang denyutan menjalar dalam bentuk persamaan pembezaan biasa atau persamaan pembezaan separa. Kami perlu menyelesaikan persamaan pembezaan dengan mengunakan kaedah analisis atau berangka untuk mencari penyelesaian kepada soliton.

Dalam tesis ini, penyelesaian soliton seperti pusaran, ekakutub magnet instanton akan dikaji dalam konteks teori $U(1)$ tolok Abelan dan teori tolok tak Abelan medan $SU(2)$ Yang-Mills-Higgs yang mana ia juga dikenali sebagai model $SU(2)$ Georgi-Glashow. Matlamat kajian ini adalah untuk mendapatkan informasi mengenai kewujudan dan ciri-ciri soliton topologi, struktur dan sifat-sifat mereka dengan mengkaji persamaan medan klasik.

Teori penguraian Abelan Cho diperkenalkan untuk mengkaji dan mencari penyelesaian tepat ekakutub. Kami memperolehi penyelesaian ekakutub dengan mengaktifkan bahagian terbatas penguraian, manakala bahagian valensi yang tak terbatas dinyahaktifkan. Penyelesaian ini mempunyai parameter bebas yang mana ia boleh digunakan untuk mewakili pelbagai jenis penyelesaian ekakutub, contohnya, penyelesaian ekakutub Wu-Yang, ekakutub t' Hooft dan juga ekakutub setengah.

Kami juga menyelidik sifat-sifat untuk penyelesaian pusaran tak Abelan dengan istilah Chern-Simons. Kami telah menjanakan penyelesaian berangka daripada sekumpulan persamaan pembezaan yang tak linear yang juga dikenali sebagai persamaan gerakan bagi pusaran.

Akhirnya, kami mengkaji penyelesaian ekakutub-instanton dalam teori tolok topologi berat dalam dimensi (2+1) dengan istilah jisim Chern-Simons yang telah dikaji oleh Pisarski sejak bertahun-tahun yang lalu. Kami berjaya memperolehi penyelesaian berangka yang sekata bagi nilai istilah kekuatan Chern-Simons dan kekuatan medan Higgs yang berbeza dengan menginterpolasikan antara ciri-ciri pada sempadan asalan dan infiniti .

THE STUDY OF THE SOLITON SOLUTIONS IN (2+1) YANG-MILLS-HIGGS FIELD THEORY

Abstract

The purpose of this thesis is to study the solitonic solutions in (2+1) dimensions of the Yang-Mills-Higgs field theories. Soliton is defined as solitary, traveling wave pulse solution of nonlinear ordinary differential equations (ODEs) or partial differential equations (PDEs). In order to find the soliton, we must solve the differential equations either analytically or numerically.

In this thesis, the soliton solutions such as vortex, monopole-instanton are studied in the context of $U(1)$ Abelian gauge theory and the non-Abelian $SU(2)$ Yang-Mills-Higgs field theory which is also known as the $SU(2)$ Georgi-Glashow model. The aim is to gain information on the existence and properties of these topological solitons, their structure and behaviour by studying the classical field equations.

The theory of Cho's Abelian decomposition is introduced to study and find the exact solution of the monopoles. We are able to obtain the monopole solutions by switching on the restricted part of the decomposition, when the valence part is switched off. The monopole solutions possess free parameters that can represent different types of monopole solutions, such as, Wu-Yang monopole, t' Hooft-Polyakov monopole and half monopole solutions.

We also investigate the properties of the non-Abelian vortex solutions with the Chern-Simon term. We generate the numerical solutions from a set of nonlinear differential equations which is also known as the equations of motion of the vortex.

Finally, we study the monopole-instanton solution in topologically massive gauge theory in $(2+1)$ dimensions with a Chern-Simons mass term that have been studied by Pisarski many years ago. We obtained numerical regular solutions

that smoothly interpolates between the behavior at small and large distances for various values of the Chern-Simons term strength and also Higgs field strength.

Chapter 1

Introduction

1.1 Why quantum field theories?

Quantum field theory originates from the unification of quantum mechanics and special relativity. It had emerged as the triumphal physical framework to describe the subatomic world. Quantum field theory is successful as a theory of subatomic forces because it can exactly express the standard model in particle physics. Even up until now, there are no experimental deviations that can be found in the standard model.

P.A.M Dirac gives his great contribution to quantum field theory when he wrote his first pioneering paper (Dirac, 1927) that combines quantum mechanics with classical theory of radiation in the year 1927. In his paper, he unified relativistic quantum mechanics with the theory of special relativity and electrodynamics, which lay the foundation for modern high-energy physics in the future. Enthusiastic physical intuition and bold mathematical insight lead Dirac to postulate the celebrated Dirac electron theory (Dirac, 1928a,b) in 1928. Breakthrough developments come rapidly in later when Dirac coupled his relativistic theory of electron with the theory of radiation, and hence created an elegant and powerful theory, which is quantum electrodynamics (Dirac, 1930).

However, there is a defect in the quantum electrodynamics theory. The imperfection of the quantum electrodynamics is that this theory only represents the lowest corrections in classical physics. Study of higher corrections to quantum electrodynamics will led to the divergent of integrals (Oppenheimer, 1930). Hence renormalization theory is introduced to solve this problem. These divergent integrals were absorbed into the infinite rescaling coupling constants and masses of the theory. Tomonaga (1946; 1948), Schwinger (1949a; 1949b) and Feynman (1949b; 1949a) apply this renormalization theory to extract the useful physical information from quantum electrodynamics, for which they received the Nobel prize in 1965.

Even though the theory of quantum electrodynamics relished great success in the 1950s, but the generalized quantum field theory is incapable of describing all the four fundamentals force of nature. These forces are the electromagnetic force, weak force, strong force and gravitational force, which have the coupling constants,

$$\begin{aligned} \alpha_{\text{em}} &\backsim \frac{1}{137.0359895\,(61)}; \\ G_{\text{weak}} &\backsim \frac{1.02 \times 10^{-5}}{M_p^2} \backsim 1.16639\,(2) \times 10^{-5}\text{GeV}^{-2}; \\ \alpha_{\text{strong}} &\backsim 14; \\ G_{\text{Newton}} &\backsim 5.9 \times 10^{-39}/M_p^2 \backsim 6.67259\,(85) \times 10^{-1}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \end{aligned} \tag{1.1}$$

The electromagnetic force was successfully described by quantum electrodynamics. The value of the coupling constant α_{em}, which is also denoted as "fine-structure constant" for quantum electrodynamics is approximately $1/137$. This means that physicists might successfully power expand the theory in the power of the fine-structure constant α_{em}, and this is the so called "perturbation theory". The value of coupling constant for quantum electrodynamics is very small and hence this gave physicists confidence that the perturbation theory was a reliable approximation to the theory.

Historically, weak interaction was first observed in beta decay experiments. When an electron is emitted by a radioactive nucleus, which we called β decay, the weak interaction converts the strong interacting particle, which is a nucleon

(n) into a photon (p), and the lighter particles such as electron (e) and electron antineutrino ($\bar{\nu}_e$),

$$n \longrightarrow p + e + \bar{\nu}_e. \tag{1.2}$$

Light particles like electron e, its neutrino ν, muon μ, tau τ were also known as leptons. In 1930s, Fermi construct a theory of four-fermion contact interaction (Fermi, 1930) to describe the lowest order behavior of the weak interaction. However, any attempt to apply Fermi's theory to calculate the quantum corrections of the weak interactions failed because divergent terms emerged from higher-order corrections in these interactions. The quantum field theory failed to describe the weak interactions: theory of the weak interactions was nonrenormalizable because the coupling constant of Fermi theory G_F has an order of negative dimensions, $G_F \propto {M_W}^{-2}$, where M_W is the mass of the W boson.

Unlike quantum electrodynamics, quantum chromodynamics is a theory that describes the strong interaction of colour quarks and gluons making up the hadrons (from the Greek words *hadros*, meaning "strong"). The strong forces have a large coupling constant, and perturbation theory cannot be used to predict the spectrum of the strongly interacting particles. Unfortunately, nonperturbative approaches were notoriously rough and unreliable, therefore, developments of the strong interactions were painfully slow. The most successful and well-known approach is the quark theory. Gell-Mann (1962; 1964), Nee'man(1961) and Zweig (1964) constructed the $SU(3)$ quark theory of the hadron by referring to Sakata (1956) and his collaborators' (Ikeda et al., 1959) earlier work, which tried to explain the hadron spectrum with the $SU(3)$ gauge group. However, fractional charged quarks have not yet been observed in any scattering experiments. Furthermore, the whole problem of understanding quark confinement still remains open; the dynamical force that binds the quarks together still remains unknown.

Gravitational interaction was one of the earliest of four fundamental forces to be investigated classically. Ironically, this force was the hardest one to be quantized. Normally, we can apply the general physical arguments to calculate the mass and spin of the gravitational interaction. Gravity is a long-range attractive force. Therefore this interaction should be massless and its spin must be even. Spin-0 theory is not compatible with gravity because this theory is used to describe the Higgs mechanism and predict the massive Higgs bosons has a spin quantum number equal to zero. Spin-1 theory is also not selected as a theory to explain gravity because this theory only explain the force that can be both attractive and repulsive such as electromagnetism, which is contradicting to gravity. Hence, we were left with the spin-two theory, and this theory could be coupled to all scalar fields, which was consistent with the equivalence of gravitational and inertial mass. These heuristic arguments indicated that the classical approximation to a quantum theory of gravity should be the Einstein's theory of general relativity. However, Newton's gravitational constant in quantum gravity, as seen in equation (1.1) had a dimensionful coupling constant ($6.67259\,(85) \times 10^{-1} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$) and hence the theory of gravitation cannot be renormalized.

In summary, quantum field theory plays an important role to describe all fundamental forces in the nature. Many problems of quantum field theory, such as the search for the existence of the monopole, solving the renormalization of quantum chromodynamics and gravity still remain as an open question waiting to be solved.

1.2 Solitons as particles

A new approach of quantum field theory was developed, and it became popular in 1960s and early 1970s (Manton and Sutcliffe (2004), Chapter 1, pp. 1). During 1960s and '70s, quarks and leptons are found to be the fundamental particles that experience the second nuclear force -weak force. Quantum field theory for the strong force, which is known as quantum chromodynamics was developed during the 1970s. At the same time, theorists began to invoke the concept of local gauge invariance, which postulates symmetries of the basic field equations at each point in space-time . At that time, mathematicians and physicists have developed several inspiring and complex techniques to study the classical field equations in their fully nonlinear form. In addition, they interpreted some of these classical field equations as candidates for the theory of particle. One of these is the inverse scattering method. General view of some of these techniques is accessible in the book written by Whitham (1973) and the review article by Scott et al. (1973).

Unlike elementary particles that arise from quantization of the wave-like excitations of the fields, these solitons are totally unknown and their properties are largely determined by the classical equations. New characteristics features of these soliton solutions is their topological structure that differs from the vacuum (Manton and Sutcliffe (2004), Chapter 1, pp. 1). The topological structure remains invariant under quantum excitation if we consider that excitations around the vacuum are associated with the smooth deformation of the fields. Hence, usual elementary particles such as photon in quantum field theory have no topological structure. These solitons are stable due to their topological uniqueness. They do not easily decay into a number of elementary particles.

Topological character of the field is represented by a single integer N which is also known as the topological charge. Topological charge is also known as the topological degree or generalized winding number of the field. The energy E of the soliton solutions is proportional to the topological charge N, with the energy increasing as N increases. The soliton with $N = 1$ is a stable classical solution which has the minimal field energy configurations and it cannot simply decay to the topologically trivial field. The energy density of this solution is smooth and concentrated in some finite region of space. We call this kind of field configuration topological soliton or just soliton. Soliton with a positive sign N can be reversed by a reflection symmetry and becomes an antisoliton with a sign $N = -1$. Multisoliton solution is interpreted as a solution that has a field configuration with topological charge $N > 1$. Soliton-antisoliton pairs can be pair-produced or annihilated.

Coupling constants that appear in the Lagrangian and field equations play an important role because they can influence the length scale and energy of a soliton. The energy of a soliton is designated as its rest mass in a Lorentz invariant theory where speed of light is unity $c = 1$. In contrast, the mass of elementary particles are proportional to the Planck constant $\hbar$ and quantum effects become small because the value of the Planck constant $\hbar \approx 6.626068 \times 10^{-34}$ m^2 kgs^{-1} is too small. Elementary particle's mass goes to zero whereas the mass of a topological soliton is finite.

There exists a crucial relationship between the theory of a solitons and wave like fields that satisfy the linearized field equations. The quantization of the wave-like fields provides the elementary particles state, with the nonlinear terms being responsible for interactions between these particles. First, the field approaches the vacuum in the region of space far from the soliton and the linearized field equation is used to determine the rate of approach. Thus, elementary particles are massless if the mass term does not appear in the linearized equation, then the soliton's tail will be long range and falling off with an inverse power of distance. On the other hand, soliton field approaches the vacuum exponentially fast when linearized equation possesses the positive mass coefficient m, inferring that the mass of the elementary particle is $m\hbar$ to first approximation. The difference being $\exp(-mr)$ is corrected by the power of r at a distance r from the soliton core and this is known as the Yukawa tail (Yukawa, 1935).

Secondly, interaction energy of two well separated solitons depends on their separation distance. The reason one can explain in this way is that the interaction energy of the separated solitons is determined by the asymptotic and linearized field of the solitons. The force between the solitons is defined as the derivative of the energy with respect to the separation distance between two solitons.

Thirdly, scattering of waves off the solitons can be described by the linearized field theory. The equations are linearized around the soliton solution instead of the vacuum solutions. Partial wave analysis that involved two or three dimension, which shows that the incoming plane waves emerge after the collisions with the soliton are radially scattered waves. This quantum interpretation (Rajaraman, 1982) is given by,

$$\text{Soliton} + \text{elementary particle} \longrightarrow \text{Soliton} + \text{elementary particle}. \tag{1.3}$$

The properties of the solitons are superficially similar to the elementary particles. Solitons do not change its form under propagation, and it may be regarded as a local confinement of wave fields. Similar to elementary particles, solitons also can be pair-produced and annihilated. Hence, the interpretation (1.3) shows that when a soliton and an elementary particle are collided with each other and this also means that soliton and the elementary particles are conserved as it had before the collision.

Finally, linearized waves play crucial roles in soliton-soliton scattering. Even though conservation of topological charge indicates that solitons do not vanish during the collision, however a part of the kinetic energy from the soliton can be converted into radiation during the scattering process, mainly in a high energy relativistic collision. The radiation disperses into space, and it can be described by the linearized field equations at low amplitude. We can occasionally use the linearized theory to estimate the total amount of radiation, treating the moving solitons as sources, but this only works at moderate collision speeds and this gives another quantum interpretation,

$$\text{Soliton} + \text{Soliton} \longrightarrow \text{Soliton} + \text{Soliton} + \text{elementary particles}. \tag{1.4}$$

The interpretation (1.4) in a quantized field theory showing that soliton-soliton collisions can produce elementary particles.

1.3 A Brief History of Soliton

In mathematics and physics, a self-reinforcing solitary wave that maintains its shape while it travels at a constant speed is known as a soliton. Cancellation of nonlinear and dispersive effects in the medium is the reason that is used to explain the existence of soliton. The soliton is a fabulous scientific discovery that is made by a young Scottish engineer named John Scott Russell (1808-1882) when he was conducting experiments to determine the most efficient design for canal boats.

In 1834, John Scott Russell describes his wave of translation (Scott, 1844) by using his own words, "*I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation*" (Scott, 1844).

Russell believed that his solitary wave (the "Wave of Translation") is fundamentally crucial in his whole life, but nineteenth and early twentieth century scientists thought otherwise. His reputation has rested on other achievements. To mention some of his varied and many activities, he developed the "wave line" system of hull construction, which revolutionized the nineteenth century naval architecture. Russell's early ideas began to be appreciated until the mid 1960's when applied scientists started to use modern digital computers to learn nonlinear wave propagation. He observed the solitary wave as a self-sufficient dynamic entity, a "thing" that can exhibit numerous properties of a particle.

Historically, skrymion was found as the first example of topological soliton model of a particles. Skrymion was originated by Tony Skryme as a mathematical model to study and model the subatomic particles - baryons. The skrymion is proven to emerge from the Yukawa model, a field theory consists of spin 1/2 nucleons as a dynamical soliton in a nonlinear field theory of three types of spinless pions (π^+, π^0, π^-). By considering the pion interaction term, Skryme proposed a symmetry argument, which led to a particular form of Lagrangian for the three-component pion field. This three-component pion field with a topological structure admits a topologically stable solution of the classical field equation, which is distinct from the vacuum. However, Skryme face difficulties to analyse his pion theory in three dimensional spaces. Thus, he developed a toy model (Skryme, 1961) by introduced a Lorentz invariant field theory in one- dimensional space, where the field has a value on a circle, and this is the sine-Gordon theory. Sine-Gordon equation in this theory has an interesting feature which known as the one dimensional topological soliton solution.

Two-dimensional soliton that we discussed here often appears in the condensed matter system. However, the historical development of the two-dimensional soliton from condensed matter theory is limited because condensed matter systems usually are quantal, nonrelativistic, and entail complicated, many electron states in nature. Hence, quantum field theory is introduced to become the natural and quantitative language of condensed matter physics. To solve these problems stated above, several phenomenological methods were developed based on the classical field theory. Ginzburg-Landau approach is one among these phenomenological approaches that is used to describe the properties of superconductors. In Ginzburg-Landau theory, the complex scalar field Φ represents the density and phase of a superconducting paired electrons. In 1957, Abrikosov discovered the existence of a topological soliton in the Ginzburg-Landau energy function. He found that the vacuum manifold is a circle and gives rise to a topological structure. These solitons are called magnetic flux vortices, or simply vortices are the solitons that exist in two dimensional spaces. The phase of Φ changes by 2π along a circle surrounding the basic vortex. Vortices can carry a magnetic flux through the superconductor when there are extended into flux tubes in three-dimensional spaces.

Monopole was first discovered by Dirac in 1931. He found a singular solution of the typical electromagnetic equation with a net magnetic charge. This solution has a string singularity and its energy is infinite. Nevertheless, the fields at a fixed distance r from the origin are topologically interesting and their topology is related to magnetic charge, which follow the quantization condition, $g = n\hbar c/2e$, where n is an integer and e is the electric charge. Other gauge theories with symmetry breaking were considered only when Glashow-Weinberg-Salam (Georgi and Glashow, 1972; Salam and Ward, 1964) theory was experimentally established. The Georgi-Glashow model with a symmetry breaking of $SO(3)$ to $U(1)$ is interesting because it gives a Higgs vacuum manifold structure, which is in the form of a 2-sphere. This is an important structure that admit soliton solutions in three dimensions. Similar to the Landau-Ginzburg vortex, this topological structure is associated with a non-trivial magnetic field, which points radially inwards or outwards. Undoubtedly, this is a magnetic monopole and it was discovered by 't Hooft (1974) and Polyakov (1974, 1975b,a) as a three-dimensional topological soliton in the Georgi-Glashow model independently in 1974.

Quantum chromodynamics is an indispensable ingredient in the standard model of particle physics. Quantum chromodynamics is a pure Yang-Mills gauge theory without Higgs fields, and it coupled to the fermionic quarks fields. If the quarks are ignored, quantum chromodynamics is just a pure gauge theory with a $SU(3)$ gauge group and its classical field equations are the Yang-Mills equations.

The three-dimensional Yang-Mills equation does not acquire soliton solutions. However, four- dimensional Yang-Mills equations do have topological solition solutions, which is known as instantons. The name instantons is given because it can simultaneously be localized in three-dimensional space and in Euclidean time. Hence, it corresponds to a spatially localized event happening in an instant.

1.4 Bogomol'nyi equations and soliton dynamics

The Bogomol'nyi equation, which helps in the study of topological soliton in many field theories, is the reduction of the field equations from second order to first order partial differential equations. Quite a few examples were revealed in a seminar paper of Bogomol'nyi in 1976, and many other examples are exposed later. Bogomol'nyi equations are first order equations and it never involves time derivatives, their solutions are in the soliton or multi-soliton configurations.

Bogomol'nyi proposed that an energy of the field is bounded below by a numerical multiple of $|N|$, with equality if the field satisfies the Bogomol' nyi equations. Hence, all the solutions of the Bogomol'nyi equations of a given charge possess the same amount of energy; and the solutions are automatically stable since the fields minimize the energy. Here, $|N|$ is denoted as a modulus of the topological charge. Generally, second variation of the energy in the background of a static solution has a spectrum which consists of a finite number of zeros and negative eigenvalues. The normalizable eigenfunctions are known as zero and negative modes, respectively. However, spectrum of the variation energy also consists of infinitely many positive eigenvalues. Solitons with zero modes may either shrink into a thin spike or expand without limit under a small perturbation and this is the so-called rolling instability. Although Bogomol'nyi solitons with zero modes may still lead to rolling instabilities, but they are considered stable in the sense that they only have positive and zero modes (Manton and Sutcliffe (2004), Chapter 1, pp. 7), (Jackiw and Weinberg, 1990; Choonkyu et al., 1991). Other than that, Bogomol'nyi solitons also satisfy the Euler equation, which usually only imply that a stationary point of the energy is a static solution.

One-dimensional topological solitons like kinks are the solutions of a Bogomol'nyi equation, even though this is a rather trivial case. For gauged vortices, Ginzburg-Landau equations are reduced to a coupled pair of Bogomol'nyi equations that happen at the critical value of the coupling constant. The critical value of the coupling constant is used to determine the Type I and Type II superconducting systems. Monopoles satisfy a Bogomol'nyi equation when it is considered in the Bogolmol'nyi-Prasad-Sommerfeld limit where the strength of the Higgs potential is $\lambda = 0$. Instantons are just like a Bogomol'nyi equation in four-dimensional space, which satisfy the self-dual Yang-Mills equation.

In this thesis, the solitons that we discussed in various field theories are completely static and we do not consider the dynamics and the interactions of the soliton in our study. However, dynamical equations play a crucial role in a relativistic field theory because they are necessarily determined as the relativistic generalization of the Euler-Lagrange equation for a static field. A soliton can be enhanced to travel at an arbitrary speed less than the speed of light. In a non-relativistic field theory such as Ginzburg-Landau theory of superconductors, it is difficult to determine the exact equations for time dependent fields and experimental input is required.

Solitons can be assumed as a point like object carrying charges or may be a more complicated internal structure when they are well separated. The charges of the solitons Q are defined in terms of the asymptotic form of the fields $\Phi(\infty), \Phi(-\infty)$ encircling the soliton, $Q = [\Phi(\infty) - \Phi(-\infty)]/2\pi = N$, where N is an integer. The result of the relative motions and the forces between well-separated solitons are interpreted in terms of the charges. For example, a magnetic charge that is carried by Bogomol'nyi monopoles. Corresponding forces of the static monopoles are exactly cancelled, whereas for the monopoles in relative motion, their corresponding forces do not cancel, and this will lead to the velocity dependent forces and hence accelerations. The forces of the monopole can be calculated by considering derivatives of the energy momentum tensor or directly from the time-dependent field equations.

1.5 Developments of Solitons-experimental status

Soliton can be viewed as a self-sufficient dynamic entity, which displays many properties of a particle. Optical fibres (Drazin and Johnson, 1989) and the narrow water channels are the physical systems, which carry one-dimensional solitons. Near-integrable variant or integrable model can be used to explain the properties of a soliton. Static soliton solutions are mathematically interesting because they are rational functions of a single complex variable, and therefore soliton solutions can be easily written down explicitly. Furthermore, rational mapping play a crucial role in the theory of monopole.

Sigma model lumps occur as two-dimensional solitons and they can be used to implement the study of ferromagnetic and antiferromagnetic systems in the continuum approximation. Another example of two dimensional solitons are the vortices. Landau-Ginzburg vortices are observed as topological solitons in thin superconductors, and it can be extended into flux tubes in three-dimensional superconductors (Parks, 1969; Ketterson and Song, 1999). The study of Chern-Simon vortex dynamic may be useful to understand the phenomenon of the quantum Hall system (Zhang et al., 1989), but this requires clarifications. Recently, a new breakthrough in the study of two-dimensional soliton is that global vortices have been created as a extended string in three-dimensional Bose-Einstein condensates experiment (Matthews et al., 1999), which is composed of trapped dilute alkali gases.

Magnetic monopole is defined as a particle, which carries an isolated north or south magnetic pole. Despite the idea of the magnetic monopole proposed by Dirac (1931) is beautiful, but all the experiments searching for the magnetic monopole show fruitless results. Fortunately, monopole solutions are not found in the standard model of elementary particles. However, certain Grand Unifies

Theories do have the monopole solutions, and some of these models (Zeldovich and Khlopov, 1978; Preskill, 1973) are severely constrained without the existence of the monopoles. The essential challenges before searching for the monopole is to find the Higgs particle. This is because the mathematical structure leading to monopoles is in doubt without the Higgs particle.

In one-dimensional quantum mechanics, instanton or pseudoparticle (an alternative name suggested by Alexander Polyakov) can be used to calculate the transition probability for a quantum-mechanical particle tunneling through a potential barrier. Worldsheet instantons in two dimensional Abelian gauge theories are known as the magnetic vortices with extra flavors. 't Hooft-Polyakov monopole's play the role of instantons in three dimensional gauge theories with Higgs fields. Alexander Polyakov demonstrated that instanton effects in three-dimensional quantum electrodynamics coupled to a scalar field lead to a mass for the photon in his paper (Polyakov, 1977). The instanton effects plays a crucial role in understanding the formation of condensates in the vacuum of quantum chromodynamics and the study of quantum chromodynamics lattice (Scott et al., 1973).

1.6 Outline

In this thesis, we will discuss some examples of topological solitons that have been mentioned so far. In chapter 2, we shall present some of the essential and fundamental background ideas, concerning Lagrangian and field equations, the roles of group theory and symmetry, and structure of group theory and describe the phenomenon of spontaneous symmetry breaking. In chapter 3, we present the three-dimensional and four-dimensional topological solitons, which are monopoles and instantons. In our work, we also discuss and apply the theory of Cho's Abelian Decomposition to find out the exact solution of the topological soliton especially monopoles. In chapter 4, we show the importance of Chern-Simon terms in the study of vortices as a two-dimensional soliton. In our work, we numerically study non-Abelian vortex solutions by choosing different values of boundary condition, Chern-Simon term strength and Higgs field strength. In Chapter 5, we work on the numerical study of one monopole-instanton solution for the massive $SU(2)$ Yang-Mills Higgs theory. Further works and conclusions are discussed in the final chapter. In this thesis, we only concentrate on the classical solutions, which can be used to describe the topological solitons.

Chapter 2

Gauge Fields

2.1 Gauge theory

Lagrangian of a field theory that remains invariant under a continuous group transformation is known as gauge theory. Gauge theory is important because it can be used to explain and study the dynamics of elementary particles. A gauge transformation is a transformation of the fields which leaves the physical observables invariant. The idea of a gauge transformation comes from the studies of electromagnetism where we can change the vector and scalar potential without changing the field equations. Gauge transformations can form a Lie group, which is referred to as the gauge group or the symmetry group of the theory. Lie algebra of group generators is associated with any Lie group. A corresponding vector field that is necessarily originated from each group generator is known as gauge field. Gauge invariance is defined in the sense of the gauge fields remaining invariant under the local group transformations.

In physics, Lagrangians are very helpful in describing the theories consistently with either general relativity or special relativity when they are invariant under some symmetry transformation groups. Lagrangians are said to possess a global symmetry when they are invariant under a transformation identically performed at all the points in spacetime. In another way, Lagrangians are said to have a local symmetry if it can undergoes local gauge transformation. In quantum electrodynamics, the theory is invariant under a local change of phase or in other words the phase of all wave functions are shifted differently at every point in space-time. This is known as local symmetry.

Historically, the ideas of gauge theories were first reported in the context of classical electromagnetism and later in general relativity. However, the modern field of physics in which the gauge symmetries contribute an earliest important significance is quantum electrodynamics. Nowadays, gauge theories are playing crucial roles in several fields of physics such as high energy, condensed matter and nuclear physics.

2.1.1 Abelian Gauge Theory

The Lie group $G = U(n)$ is denoted as unitary group of degree n, which is the group of $n \times n$ unitary matrices with the group operation that of matrix multiplication. In mathematics, $U(1)$ is denoted as a unitary group that consisting of all complex numbers with absolute value 1 under multiplication, which is commutative. Hence gauge theory with $U(1)$ gauge group is known as an Abelian gauge theory. Physcially, a gauge theory is used to describe the electromagnetic interactions. Let us start with the basic example of one complex scalar field is scalar electrodynamics (Manton and Sutcliffe, 2004), which also is an ungauged (scalar) theory with the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - V\left(\Phi^* \Phi\right). \tag{2.1}$$

Obviously, the Klein-Gordon equation can be derived from the Lagrangian (2.1) when $V\left(\Phi^*\Phi\right) = \left(m^2/2\right)\Phi^*\Phi$. Here $\Phi = \Phi(x)$ in the equation is a complex valued scalar field. The scalar field $\Phi(x)$ has two components Φ_1 and Φ_2, which can be put in the form of,

$$\Phi = \Phi_1 + i\Phi_2, \quad \Phi^* = \Phi_1 - i\Phi_2. \tag{2.2}$$

The potential V only depend on $|\Phi|^2 = \Phi^*\Phi$. The internal symmetry unitary group of degree n equal to 1-$U(1)$, or equivalently the special orthogonal group of degree n equal to 2- $SO\,(2)$,as the group of global phase rotations,

$$\Phi \longrightarrow \Phi \exp\left(i\alpha\right) \tag{2.3}$$

leaving the Lagrangian invariant. Noether's theorem states that any differentiable symmetry of an action principle can lead to a conservation law of that particular physical system. Here, a conserved current is then given by Noether's theorem, which is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\left(-i\Phi\right) + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi^*\right)}\left(i\Phi^*\right) \tag{2.4}$$

Substitute Lagrangian (2.1) into the current (2.4) and get

$$J^\mu = \frac{i}{2}\left(\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right). \tag{2.5}$$

To attain a $U(1)$ gauge theory, the Lagrangian is necessarily to be invariant under

$$\Phi\left(x\right) \longrightarrow \Phi\left(x\right)\, \exp\left(i\alpha\left(x\right)\right), \tag{2.6}$$

where $\alpha\left(x\right)$ is an arbitrary function of space-time. The $\partial^\mu\phi^*\partial_\mu\phi$ term are changeable when they are involving derivatives of Φ, whereas the term $V\left(\Phi^*\Phi\right)$ is invariant. Hence, we need to introduce the electromagnetic gauge potential $A_\mu\left(x\right)$, which is the new, independent fields with space-time components (A_0, A_1, A_2, A_3) to remedy this problem.

Gauge covariant derivative of the scalar field Φ is defined as,

$$D_\mu\Phi = \partial_\mu\Phi - iA_\mu\Phi, \tag{2.7}$$

and the gauge potential A_μ transforms under the $U(1)$ gauge transformation (2.6) as

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha. \tag{2.8}$$

The covariant derivative $D_\mu \Phi$ changes with a factor $\exp(i\alpha)$ under a gauge transformations (2.3) and (2.8),

$$\begin{aligned} D_\mu \Phi = \partial_\mu \Phi - iA_\mu \Phi \longrightarrow & \, \partial_\mu (\Phi \exp(i\alpha)) - i(A_\mu + \partial_\mu \alpha) \Phi \exp(i\alpha) \\ &= (\partial_\mu \Phi - iA_\mu \Phi) \exp(i\alpha) \\ &= \exp(i\alpha) D_\mu \Phi. \end{aligned} \tag{2.9}$$

Complex conjugate of the covariant derivatives of Φ, which is the covariant derivatives of Φ^* is $D_\mu \Phi^*$, since the gauge fields A_μ are real fields. Similar to Φ, $D_\mu \Phi^*$ transforms as Φ^* itself under the gauge transformation,

$$D_\mu \Phi^* \mapsto \exp(-i\alpha) D_\mu \Phi^*. \tag{2.10}$$

The term $D^\mu \Phi D_\mu \Phi^*$ could appear in the Lagrangian since it is gauge invariant.

One needs to take account of the expressions involving derivatives of fields in the Lagrangian in order to make the gauge fields A_μ to be dynamical. This problem can be solved by applying the field tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.11}$$

By using the symmetry property of double partial derivatives $\partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha$, the field tensor is gauge invariant under the gauge transformation (2.8),

$$\begin{aligned} F_{\mu\nu} \mapsto & \, \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha) \\ &= F_{\mu\nu}. \end{aligned} \tag{2.12}$$

The components of the field tensor are the electric components and the magnetic components,

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0, \tag{2.13}$$

$$B_k = -\frac{1}{2}\epsilon_{ijk} F_{ij} = -\frac{1}{2}\epsilon_{ijk}\left(\partial_i A_j - \partial_j A_i\right), \tag{2.14}$$

respectively. The electric components of $F_{\mu\nu}$ consist of a 1-form in space, whereas the magnetic component of $F_{\mu\nu}$ comprises a 2-form in space.

By using all these important elements, we can now construct a Lorentz invariant Lagrangian density for scalar electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^{\mu}\Phi D_{\mu}\Phi^{*} - V(|\Phi|^2). \tag{2.15}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$, μ, ν are the space-time indices that runs from 0 to 3 and the metric $g_{\mu\nu}$ that we used is $(-,+,+,+)$. When space and time parts are separated explicitly, we obtain

$$\mathcal{L} = -\frac{1}{2}E_i E_i + \frac{1}{2}D_0\Phi D_0\Phi^{*} - \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}D_i\Phi D_i\Phi^{*} - V\left(|\Phi|^2\right), \tag{2.16}$$

where both A_0 and ∂_0 terms are contained as "time" parts. The Lagrangian is defined by

$$\mathcal{L} = T - V, \tag{2.17}$$

where the kinetic and potential energies with n dimensions are given by

$$T = \int \left(\frac{1}{2}E_i E_i + \frac{1}{2}D_0\Phi D_0\Phi^{*}\right) d^n x, \tag{2.18}$$

$$V = \int \left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}D_i\Phi D_i\Phi^{*} + V\left(|\Phi|^2\right)\right) d^n x. \tag{2.19}$$

Notice that the choice of signs in the Lagrangian density (2.15) is to ensure that the kinetic energy T is positive definite.

Here, we obtain the field equations of scalar electrodynamics with Lagrangian density (2.15), which are Lorentz invariant. Kinetic energy will only be considered when the Maxwell term $1/2E_iE_i$ is drop in favour with Chern-Simons term, and the covariant derivatives of Φ exists in the Lagrangian as $i(\phi^* D_0\phi - \phi D_0\phi^*)$, which is real and gauge invariant.

The Euler-Lagrange equations for the scalar field Φ and the gauge potential A_μ are given by

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi \right)} \right) = 0, \tag{2.20}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} \right) = 0. \tag{2.21}$$

Substituting the Lagrangian density given by (2.15) into the Euler-Lagrange equations (2.20) and (2.21), we get the equations of motion in the form of nonlinear ordinary differential equations,

$$\partial^\mu F_{\mu\nu} = \frac{i}{2} \left(\Phi^* \partial_\nu \Phi - \Phi \partial_\nu \Phi^* \right) + A_\nu \left| \Phi \right|^2 \tag{2.22}$$

$$D^\mu D_\mu \Phi = -2V' \left(\left| \Phi \right|^2 \right) \Phi, \tag{2.23}$$

where V' is the derivative of V respect to $\left|\Phi\right|^2$. The right hand side of the equation (2.22) is the Noether current and its conservation, $\partial^\nu J_\nu = 0$, associated with the $U(1)$ global symmetry $\Phi \longrightarrow \Phi \exp\left(i\alpha \right)$. The equation (2.22) with $\nu = 0$ is known as Gauss' law, and reads

$$\partial^i E_i = \frac{i}{2} \left(\Phi^* \partial_0 \Phi - \Phi \partial_0 \Phi^* \right) + A_0 \left| \Phi \right|^2. \tag{2.24}$$

The conclusion of this section is that kinetic energy T and potential energy V in a $U(1)$ gauge theory are well defined and gauge invariant, provided one applies Gauss's law to determine A_0. These considerations will be crucial in the discussion of the dynamics of the soliton in $U(1)$ gauge theories.

2.1.2 Non-Abelian Gauge Theory

In the previous section, we have already studied the Abelian gauge theory with the $U(1)$ symmetry group, which is also a commutative group. The group $U(1)$ is the gauge group of the theory of the electromagnetism. However, there are some Lagrangians that possess a higher symmetry than $SO(2)$ or $U(1)$. One of the easiest examples is the generalization to the $SU(2)$ group, as well as the more complicated ones which are non-Abelian groups.

$SU(2)$ group is a non-commutative group, which can be used to determine the general properties for one of the four fundamental forces of natures - the weak interaction, which also known as the weak nuclear force. The weak interaction is a non-contact force because it is a consequence of the exchange (i.e. emission or absorption) of $W_{\pm}$ and Z bosons. Due to this reason, the weak interaction is theorised in the Standard Model of particle physics. Here, a non-contact force is defined as a force applied to an object by another body, which is not in direct contact with it.

All known fermions of the Standard Model, as well as the hypothetical particles - the Higgs boson are affected by the weak interactions. Neutrinos only interact through the weak interactions. There are two types of weak interaction, charged current interaction and neutral current interaction. Charged current interaction is mediated by the electrically charged particles W^+ or W^- bosons, which can be used to described the phenomenon of beta decay. Neutral current interaction is mediated by the neutral particles, Z boson.

Yang and Mills (1954) first advanced the principle of isotopic gauge invariance and the hypothesis that isospin symmetry is a local symmetry in 1954. Unfortunately, some defects appear on their work because there is no evidence to show that the isospin is related to the gauge theory. However, the idea of Yang and Mills is resurrected once again to apply to the unification of the electromagnetic and weak interactions (Glashow, 1961; Salam and Ward, 1964) and the strong interactions between quarks (Gell-Mann, 1964; Zweig, 1964), which is also known as colour gauge symmetry.

The $SU(2)$ gauge theory of weak interaction has several number of uniqueness. It is only the interaction, which is capable of changing the quarks' flavor. Weak interaction is also the only interaction that undergoes P (Lee and Yang, 1956) (parity) and CP (charge parity) symmetry violations . It is propagated by carrier particles that are known as gauge bosons, which have significant masses and this unusual feature can be explained by the Higgs mechanism in the Standard Model (Higgs, 1964; Guralnik et al., 1964).

Although weak interaction has its own exclusive properties in particle physics, however, this interaction also has its weakness. The most fatal weakness of the $SU(2)$ weak interaction is that the three gauge bosons W^+, W^- and Z that should be massless are found to be massive bosons, and photon is the massless boson that can be found in this interaction. Another problem of the weak interaction is the gauge theory of this interaction alone cannot be made renormalizable. To solve this problem, Weinberg and Salam proposed that the electromagnetic and weak interactions must be unified to form an electroweak interaction with the $SU(2) \times U(1)$ electroweak gauge group. The spontaneous symmetry breaking of the $SU(2) \times U(1)$ electroweak gauge symmetry introduces a mass to the bosons. Weinberg and Salam subsequently also prove the renormalizability in their work to rectify the unpleasant omissions in the $SU(2) \times U(1)$ Glashow's model.

The gauge group $SU(2)$ is still widely used by physicists to construct the Lagrangian in a system even though there is a defect in the electroweak interaction of $SU(2)$ gauge theory. This is because $SU(2)$ gauge group is the most fundamental and easiest gauge group in comparison with others and hence it is suitable for use in study of toy model of solitons in different dimensions. Now consider the gauge potential, the Higgs field and the field tensor in the $SU(2)$ group are

$$A_\mu = -ig\frac{\sigma^a}{2}A^a_\mu = g\frac{\sigma^a}{2i}A^a_\mu, \quad \Phi = -ig\frac{\sigma^a}{2}\Phi^a = g\frac{\sigma^a}{2i}\Phi^a, \tag{2.25}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{2.26}$$

By using the expression (2.25) and we obtain (Rubakov, 1999)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu \left(g\frac{\sigma^a}{2i} A^a_\nu \right) - \partial_\nu \left(g\frac{\sigma^a}{2i} A^a_\mu \right) + \left[g\frac{\sigma^b}{2i} A^b_\mu, g\frac{\sigma^c}{2i} A^c_\nu \right] \\ &= g\frac{\sigma^a}{2i} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) + \frac{g^2}{i^2} A^b_\mu A^c_\nu \left[\frac{\sigma^b}{2}, \frac{\sigma^c}{2} \right] \\ &= g\frac{\sigma^a}{2i} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) - g^2 A^b_\mu A^c_\nu i\epsilon_{abc} \frac{\sigma^a}{2} \\ &= g\frac{\sigma^a}{2i} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu \right). \end{aligned} \tag{2.27}$$

By observing the expression (2.27) and we can write the field strength $F_{\mu\nu}$ in the form of

$$\begin{aligned} F_{\mu\nu} &= g\frac{\sigma^a}{2i} F^a_{\mu\nu}, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu, \end{aligned} \tag{2.28}$$

where a, b, c are the Roman indices, which possess the values 1, 2 and 3. The constant g is the gauge coupling constant. Gauge fields A^a_μ are real fields, which are expressed as real components in the field tensor $F^a_{\mu\nu}$. The Hermitian generators of the $SU(2)$ gauge group are $-i\sigma^a/2$, and σ^a is the Pauli matrices where $a = 1, 2, 3$. Pauli matrices are Hermitian in the sense that they have real eigenvalues and are traceless, They are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.29}$$

Pauli matrices obey the following commutation and anticommutation relations,

$$\left[-i\frac{\sigma^a}{2}, -i\frac{\sigma^b}{2} \right] = -\frac{1}{2} i\epsilon_{abc}\sigma^c, \quad \left\{ -i\frac{\sigma^a}{2}, -i\frac{\sigma^b}{2} \right\} = \frac{1}{2}\delta_{ab} I \tag{2.30}$$

where I is the identity matrix, δ_{ab} is the Kronecker delta and ϵ_{abc} is the Levi-Civita symbol. Pauli matrices also have some following properties such that

$$\sigma^a \sigma^b = i\epsilon_{abc}\sigma^c, \quad \mathrm{Tr}\left(\sigma^a\right) = 0, \quad \mathrm{Tr}\left(\sigma^a \sigma^b\right) = 2\delta_{ab}. \tag{2.31}$$

Let us recall that under SU(2) finite gauge transformations $\Gamma(x)$, the non-Abelian gauge potential A_μ and Higgs field Φ transform as,

$$A_\mu \to A'_\mu = \Gamma A_\mu \Gamma^{-1} - \Gamma\left(\partial_\mu \Gamma\right)^{-1}, \tag{2.32}$$

$$\Phi_\mu \to \Phi'_\mu = \Gamma \Phi_\mu \Gamma^{-1}. \tag{2.33}$$

The $SU(2)$ non-Abelian gauge transformation has the infinitesimal form, which also depend on space-time 4-vectors, and it can be written as

$$\begin{aligned} \Gamma(x) &= \exp\left(-\varphi_a(x) T^a\right) \\ &= I\cos\left(\frac{\varphi(x)}{2}\right) + i\hat{n}_a(x)\,\sigma_a \sin\left(\frac{\varphi(x)}{2}\right) \end{aligned} \tag{2.34}$$

where $T^a = -i\sigma^a/2$ are the SU(2) gauge group generator. $\hat{n}^a(x)$ is the unit vector in the internal group space that transform covariantly, and $\varphi(x)$ is an arbitrary function of x, such that

$$\varphi_a(x) = \hat{n}_a(x)\,\varphi(x). \tag{2.35}$$

In equation (2.35), $\varphi_a(x)$ are the functions of the $SU(2)$ transformation, which are complex for the complex $SU(2)$ gauge transformation and real for the real $SU(2)$ gauge transformation. The equation (2.32) with the pure gauge $A^\mu = 0$ is given by

$$A'_\mu\,(\mathrm{pure}) = -\Gamma\left(\partial_\mu \Gamma\right)^{-1}. \tag{2.36}$$