SPLINES FOR LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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by

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SPLIN UNTUK MASALAH NILAI SEMPADAN DUA TITIK LINEAR

ABSTRAK

Masalah nilai sempadan dua titik linear berdarjah dua diselesaikan menggunakan kaedah interpolasi Splin-B trigonometri kubik, Splin-Beta kubik dan Splin-B kubik lanjutan. Splin-Beta kubik mempunyai dua parameter bentuk, iaitu β_1 dan β_2 manakala Splin-B kubik lanjutan mempunyai satu parameter bentuk sahaja iaitu λ. Dalam kaedah-kaedah ini, parameter-parameter tersebut divariasikan dan penyelesaian anggaran yang diperoleh dibandingkan dengan penyelesaian tepat untuk mendapatkan nilai-nilai terbaik bagi β_1, β_2 dan λ. Empat masalah diuji menggunakan kaedah-kaedah ini dan penyelesaian anggaran yang didapati dibandingkan dengan penyelesaian anggaran dari kaedah interpolasi Splin-B kubik. Hasil daripada ujikaji tersebut, Splin-B trigonometri memberi anggaran yang lebih baik daripada Splin-B bagi masalah-masalah berbentuk trigonometri manakala Splin-Beta dan Splin-B kubik lanjutan juga menghasilkan anggaran yang lebih baik untuk beberapa nilai β_1, β_2 dan λ.

Secara keseluruhannya, Splin-B kubik lanjutan menghasilkan anggaran yang paling tepat di antara ketiga-tiga splin. Walau bagaimanapun, kaedah yang digunakan untuk mencari nilai λ sebelumnya tidak boleh diaplikasikan kepada permasalahan sebenar kerana penyelesaian tepat tidak diketahui. Kaedah itu hanya dijalankan untuk memastikan bahawa nilai-nilai λ yang menghasilkan penyelesaian anggaran yang lebih baik wujud. Jadi, satu pendekatan untuk mencari nilai-nilai λ yang optimum dibangunkan dan kaedah Newton digunapakai dalam pendekatan tersebut. Pendekatan ini memberi keputusan yang lebih baik berbanding kaedah interpolasi Splin-B kubik.

SPLINES FOR LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

ABSTRACT

Linear two-point boundary value problems of order two are solved using cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline interpolation methods. Cubic Beta-spline has two shape parameters, β_1 and β_2 while extended cubic B-spline has one, λ. In this method, the parameters were varied and the corresponding approximations were compared to the exact solution to obtain the best values of β_1, β_2 and λ. The methods were tested on four problems and the obtained approximated solutions were compared to that of cubic B-spline interpolation method. Trigonometric B-spline produced better approximation for problems with trigonometric form whereas Beta-spline and extended cubic B-spline produced more accurate approximation for some values of β_1, β_2 and λ.

All in all, extended cubic B-spline interpolation produced the most accurate solution out of the three splines. However, the method of finding λ cannot be applied in the real world because there is no exact solution provided. That method was implemented in order to test whether values of λ that produce better approximation do exist. Thus, an approach of finding optimized λ is developed and Newton's method was applied to it. This approach was found to approximate the solution much better than cubic B-spline interpolation method.

CHAPTER 1

INTRODUCTION

1.1 Two-Point Boundary Value Problems

This thesis deals with two-point boundary value problems. These problems occur widely in the fields of physics, chemistry and engineering. Therefore, it is deemed appropriate to start off with the definition of these problems.

1.1.1 General Two-Point Boundary Value Problems

Given an ordinary differential equation,

$$F\left(u(x), u'(x), ..., u^{(k)}(x)\right) = r(x), \quad x \in [a,b], \tag{1.1}$$

with n boundary conditions at both end points,

$$u^{(i)}(a) = \alpha_i, \quad u^{(i)}(b) = \beta_i, \tag{1.2}$$

for some

$$i,k \in \mathbb{N}, \quad i \in [0,k], \quad k \geq 2,$$

the problem of solving for $u(x)$ is called a two-point boundary value problem of order k. The 'two-point' term refers to the two end points of x where the boundary conditions are specified [1, 15].

The problems arise profusely from representations of physical situations by some equations that are related to solving for position functions. Usually, the position of any matter would be under the influence of certain forces, such as gravity and magnetic forces. These forces can be represented as a linear combination of the acceleration or gravity, which involve the derivative expression of the position function. From there, a differential equation can be derived using the law of physics. Furthermore, the position dependent problems are frequently represented by differential equations with boundary conditions specified at more than one point, as opposed to the time dependent problems which normally have only one boundary condition [16].

Two-point boundary value problem can generally be solved without difficulty if the differential equation in 1.1 can be solved analytically. Unfortunately, this is not the case for most of the problems arising from the computational world. Therefore, numerical treatments are needed to obtain the approximation of the solution. Some of the standard numerical methods to solve these problems are shooting, finite difference, Rayleigh-Ritz, collocation and variational methods [7, 16].

1.1.2 Second Order Linear Problems

Linear two-point boundary value problems of order two are the simplest form of two-point boundary value problems. The general formula of these problems can be simplified from (1.1) and (1.2) into (1.3) [16].

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \quad x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta. \tag{1.3}$$

The existence and uniqueness of the solution for (1.3) have been discussed extensively in many numerical analysis books such as [14] and [16]. Theorem 1.1 summarizes the prerequisites to have a unique solution.

Theorem 1.1 [16, 24]

For $x \in [a,b]$, if

(i) $p(x)$, $q(x)$ and $r(x)$ are continuous, and

(ii) $q(x) < 0$,

then (1.3) has a unique solution. ■

However, linear two-point boundary value problems of order two can also be formulated in another form and the issue of existence and uniqueness is addressed in the next theorem.

Theorem 1.2 [1, 17]

Given a two-point boundary value problem of the form

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = r(x), \quad x \in [a,b], \quad u(a) = u(b) = 0. \tag{1.4}$$

If

(i) $p(x)$ and $r(x)$ are C^1 continuous, and

(ii) $p(x) > 0$,

then (1.4) has a unique solution. ■

As an example for linear two-point boundary value problem of order two, suppose we have a beam of rectangular cross section that is supported in such a way that any uniform loading placed on it would not affect the end's positions [16]. We want to know $w(x)$, the deflection function of the beam at position x. The mathematical equation governing the system is

$$w'(x) - \frac{S}{EI(x)}w(x) = \frac{qx}{2EI(x)}(x-l), \quad w(0) = 0, \quad x(l) = 0,$$

where

$l =$ the length of the beam,

$q =$ the intensity of the uniform load,

$E =$ the modulus of elasticity

$S =$ the stress at the endpoints,

$I(x) =$ the moment of inertia at position x.

1.2 Motivations of Study

The notion of splines is very common in the field of Computer Aided Geometric Design (CAGD). Splines essentially mean piecewise equations. Since 1940s, the study of splines has advanced steadily where numerous types of splines emerged one after another. These splines were employed as curves and surfaces generator in the designing world due to some nice properties such as continuity and convex hull properties. In 2006, Caglar et al. proposed the use of cubic B-spline to solve two-point boundary value problems of order two [17]. B-spline is one of the widely used splines in the literature. The results of this method were comparable to other numerical methods available hitherto.

Since many other splines have been actively developed over the years after B-spline, the idea of using some of these more sophisticated splines in place of cubic B-spline became the main motivation of this study. Through some tests on many types of splines from the literature, three of them were found suitable as the replacement splines, i.e. the splines to replace B-spline. These splines as well as the justification of their selection are presented in Chapter 3.

Besides, other motivation includes the vast applications of two-point boundary value problems in the field of science that stimulate the need for better approximations of the solutions.

1.3 Problem Statement and Scopes of Study

This thesis considers the simplest form of the problem which is linear two-point boundary value problem of order two as in (1.5) and (1.6).

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \quad x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta. \tag{1.5}$$

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = r(x), \quad x \in [a,b], \quad u(a) = u(b) = 0. \tag{1.6}$$

The reason for that is because of their simple and direct nature. Since the methods proposed in this thesis are prototypes, it is wise to first test them on the simplest problems. Therefore, should the methods produce relatively accurate results, they can be applied on the more complicated problems such as problems of higher order or problems that are nonlinear.

Moreover, the problems are also limited to the ones conforming to Theorems 1.1 and 1.2. Thus, for (1.5),

$$p, q, r \in C^0[a,b] \text{ and } q(x) < 0 \text{ on } [a,b],$$

whereas for (1.6),

$$p, r \in C^1[a,b] \text{ and } p(x) > 0 \text{ on } [a,b].$$

This is to ensure that the problems are well-posed problems.

According to the renowned French mathematician, Jacques Hadamard, well-posed problems possess three criteria, namely the existence of solution, the uniqueness of solution and the continuous dependence of the solution to the data. The existence and uniqueness criteria are already fulfilled by Theorems 1.1 and 1.2. Both criteria confirm that we are solving problems that have solutions and are looking for the right ones. The third criterion is also equivalent to having solutions that are not sensitive to small changes in the coefficients of the differential equations as well as the boundary conditions. This is fulfilled by having Dirichlet boundary condition, which is

$$u(a) = \alpha \text{ and } u(b) = \beta.$$

This criterion is important because we are doing approximation of solution, which requires stability [7].

On the other hand, three types of splines are used in this study to approximate the solution for linear two-point boundary value problems by splines interpolation. They are cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline. The details on the splines as well as the reasons for choosing the splines are discussed in Chapter 3. Up to our knowledge, no work has been published pertaining to using these three splines in solving two-point boundary value problems. Therefore, this study is a fresh start in this direction. For that reason, the experimental results are only compared with the results from cubic B-spline interpolation method, but not with other methods that do not use splines interpolation.

Above all, MATLAB 7.6.0 and Mathematica 7.0 were used interchangeably to run the experiments and produce the figures.

1.4 Aims and Objectives of Study

The aims of this thesis are to solve linear two-point boundary value problems of order two using three kinds of splines mentioned above and analyze the results. The results obtained from cubic B-spline interpolation method were used as the benchmark. Moreover, the objectives of this thesis are:

1. to solve linear two-point boundary value problems of order two using cubic trigonometric B-spline interpolation method,
2. to analyze the results from item 1,
3. to solve linear two-point boundary value problems of order two using cubic Beta-spline interpolation method,
4. to analyze the results from item 3,
5. to solve linear two-point boundary value problems of order two using extended cubic B-spline interpolation method,
6. to analyze the results from item 5,
7. to compare the results of item 1, 3 and 5 and pick the best approach, and
8. to refine the best approach and analyze the results.

Upon achieving the aims and objectives, a better approximation of the solutions for linear two-point boundary value problems can be developed.

1.5 Methodology

In order to achieve objectives 1 to 6, that is, solving linear two-point boundary value problems using cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline, the approach proposed by Caglar et al. in [17] is followed closely. Briefly, each spline and its derivatives are simplified at the collocation points. Then, the simplifications are substituted in the problems, resulting a system of linear equations which requires solving for the unknown constants. These constants are then substituted back in the respective splines, which are the approximated analytical solutions for the problems. Two of the splines contain some free variables. Thus, the variables are varied to find values that produced the most accurate results.

After completing objective 7, objective 8 is achieved using minimization theory from the calculus and Newton's method. Since the best results are obtained from the spline having one free variable, minimization is needed to look for its best value. Having completed all the objectives, the aim of this study is also achieved.

1.6 Structure of Thesis

This thesis contains six chapters altogether. Chapter 1 provides an overview and key factors of the study. Chapter 2 covers a survey of recent numerical methods and a brief history on the application of splines interpolation for solving linear two-point boundary value problems. This chapter also identifies an important work by Caglar et al. which is the basis of this research. Chapter 3 discusses on the definition and some relevant properties of cubic B-spline, cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline. This chapter provides simplifications of the spline functions and curves that are useful in Chapter 4.

Chapter 4 continues explaining the B-spline interpolation method and applies the same approach to the other three splines. A numerical experiment consisted of four problems were carried out and the results are discussed. One of the three splines showed the biggest potential to approximate the solution for the problem better than B-spline. Chapter 5 elaborates more on the spline and another experiment was conducted. Chapter 6 concludes this study and mentions briefly on the possible future work.

CHAPTER 2

A SURVEY OF METHODS

2.1 Introduction

This chapter presents some of the recently developed methods of solving linear two-point boundary value problems. From there, one work is identified to be the major reference to this research. This work involves the use of spline interpolation method. Thus, the following section covers a brief history of the application of spline interpolation in solving these problems. This chapter concludes with several issues that would be addressed throughout this thesis.

2.2 Recent Methods

Some of the standard methods of solving two-point boundary value problems are shooting, finite difference, collocation and variation methods. All these methods can be found in most of the numerical analysis textbooks. In this section, the survey of methods is made starting with the year of 2002 onwards.

In 2002, Fang et al. had applied **Finite Difference (FD)**, **Finite Element (FE)** and **Finite Volume (FV)** methods in solving linear two-point boundary problem of the form

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = r(x), \quad x \in [a,b], \quad u(a) = u(b) = 0.$$

FD essentially replaces the differential terms in the problem with the equivalent expressions containing just $u(x)$. This is done by Taylor's expansion of order two. The equation is then evaluated at x_i for $i = 1,2,...,n$ resulting a linear system of order $(n \times n)$, where $u(x_i)$ can be solved [16].

By calculus of variation, **FE** solves the problem implicitly by finding $u(x)$ that minimizes a quadratic functional constructed from the differential equation. This is done by assuming any form of equation, say, a straight line, to be the solution of the problem and substituting it into the functional formula. Then, partial derivatives of the functional are taken to be zero which in the end can solve a linear system for $u(x_i)$ [9]. On the other hand, **FV** replaces the differential terms in the problem with their corresponding central difference formulas. The integral conservation law is imposed on each subintervals, resulting, again, a linear system of order $(n \times n)$, which can be solved for $u(x_i)$ [19].

All the three methods involve solving a tridiagonal linear system. Fang et al. applied the inversion formula of a nonsingular tridiagonal matrix to directly obtain the explicit expression of the solution. According to them, this approach provided a unified understanding of the method as well as the error estimates. All three methods produced almost similar results if f was "sufficiently smooth". Otherwise, finite element method produced slightly better results than the other two [19].

In the same year, Taiwo proposed the **Exponential Fitting (EF)** approach with cubic spline collocation tau method. This work was a continuation of his work using perturbed tau method, where (1.3) became

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x) + \tau_1 T_N(x), \quad x \in [x_0, x_N],$$

where $T_N(x)$ was the shifted Chebyshev polynomial of degree N defined in (2.1). Furthermore, an exponential was fitted in the boundary conditions, resulting (2.2) and (2.3).

$$T_N(x) = \cos(N \arccos(x)), \quad x \in [a,b]. \tag{2.1}$$

$$u(a) + \tau_2 e^a = \alpha, \tag{2.2}$$

$$u(a) + \tau_2 e^b = \beta. \tag{2.3}$$

Using the definition and recurrence relation of cubic spline, this perturbed problem was converted into a system of linear equation of order $(n+4) \times (n+4)$, and solved for

$$\{u_0, u_1, \ldots, u_{N+1}, \tau_1, \tau_2\}.$$

This method was claimed to be better than the standard tau and perturbed tau methods, proposed by him in the early nineties [29].

Up to this point, the methods discussed produced the approximated solution only at some discrete points. In some cases, the approximated solution is preferred to be defined in a continuous fashion, i.e. an explicit equation of x. Some of the following methods provide just that.

In 2005, Attili made use of **Adomian Decomposition Method (ADM)** developed almost twenty years before to solve Sturm-Liouville two-point boundary value problem, which is a linear problem. The differential equation was expressed in the operator form,

$$Lu = Nu + r(x), \tag{2.4}$$

where L was chosen to be the simplest form of derivative that is easily invertible whereas N covered the rest of the expression containing u. In order to solve for u, the inverse operator,

L^{-1} was applied on both sides of (2.4) resulting

$$u = g + L^{-1}(Nu) + L^{-1}(r(x)).$$

g is an added term appearing after the inversion which comes from the boundary conditions. $N(u)$ can be decomposed into an infinite series of Adomian polynomials, A_n, where

$$N(u) = \sum_{i=0}^{\infty} A_n = \frac{1}{n!}\left[\frac{d^n}{d\chi^2} N\left(\sum_{i=0}^{n} u_i \chi^i\right)\right]_{\chi=0}.$$

In short, using this decomposition and an initial guess of u_0,

$$u_{i+1} = L^{-1}(Nu_i), \quad i = 0, 1, ..., k-1.$$

The approximated analytical solution of the problem in k^{th} term is the sum of u_i for all i. It was claimed that the method was "very successful and powerful". The only drawback of this method is that a "wise" guess is needed at the beginning of the iteration [8, 21].

About one year later, Caglar et al. developed **Cubic B-spline Interpolation Method (CBIM)**. In **EF**, spline function is used to represent $u''(x)$, but here, B-spline function was assumed to be the approximated solution itself, $u(x)$. Then, derivatives were applied to the B-spline function according to the differential equation and solved for the unknown constants. Hence, the approximated solution was also defined continuously. This method was compared to **FD**, **FE** and **FV**, which was concluded to be more accurate [17].

Another type of spline, **Weighted-Extended B-spline (WEB)** was applied in **FE** by Apaydin et al. in 2007. Recall that **FE** needed a form of assumed solution. Thus, generated function from **WEB** was taken to be the status quo. The results of this approach improved tremendously compared to the **CBIM**'s results for the cubic **WEB** [6].

In the same year, Lu proposed **Variational Iteration Method (VIM)**, which use the same operator form in (2.4). This method introduced a restricted variation to any singular term in N. Since only linear problems are considered here, there is no restricted variation. Thus, the corresponding recursive term was

$$u_{i+1}(x) = u_i(x) + \int_0^x \lambda \left\{ Lu_i(\xi) + Nu_i(\xi) + g(\xi) \right\} d\xi,$$

where λ is a general Langrangian multiplier. Similar to the **ADM**, an arbitrary initial approximation, $u_0(x)$, is needed. But, different from **ADM**, the k-term approximated solution for the problem was just $u_k(x)$. This method was claimed to be superior than the **ADM** and many other methods because the convergence rate was faster. Moreover, it is possible to derive the exact solution using only one iteration, even with an arbitrary initial guess [25].

In 2008, Jang presented the **Extended Adomian Decomposition Method (EADM)** to solve two-point boundary value problems. This method dealt with the drawback of **ADM**, the initial approximation. For **ADM**,

$$L^{-1} = \int_a^x dx' \int_a^{x'} dx'',$$

a common operator to invert the second derivative term. But this definition lead to having the $u'(a)$ term, which was unknown. Thus, **EADM** used

$$L^{-1} = \int_a^x dx' \int_b^{x'} dx'',$$

and applied the end condition beforehand. The recursive scheme for this approach is then

$$u_0 = \alpha + s(x)(\beta - \alpha) + L^{-1}r - s(x)\left[L^{-1}r\right]_{x=b},$$

$$u_{i+1} = L^{-1}A_i - s(x) - s(x)\left[L^{-1}A_i\right]_{x=b}, \qquad i \geq 0.$$

EADM was claimed to be better than **ADM** because the approximated solution, which was the partial sum, already satisfies the boundary conditions [21].

Lastly, the most recent work is done by Chun and Sakthivel in 2010. They proposed the **Homotopy Perturbation Method (HPM)** in solving general two-point boundary value problem. Firstly, a homotopy was constructed from the differential equation,

$$u''(x) + Hp(x)u'(x) + Hq(x)u(x) - r(x) = 0, \tag{2.5}$$

where $H \in [0,1]$ was the embedding parameter. This homotopy met certain criteria discussed further in their literature. Then,

$$u = u_0 + Hu_1 + H^2u_2 + \ldots + H^ku_k \tag{2.6}$$

was assumed to be the solution of (2.5). Substituting (2.6) into (2.5), and collecting the term according to the power of H, a list of $(k+1)$ initial value problems can be produced, as in (2.7).

$$\begin{aligned}
H^0: &\quad u_0''(x) - r(x) = 0, &\quad u_0(0) = \alpha, &\quad u_0'(0) = \gamma,\\
H^1: &\quad u_1''(x) + p(x)u_0'(x) + q(x)u_0(x) = 0, &\quad u_1(0) = 0, &\quad u_1'(0) = 0,\\
H^2: &\quad u_2''(x) + p(x)u_1'(x) + q(x)u_1(x) = 0, &\quad u_2(0) = 0, &\quad u_2'(0) = 0,\\
&\vdots\\
H^k: &\quad u_k''(x) + p(x)u_{k-1}'(x) + q(x)u_{k-1}(x) = 0, &\quad u_{k-1}(0) = 0, &\quad u_{k-1}'(0) = 0.
\end{aligned} \tag{2.7}$$

u_i was solved for $i = 0, 1, ..., k$ and and the k-term approximation of $u(x)$, $S_k(x)$, is

$$S_k(x) = \sum_{i=0}^{k} u_i(x).$$

γ can be solved by applying the end condition,

$$S_k(b) = \beta.$$

HPM was claimed to approximate the solution better than **EADM** and **EF** and converged faster to the solution [18]. Thus, **HPM** is the best method so far for solving linear two-point boundary value problems.

2.3 Splines Interpolation Methods

The use of cubic spline interpolation in solving linear two-point boundary problems of order two was first proposed by Bickley in 1968. Cubic spline function, $S(x)$, as in (2.8), was assumed to be the approximated solution,

$$\begin{aligned} S(x) = c + d(x - x_0) + e(x - x_0)^2 + f_0(x - x_0)^3 \\ + f_1(x - x_1)^3 + f_2(x - x_2)^3 + ... + f_{n-1}(x - x_{n-1})^3, \end{aligned} \quad (2.8)$$

where c, d, e and f_i for all i are the unknown coefficients. Then, first and second derivatives were applied to the spline function according to the differential equation. These equations were arranged in such a way that resulted in a matrix of the coefficients of Hessenberg form, which is an upper triangle with a single sub-diagonal. This system was solved for the unknown constants using forward elimination and back substitution. Then, the constants were substituted back in (2.8). The results were claimed to be encouraging [12].

Immediately after this work, Albasiny and Hoskins modified the approach in 1969. They made use of the properties of cubic spline, that can be reduced to three-term recurrence relationship at the collocation points. Hence, instead of a Hessenberg matrix, a tridiagonal matrix was constructed to represent the system of equations which was much easier to solve [4]. In the same year, Fyfe also contributed to this development by examining Bickley's method, obtaining error estimates and investigating the method if unequal intervals were used. He claimed that Bickley's method was more accurate than finite different method of the same knots and asserted that very little advantage was gained by using unequal intervals [20]. Following these development, many other analysis and improvements were made throughout the years as in [3, 22] and the references therein. However, the methods proposed and analyzed revolved around the use of monomial cubic spline.

In 2006, Caglar et al. proposed the use of cubic B-spline, a better representation than monomial cubic spline, to solve the problems. The approach from Albassiny and Hoskins [4] was adopted and thus the resulting coefficient matrix was in tridiagonal form, which can be solved very quickly using any software such as MATLAB or Mathematica [17]. Continuing with this work, we applied the same procedure using three types of splines. These splines are cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline. The discussion on the reasons for choosing these splines as well as their definition and properties are presented in Chapter 3.

The most recent literature in solving the problems using spline is the article of the **WEB** method mentioned in the previous section. However, **WEB** method did not apply spline interpolation method directly. Instead, the method used finite element as the main algorithm and inserted weighted extended B-spline as the approximating function. As of now, we have not found any work similar to what we have done and are doing.

2.4 Summary

In short, a survey of recent methods of solving linear two-point boundary value problems is done in the beginning of this chapter. Out of all methods, **HPM** was claimed to be the most accurate method. Then, **CBIM** is mentioned to be the main reference in this study, followed by a short history of the spline interpolation method in solving the problems. The technique dated back in 1968 and 1969 but became inactive for quite a long time before emerging again in recent years. Lastly, three splines are specified to be the main part of this research to replace the cubic B-spline.

CHAPTER 3

SEVERAL TYPES OF CUBIC SPLINES

3.1 Introduction

This chapter covers the definition, some properties and simplifications of four types of cubic splines, namely cubic B-spline, cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline. These details are pertinent to solving linear two-point boundary value problems by spline interpolation method, which is discussed in Chapter 4. As mentioned in the previous chapters, this research is the continuation of work by Caglar et al. [17]. In this work, cubic B-spline was manipulated to approximate the solutions of two-point boundary value problems. Therefore, the definition and properties of B-spline would be the head-start of this chapter so that the following discussions on the three splines can include some comparisons to B-spline. Some insights on the reasons for choosing the three splines are discussed towards the end of this chapter. These discussions are delayed in order to provide deeper understandings on the splines first before presenting the arguments.

Generally, spline is a piecewise equation or polynomial of some degrees that are smoothly joined at certain points called knots. A definition of spline function, a more restricted form of spline is given in Definition 3.1.

Definition 3.1 [2, 7]

Let $\{x_i\}_{i=0}^{n}$ be a uniform partition of a finite interval $[a,b]$ with $n \in \mathbb{Z}^+$ such that

$$a = x_0 < x_1 < \ldots < x_n = b.$$

A function $S : [a,b] \to \mathbb{R}$ is called a spline function of degree 3 and order 4 or a cubic spline function if

(i) S is a polynomial of degree ≤ 3 on $[x_i, x_{i+1}]$, for $i = 0, 1, ..., n-1$, and

(ii) S is C^2 on $[a,b]$.

Furthermore, if S is a cubic spline function, then the points $x_0, x_1, ..., x_n$ are called knots of S. ■

From the definition, a cubic spline function, S, may consist of several polynomials of degree three at most, but connected to each other with at least C^2 continuity. Figure 3.1 shows an example of a spline function, $S(x)$, with knots $x = \{0, 1, 2, 3, 4, 5\}$. From the figure, $S(x)$ is a piecewise polynomial as shown in (3.1).

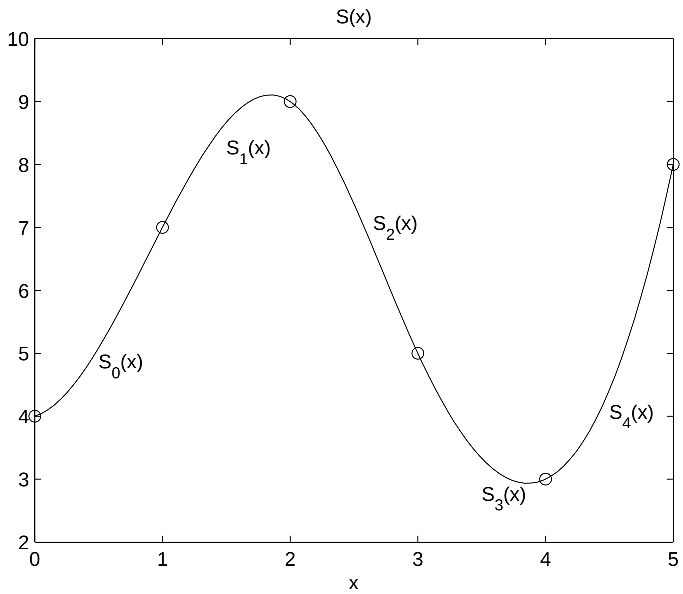

Figure 3.1: An example of a cubic spline function

$$S(x) = \begin{cases} S_0(x), & x \in [0,1], \\ S_1(x), & x \in [1,2], \\ S_2(x), & x \in [2,3], \\ S_3(x), & x \in [3,4], \\ S_4(x), & x \in [4,5]. \end{cases} \tag{3.1}$$

Out of the four splines mentioned in the introduction, only cubic B-spline and extended cubic B-spline are spline functions. On the other hand, cubic trigonometric B-spline and cubic Beta-spline are constructed from piecewise trigonometric functions with C^2 continuity and piecewise polynomial functions with G^2 continuity, respectively. Thus, cubic B-spline and extended cubic B-spline are more restricted compared to cubic trigonometric B-spline and cubic Beta-spline.

All the four splines are constructed from a linear combination of their respective basis. The extension of the knots in Definition 3.1 is needed in the definition of some bases and hence can be calculated using (3.2).

$$x_i = x_0 + ih, \qquad i = \pm 1, \pm 2, \ldots \qquad h = \frac{b-a}{n}. \tag{3.2}$$

3.2 Cubic B-spline

B-spline function was first developed in 1940s and has undergone a lot of development since. The function has been used extensively in Computer Aided Geometric Design (CAGD) field because it has many useful properties for designing such as convex hull and continuity properties. The derivation and properties of B-spline can be found in many Curves and Surfaces books. In this section, [2, 17, 27, 28] were used as the main references.

3.2.1 Cubic B-spline Basis

The basis of B-spline of order 1 can be calculated using (3.3).

$$B_i^1(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

Subsequently, B-spline basis of order can be calculated using the following recursive equations,

$$B_i^k(x) = \frac{x - x_i}{x_{i+k-1} - x_i} B_i^{k-1}(x) + \frac{x_{i+k} - x}{x_{i+k} - x_{i+1}} B_{i+1}^{k-1}(x). \tag{3.4}$$

In order to obtain the basis of cubic B-spline, (3.4) was calculated recursively up to order 4. The resulting basis, $B_i^4(x)$, is shown in (3.5) [27].

$$B_i^4(x) = \frac{1}{6h^3} \begin{cases} (x - x_i)^3, & x \in [x_i, x_{i+1}], \\ (x - x_i)\left[(x - x_i)(x_{i+2} - x) + (x_{i+3} - x)(x - x_{i+1})\right] + & \\ (x_{i+4} - x)(x - x_{i+1})^2, & x \in [x_{i+1}, x_{i+2}], \\ (x - x_i)(x_{i+3} - x)^2 + & \\ (x_{i+4} - x)\left[(x - x_{i+1})(x_{i+3} - x) + (x_{i+4} - x)(x - x_{i+2})\right], & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4} - x)^3, & x \in [x_{i+3}, x_{i+4}]. \end{cases} \tag{3.5}$$

$B_i^4(x)$ is a piecewise polynomial of degree 3 with C^2 continuity. A plot of $B_i^4(x)$ is shown in Figure 3.2.

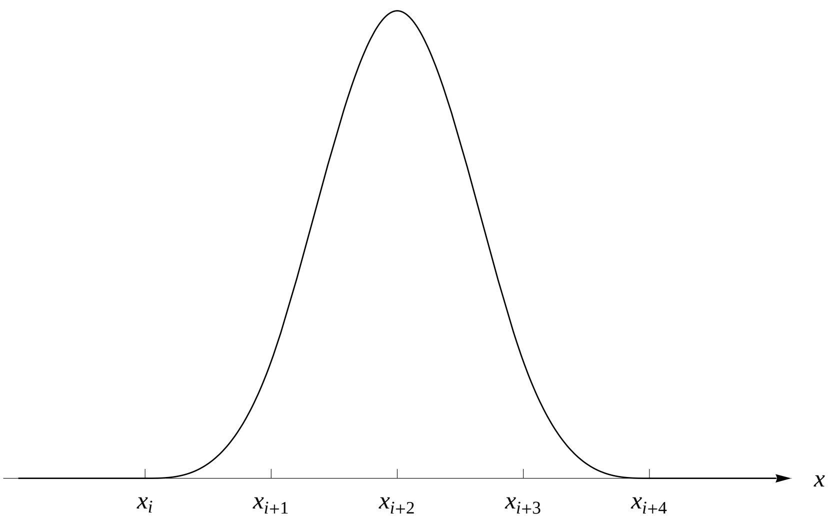

Figure 3.2: Cubic B-spline basis, $B_i^4(x)$

3.2.2 Cubic B-spline Function

Cubic B-spline function, $S_B(x)$, is generated from a linear combination of the cubic B-spline basis, as in (3.6).

$$S_B(x) = \sum_{i=-3}^{n-1} C_i B_i^4(x), \qquad x \in [x_0, x_n], \qquad C_i \in \mathbb{R}, \qquad n \geq 1. \tag{3.6}$$

Therefore, similar to $B_i^4(x)$, $S_B(x)$ is a piecewise polynomial function of degree 3 with C^2 continuity. The calculation and some properties of $S_B(x)$ are illustrated in Example 3.1.

Example 3.1

Suppose we have $x_i = i$ for all i, $n = 1$ and $\{C_{-3}, C_{-2}, C_{-1}, C_0\} = \{0, 5, 2, 10\}$. Using (3.6),

$$\begin{aligned} S_B(x) &= \sum_{i=-3}^{0} C_i B_i^4(x), \qquad x \in [x_0, x_1], \\ &= C_{-3}B_{-3}^4(x) + C_{-2}B_{-2}^4(x) + C_{-1}B_{-1}^4(x) + C_0 B_0^4(x), \qquad x \in [x_0, x_1]. \end{aligned} \tag{3.7}$$

In order to obtain the expression for $S_B(x)$, $B_{-3}^4(x)$, $B_{-2}^4(x)$, $B_{-1}^4(x)$ and $B_0^4(x)$ on $x \in [0, 1]$ are needed. From (3.5),

$$B^4_{-3}(x) = \frac{1}{6}\begin{cases} (x+3)^3, & x \in [-3,-2], \\ (x+3)\left[(x+3)(-1-x)+(-x)(x+2)\right]+(1-x)(x+2)^2, & x \in [-2,-1], \\ (x+3)(-x)^2+(1-x)\left[(x+2)(-x)+(1-x)(x+1)\right], & x \in [-1,0], \\ (1-x)^3, & x \in [0,1]. \end{cases}$$

$$B^4_{-2}(x) = \frac{1}{6}\begin{cases} (x+2)^3, & x \in [-2,-1], \\ (x+2)\left[(x+2)(-x)+(1-x)(x+1)\right]+(2-x)(x+1)^2, & x \in [-1,0], \\ (x+2)(1-x)^2+(2-x)\left[(x+1)(1-x)+(2-x)(x)\right], & x \in [0,1], \\ (2-x)^3, & x \in [1,2]. \end{cases}$$

$$B^4_{-1}(x) = \frac{1}{6}\begin{cases} (x+1)^3, & x \in [-1,0], \\ (x+1)\left[(x+1)(1-x)+(2-x)(x)\right]+(3-x)(x)^2, & x \in [0,1], \\ (x+1)(2-x)^2+(3-x)\left[(x)(2-x)+(3-x)(x-1)\right], & x \in [1,2], \\ (3-x)^3, & x \in [2,3]. \end{cases}$$

$$B^4_{0}(x) = \frac{1}{6}\begin{cases} (x)^3, & x \in [0,1], \\ (x)\left[(x)(2-x)+(3-x)(x-1)\right]+(4-x)(x-1)^2, & x \in [1,2], \\ (x)(3-x)^2+(4-x)\left[(x-1)(3-x)+(4-x)(x-2)\right], & x \in [2,3], \\ (4-x)^3, & x \in [3,4]. \end{cases}$$

Substituting the bases and C_i into (3.7),

$$\begin{aligned} S_B(x) &= 0\left[\frac{(1-x)^3}{6}\right]+5\left[\frac{(x+2)(1-x)^2+(2-x)[(x+1)(1-x)+(2-x)x]}{6}\right] \\ &\quad +2\left[\frac{(x+1)[(x+1)(1-x)+(2-x)x]+(3-x)x^2}{6}\right] \\ &\quad +x\left[\frac{(x+1)[(x+1)(1-x)+(2-x)x]+(3-x)x^2}{6}\right]+10\left[\frac{x^3}{6}\right], \\ &= \frac{11}{3}+x-4x^2+\frac{19x^3}{6}, \qquad x \in [0,1]. \end{aligned}$$

The plot of $S_B(x)$ is shown in Figure 3.3.

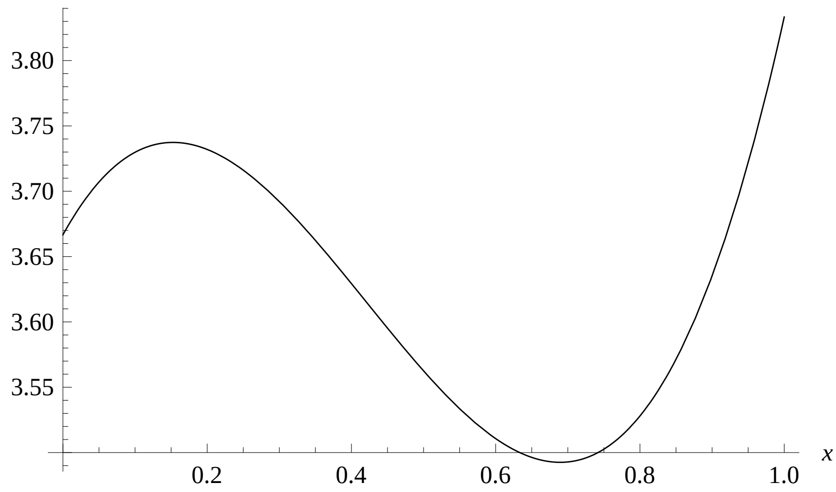

Figure 3.3: Cubic B-spline function, $S_B(x)$ from Example 3.1

One nice property of B-spline function is that it satisfies convex hull property. In other words, the range of B-spline function always lies within the convex hull of the function, which is $[\min_i C_i, \max_i C_i]$. Hence, for this example, the range of $S_B(x)$ is $[0, 10]$. ■

The coefficient, C_i, is called the control point of the function. (3.6) is defined as a functional equation, where x, $S_B(x)$, $C_i \in \mathbb{R}$. On the other hand, (3.6) can also be defined as a parametric equation, where x, $S_B(x)$, $C_i \in \mathbb{R}^2$. However, the functional form is the form of interest in this study since it was used to solve linear two-point boundary value problems of order two in [17].

3.2.3 Simplifications of B-spline Function and Its Derivatives

The simplifications of B-spline function and its derivatives was mentioned briefly by Caglar et al. in his paper [17]. These simplifications made degenerating linear two-point boundary value problems into squared linear systems of equations possible. This section will elaborate on these simplifications in details.