APPROXIMATING GENERALISED CORNU SPIRAL WITH LOW ENERGY CURVE

by

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ABSTRAK

Pembinaan lengkung yang kelihatan cantik and licin dari segi matematik memerlukan usaha yang berterusan. Dalam ulasan karya awal penyelidik-penyelidik telah mempelajari lengkung licin melalui pendekatan fizik yang dikenali sebagai elastica. Elastica merujuk kepada jalur nipis yang elastik. Danial Bernoullin telah mencadangkan elastica sebagai masalah ketaksamaan dalam bentuk tenaga tegangan dan Malcom (1977) menyatakan bahawa penyataan bentuk lengkung secara matematik yang paling mudah ialah ia mengambil bentuk yang meminimumkan tenaga tegangannya.

Dalam ulasan karya baru-baru ini, pembinaan lengkung lincin diberi tunpuan pada bentuk profil kelengkungan. Merujuk kepada Farin (1988), sesuatu lengkung dikatakan licin jika profil kelengkungannya mengandungi sikit cebisan monoton. Banyak pendekatan-pendekatan telah diperkenalkan untuk membina lengkung licin. Ia termasuk mengawal lengkung secara langsung, iaitu menguatkuasakannya supaya memuaskan variasi kelengkungan monoton dan yang kedua - menggunakan spiral yang mempunyai tarifan variasi kelangkungan monoton. Pendekatan pertama akan menyebabkan lengkung tersebut kehilangan keluwesan. Malakala, bagi pendekatan yang kedua, spiral ditakrifkan dalam bentuk bukan polinomial, menyebabkannya tidak sesuai untuk menyepadu dalam system CAD. Oleh sebab demikian, kita perlu menghampiri spiral dengan polinomial. Pelbagai jenis pendekatan mengenai

penghampiran spiral ditermui dalam ulasan karya, antaranya kekurangan perbincangan yang berkaitan dengan tenaga tegangan. Sedangkan tenaga adalah punca kajian bentuk lengkung, ia adalah munasabah untuk menghampiri spiral melalui konsep ini.

Maklamat kajian ini ialah untuk menghampiri Cornu spiral teritlak (GCS) dengan menggunakan konsep tenaga tegangan. Kaedah berangka digunakan untuk menyelaras tenaga tegangan polinomial penghampiran. Dalam kajian ini, kita menggunakan lengkung quintic Hermite sebagai lengkung penghampiran. Lengkung ini ditakrifkan oleh lokasi, pembezaan pertama dan kedua pada titik-titk hujung. Kita selaraskan lengkung ini dengan mengawal magnitud-magnitud pembezaannya. Ralat nipis kelengkungan diguna untuk menunjuk ketepatan kaedah penghampiran yang dicadangkan. Hasil kerja menunjukkan bahawa lengkung yang bertenaga rendah boleh menghampiri lingkaran spiral dengan baik.

APPROXIMATING GENERALISED CORNU SPIRAL WITH LOW ENERGY CURVE

ABSTRACT

The computations of visual pleasing and mathematically fair curve are an ongoing process. In the earlier literature, researchers have been studying the smooth curve, or in a more technically term, the fair curve through the physical approach known as elastica, Elastica means a thin strip of elastic material. Daniel Bernoulli stated the elastica as variational problem in terms of strain energy and Malcom (1977) mentioned that the simplest way to characterize a spline mathematically is with the fact that a spline assumes a shape which minimizes its elastic strain energy.

In the recent literature, curvature profile was the main focus of fair curve computation. Refer to Farin (1989), a curve is fair if it contains relatively few pieces of monotone curvature profile. The most used approaches in computing fair curve included: i. controlling the curve directly to enforce it to satisfy the monotone curvature variation and ii. using curve with define monotone curvature profile such as the spiral. The first approach will cause loss of flexibility to the curve. Whilst for the second approach, the spiral is non-polynomial, making it unsuitable to integrate with the existing computer aided design (CAD) system. Thus, we need to approximate the spiral with polynomial. There are various approximations approaches found in the literature. Among them, there is a lack of discussion about the strain energy of the approximating polynomial. Since strain energy is the basis of

curve shape study, it makes sense for us to determine the approximating curve through this concept.

The objective of this study is to approximate the generalized Cornu spiral (GCS) using the concept of strain energy. Numerical method is use to adjust the strain energy of the approximating polynomial. Here, we adopted the quintic Hermite curve which is the simplest solution involving quintic polynomial. The Quintic Hermite curve is defined by end points position, end points first and second derivatives. We adjusted the curve by varying the magnitude of derivatives. The relative curvature error is use to demonstrate the accuracy of approximation. Result had shown that low energy polynomial curve can form good approximation of spiral segment.

CHAPTER 1

INTRODUCTION

We will begin this chapter with the overview of curve study. This includes the motivation to improve the curve quality, followed by literature outline of elastica, nonlinear spline, MEC, and spiral approximation. Next we discuss the objective of this study in Section 2. The organization of this study in other chapters is shown in Section 3 and a research framework is given in Section 4.

1.1 Overview

The appearance of a product is always the most crucial consideration aspect among the potential buyers. Thus, there is a need to improve the visual properties of the product. Product design started with drawing smooth curve on the plane. The quality of the curve plays a significant role in affecting the product's beauty appearance. In general, a curve is high quality if it is fair. Computation of a mathematically fair and yet eye pleasing curve is not an easy task. Throughout the century, mathematicians have presented various approaches to computer the fair curve. Despite the different kinds of computation methods and curves used, there are many different definitions for fair curve. In this study, we will concentrate on the fairness definition regarding strain energy and curvature profile.

In the 17th century, mathematicians had started the studies of curve shape through the physical approach known as elastica. Elastica refers to the shape assumed by a uniform elastic rod when it is bent under certain stress. Galileo initiated the classic study of elastica. A number of researchers further developed it, and Euler contributed the most of it. From the study, Bernoulli pointed out that the elastic strain energy of bending beam assumes the beam's shape. Based on Bernoulli's idea, Euler then verified the equivalence of physical approach with variational approach. This brings the plausible suggestion for smoothness or fairness as: the elastic strain energy of the curve should attain a minimum value. Class of curve computed using this concept included the minimum energy curve (MEC) and nonlinear spline.

The paper nonlinear spline by Mehlum (1974) is a remarkable paper. Mehlum (1974) used the variational approach in computing the nonlinear spline. He defined the smooth spline as the one in which its integral of curvature square over arc length is as small as possible. Other studies of nonlinear spline found in the early literature included discussions of the existence of nonlinear spline by Jerome (1973); derivation of the algebraic and differential equation for open and closed nonlinear spline by Lee and Forstyhe (1973). Malcom (1977) presented the computation of nonlinear spline using finite difference method. In the modern literature, computation of nonlinear spline normally uses the numerical method. Both Edwars (1992) and Renka (2003) described the numerical methods for computing non-linear spline and demonstrated the finding in curve interpolation.

For MEC, Glass (1966) and Woodford (1969) compute the discrete MEC using multi-boundary value method. While Horn (1983) computes the curve of least energy using multi-arc spline. Today, it is common to adopt the concept of minimum strain energy in fair curve design. Vassilev (1996) described the fair curve interpolation through energy minimization while Zhang and Cheng (2001) discussed the curve fairing process through minimizing the strain energy. Wolberg (2002) presented the MEC for smooth monotonic C^2 cubic spline interpolation, and Yong (2004) introduced the geometrical smooth curve with minimum strain energy.

The concept of optimizing the energy function also appeared in 3D curve design; Moreton and Sequin (1991) discussed the computation of minimum energy networks with Hermite curve. Hofer and Pottmann (2004) presented the geometric optimization algorithm for the quadratic function. Hofer and Pottmann applied their result in manifolds design, and as the motion spline for Computer Graphic design.

Rather than optimizing the traditional strain energy function, Wesselink and Veltkamp (1995) minimize the total energy which is a combination of external and internal energy. They approximated the energy with a quadratic function, and solved for the optimum curves by minimizing the function. There are also researcher approaches the computation of MEC by simplify the energy function and express it in a new form; these included studies by Benoit (2010) and Ahn, Hoffmann and Rosen (2011). They use the new energy approximation expression or equation for computing the optimal curve.

Besides that, the curve's intrinsic equation – the curvature profile is also crucial for determine the shape of the curve. It defines the shape of the curve well; therefore, is an adequate measure of curve's fairness. Referred to Nutbourne, McLellan and Kensit (1987), designers prefer to use curvature profile to determine the shape of the curve. Monotone curvature profile is the most used concept for fair curve design in the recent literature. According to Farin and Sapidis (1989), a curve is fair if its curvature plot consists of relatively few monotone pieces. There are various approaches presented for computations of the curve satisfying monotone curvature variation. The three main approaches are: i. fairing the curve by adjusting the point on the curve or the control polygon (Kjellander 1983; Farin, Sapidis and

Worsey 1987; Farin and Sapidis 1989; Sapidis and Farin 1990; Poliakoff, Wong and Thomas 1999; Mullineux and Robinson 2007) ii. defining the monotone curvature variation conditions (Sapidis and Fey 1992; Frey and Field 2000; Wang et al 2004) and iii. using the curves that have define monotone curvature profile, the spiral (Meek and Walton 2004; Walton and Meek 1998; 2009; 2010). For the first two methods, enforcing the curvature profile to vary monotonically causes the curve to become less flexible. For the third method, spiral is non-polynomial, hence, unable to apply directly in the current CAD system. It needs to approximate with a polynomial function.

Spiral is a curve that turns around a centre point, either getting progressively closer or further from the centre point. Monotone curvature profile is the most striking properties of spiral. Few spirals that appeared in the literature included logarithmic spiral, Cornu spiral and generalised Cornu spiral (GCS). Logarithmic spiral, also known as equiangular spiral, possesses the self-similarity properties. Cornu spiral is well-known by its linear curvature profile. Highway and railway designs often use it as the transition curve. GCS is a family of spirals that can be further reduced to Cornu spiral and logarithmic spiral. GCS can generate a wide spectrum of the curve with the same curvature properties. Therefore, it is more flexible and suitable to apply as the decorative curve.

Researchers had approximated the spiral with various approaches and different polynomial curves. These included the approximation of Cornu spiral using arc spline by Meek and Walton (2004); s-power series by Sánchez-Reyes and Chacón (2003) and also continued fraction by Wang et al (2001). In addition, Baumgarten and Farin (1997) described the approximation of logarithmic spiral using rational cubic Bézier. Whilst Cripps, Hussain and Zhu (2010) discussed approximating GCS using quintic Bézier curve. In this study, we will discuss the approximation of GCS with low energy curve.

1.2 Objective of study

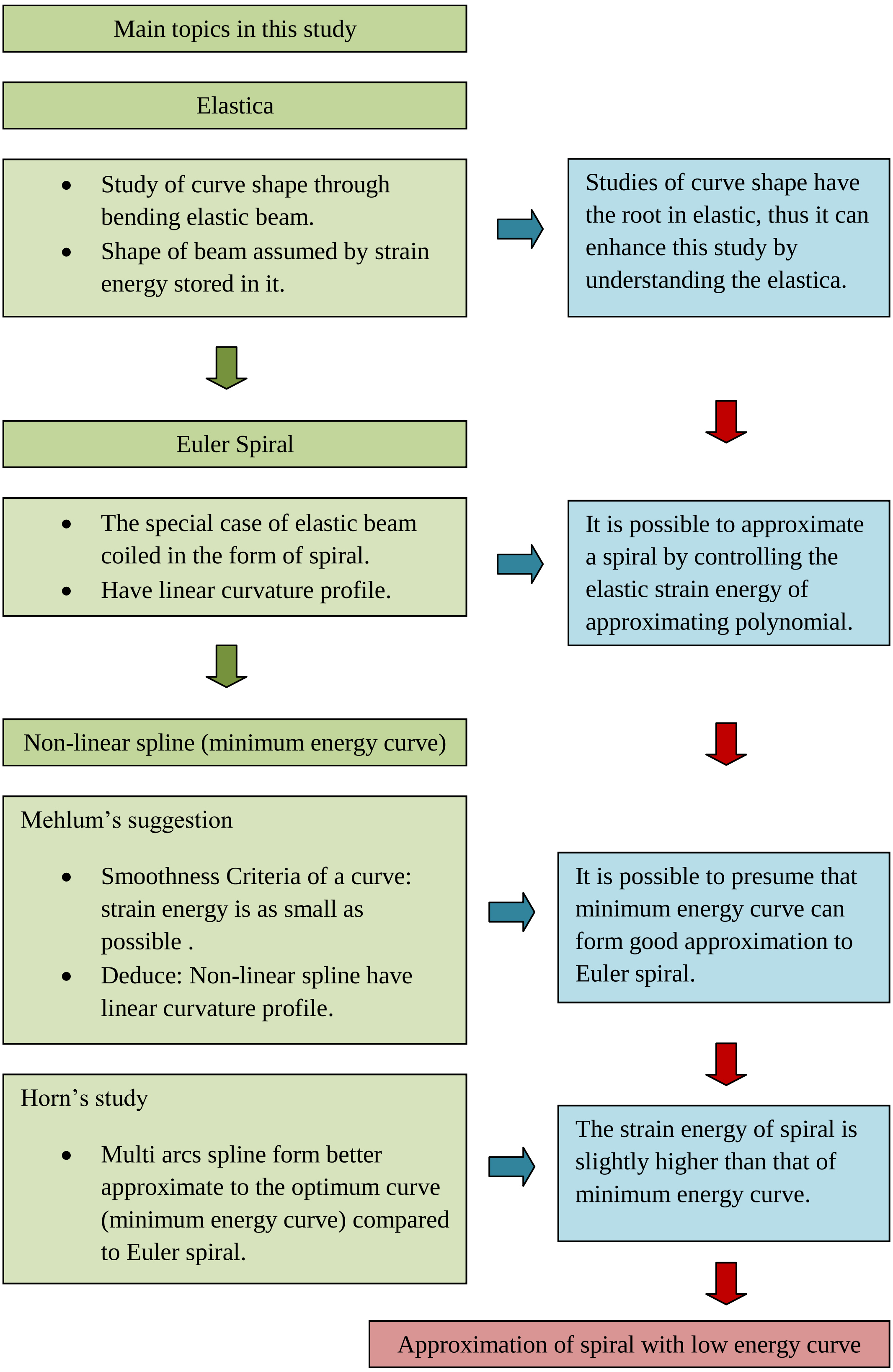
Objective of this study is to establish a new technique for approximation of generalised Cornu spiral (GCS); with emphasis on geometrical fairness specifications and physical strain energy concept. Up to present, there is a lack of discussion on strain energy of the approximating curve in literature. Looking forward for the possibility of fill up the gap, in this study we approximate the spiral by controlling the energy of the approximating polynomial. MEC is the classic strain energy curve; here we developed an alternative idea - the low energy curve for spiral approximation. The resulting curve will satisfy the given set of end point constrains and monotone curvature variation. Focus of this thesis is to solve the difficult problem of searching suitable value for the control elements of the approximating polynomial curve. This thesis presents in detail the method of incorporating the control elements into function measuring strain energy and numerical method adopted to regulate the elements.

1.3 Organization of the Thesis

Before we start the detailed discussion, it is essential for us to establish the geometrical and physical concept of curve. Chapter 2 introduces the geometric characteristic of the polynomial curve and its properties. Chapter 3 presents the development of the curve from physics approach to variational approach and derivation of strain energy. In Chapter 4, we start the discussion of incorporating control elements into function measuring strain energy and the numerical method used to adjust the control elements. Chapter 5 will be mainly about the existing spiral in literature and verifying the proposed strain energy approach by approximating the clothoid using MEC. Finally in Chapter 6, we show the approximation of GCS by low energy curve with numbers of examples and its application in curve design.

1.5 Research framework

The diagrams below indicate the framework of the research.



CHAPTER 2

THE PLANAR CURVE

Approximation of spiral with polynomial curve is useful in CAD; especially in design require high quality curves. This chapter discuss the important geometric characteristics and properties that affect a curve's utility. Section 2.1 gives the introduction for planar curve in vector form. Section 2.2 to 2.4 discussed the general properties of planar curve. These included the arc length, tangent and curvature. In Section 2.5, the discussion will be on the derivation of curve using implicit form, and Section 2.6 presents the key properties of fair curve.

2.1 Introduction

Curve can exist in 2 dimensions (2D) or 3 dimensions (3D). In this study, we will concentrate on the approximation of spiral in 2D, or we called it the planar curve. There are three common ways of defining a curve. These include the parametric form, explicit from and implicit form. We will focus on the discussion of the parametric form. Parametric representations have the overwhelming advantage for investigating the aesthetic of curves (Burchard et al, 1994). With referred to Farin (2002a), Ferguson has introduced the idea of representing the curve segment as vector function using parameter in 1960. Subsequently, this approach became influential in computer aided design (CAD). The concept of this approach can be understood through considering a moving particle in the vector space. Consider a particle X moved from a point p at the time t =0, and arrived at a point q at t =1. The location of X is the displacement at the time t. It can be written in vector form as $X(t)$, where t is the parameter. A curve is the path passing through by X as it moves from a point p to q.

A parametric curve is defined in vector form as $X = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, t \in \mathbb{R}$, f and g can be any function. For example, the f and g function of a Bézier curve is defined in term of control points and Berstein polynomials, where $t \in [0 \quad 1]$. The natural 3D extension adds advantage to the parametric from. We will start the discussion of planar parametric curve by considering three general characteristic: the arc length, tangent and curvature.

2.2 Arc length

In literature, arc length plays a decisive role in measuring the accuracy of curve approximation. In this study, we will use an alternative method for the accuracy evaluation. However, it is essential for us to understand the arc length because it is a geometrical character that reflexes the curve visual properties.

According to Burchard et al. (1994), arc length $s(t)$ is the distance travelled by the particle from a point $c(a)$ of the curve to the point $c(t)$. The arc length is defined as follows:

$$s(t) - s_a = \int_a^t \frac{ds}{dt} dt = \int_a^t \| c'(t) \| dt = \int_a^t \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \tag{2.1}$$

where $\frac{ds}{dt} = \| c'(t) \|$ is the parametric speed. Arc length parameterise $c(s)$ indicates that the parametric speed equals to constant and unity. Since arc length is independent of xy-coordinate system used and of parameter t, arc length is an intrinsic quantity. This is true as arc-length is the visual property of a curve (Buchard et al. 1994).

2.3 Tangent

The tangent $\vec{T}(t)$ of the curve $r(t)$ is the direction of the curve at t. It is define by the first derivative of the curve, $\vec{T}(t) = r'(t)$. $\hat{T}(t) = \dfrac{r'(t)}{\|r'(t)\|}$ is the unit tangent vector.

In approximating spiral with polynomial curve, it is important to compute polynomial curve that matches the end points tangent of corresponding spiral arc. However, it is not necessary for the curves to possess the same tangent magnitude. Normal vector $\hat{N}(t)$ at a point t is the vector perpendicular to the tangent vector at that point.

2.4 Curvature

Curvature is the centre of determining the shape of the curve. It measured how quickly and how much the curve is bending away from the tangent direction. It is defined as the rate of change of tangent angle as the particle moves along the curve.

$$k(s) = \frac{d\theta}{ds} \tag{2.2}$$

Osculating circle is another approach to understand the concept of curvature. Consider three non-collinear adjacent points on the plane, as the two points on both the side move closely to the middle point x a limiting circle can be generate. The resulting circle is the osculating circle; it is the best circle which approximate the curve at the point x. The centre and radius (r) of curvature is equal to the centre and radius of the osculating circle respectively. The curvature k is defined by the reciprocal of the radius of curvature $k = \pm\frac{1}{r}$. This implies that if the radius of curvature is small then the curve turns sharply at point x. The curvature sign is determinable by considering the convex and concave region of the curve. The sign depends on the orientation of the normal vector. The curvature has a positive sign, if the normal vector is pointing toward to the centre of the osculating circle. In this situation, the curve turns counter clockwise. When the normal vector is pointing in the opposite direction of the centre of osculating circle, the curvature has a negative sign, and it turns clockwise (concave). Figure 2.1 illustrates the curvature sign and the orientation of the normal vector.

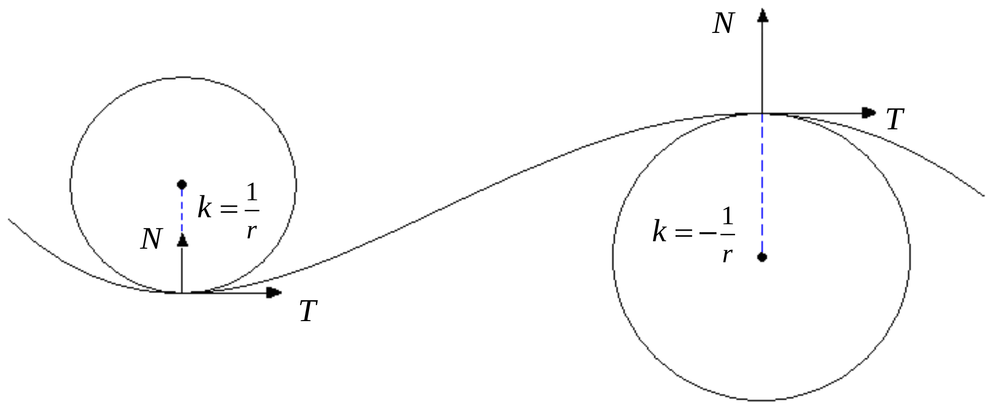


Figure 2.1 Osculating circle and the sign of curvature.

Refer Farin (2002) the curvature of parametric curve $r(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}$ is given by

$$k(t) = \frac{\dot{r}(t) \times \ddot{r}(t)}{\|\dot{r}(t)\|^3}$$

where $\| \ \|$ is the modulus, $\dot{r}(t) = \frac{dr}{dt}$ and $\ddot{r}(t) = \frac{d^2 r}{dt^2}$. For planar curve, the sign of curvature is equal to sign of z component of $\dot{r}(t) \times \ddot{r}(t)$. The signed curvature is given by

$$k(t) = \frac{\ddot{x}(t)\dot{y}(t) - \ddot{y}(t)\dot{x}(t)}{\left(\dot{x}(t)^2 + \dot{y}(t)^2\right)^{3/2}}.$$

Similar to arc length, curvature is independent of xy -coordinate system used and of parameter t . Thus, curvature is an intrinsic quantity describes the shape of the curve. Farin and Sapidis (1989) has emphasized the usefulness of curvature as a measurement for curve fairness and discussed the usage of curvature plot in analysing the shape of the curve. Curvature plot is the plot of the curvature value at every point of a curve along the arc length. In this study, curvature plot will be use widely for determining the shape of the curve.

2.5 Fundamental theorem of planar curves

Another useful way of describing a planar curve is via integrating the curvature function which provides the desired curvature profile. Consider the problem as finding the arc length parameterise curve $c(s) = \begin{bmatrix} x(s) & y(s) \end{bmatrix}$ described by a specific curvature function $k(s)$. The angle of tangent at parameter s formed with the x -axis is defined as $\tan(\theta(s)) = \frac{y'(s)}{x'(s)}$. Following the elementary of trigonometry we have

$$x'(s) = \cos(\theta(s)) \tag{2.3}$$

$$y'(s) = \sin(\theta(s)) \tag{2.4}$$

Recall that $k(s) = \frac{d\theta}{ds}$; thus, we have the general definition for angle θ:

$$\theta(t) = \theta_0 + \int_0^t k(s)ds \tag{2.5}$$

(2.5) is the function measuring angle at the time t. Where θ_0 is the initial angle at $t=0$. The function $x(s)$ and $y(s)$ can then be obtained by substituting (2.5) into (2.3) and (2.4) followed by integration,

$$x(s) = x_0 + \int_0^s \cos\left(\theta_0 + \int_0^t k(s)ds \right)dt \tag{2.6}$$

$$y(s) = y_0 + \int_0^s \sin\left(\theta_0 + \int_0^t k(s)ds \right)dt \tag{2.7}$$

(2.6) and (2.7) are the explicit solutions obtained from a given curvature function. x_0 and y_0 in (2.6) and (2.7) indicate the starting coordinate of the curve.

The above explanation is the fundamental theorem of planar curve. According to Nutbourne, McLellan and Kensit (1972) design engineers prefer to describe curves by using this approach because they like to specify the shape of curves in terms of its curvature profile. They also prefer a ready method for integrating the curvature profile to obtain the co-ordinates of any point along the curve. The generalised Cornu spiral (GCS) discussed in Section 5.1.3 is described using this approach.

2.6 Curves properties

There are many desired properties that make a curve the ideal one. Herein, we focus on those properties which are related with this thesis. The discussions were referring to documentation by Moreton (1992) and Levien (2009).

2.6.1 Continuity

The parametric curve can be defined by using various functions. However, polynomial and rational polynomial is preferred because of the ease of computation.

Nevertheless, a single piece of polynomial has limited descriptive power. In certain cases, it is insufficient to represent the desire curve's shape by using a single piece of polynomial. In this circumstance, several pieces of polynomials are joined together to represent the complicated shape. The resulting curve is called piecewise polynomial. In this case, the manner in which the polynomials are joined together is important. Continuity is the smoothness of joints between adjacent polynomials. The choice for order of smoothness is application dependent. Referring to Moreton and Séquin (1991), for application such as architectural drawing, continuity only in position is fine, but for application such as design of mechanical parts, it requires first or second order smoothness.

There are two types of smoothness which are particularly important in the current CAD application. The first type is the smoothness of speed. In other words, the speed of the particle must be continuous as it moves along the joining path. This motion is known as parametric continuity C^n. C^n can be obtained by requiring continuity of derivative vectors; such as velocity vector for first order parametric continuity C^1 and acceleration vector for second order parametric continuity C^2. C^n or n^{th} order parametric continuity is defined as first n parametric derivatives agree where the polynomial abut (Barsky and DeRose 1989). The second type of smoothness is the geometrical smoothness, know as geometric continuity G^n. First order geometric continuity G^1 is defined as continuity of tangent vector. Tangent vector has the same direction as the velocity vector but with unit magnitude. Thus, if a curve satisfies G^1 but not C^1, it implies that the particle is moving in the same direction but at a different speed. Second order geometric continuity G^2 is defined as continuity of curvature. Levien (2009) stated that a higher order of continuity would not imply a fairer curve. In fact, for construction of fair planar curve, G^2 continuity is fine. Barsky and DeRose (1989) pointed out that the parametric continuity would disallow many parameterizations, preventing the generation of geometrical smooth curve. So in this study we will compute the curves satisfying second order parametric C^2 and geometric continuity G^2 which is good enough for smooth interpolation.

2.6.2 Existence

The term existence in curve design refers to how strong a curve generation technique is. The technique is qualified by its ability to generate a desired curve for the given inputs. The curve generation technique can be improved by adding constraints in the algorithm or inputting more information.

Studying the existence of MEC is a subject of interest for many researchers. The difficulty in computing MEC is that the curve tends to degenerate as the strain energy goes smaller. Refer Levien (2009), Birkhoff and deBoor pointed out that for the MEC to exist, the arc length must be finite. However, in most of the cases, users would not know the exact arc length. Existing MEC computation methods found in the literature have different approaches in controlling the existence of curve. We will present a method to control the existence of MEC in Chapter 4. The detail discussion on the relation between arc length and energy will be present in Section 3.3.7.

2.6.3 Invariance under transformations

A curve is invariant under transformation if its shape does not change when there is a change in the coordinate system in which the data is described (Moreton 1992). The two common types of invariance transformation included the invariance under similarity transformation and the invariance under affine transformation. Similarity transformation is a transformation obtained by translation, rotation or dilation. The resulting curve is invariant under similarity transformation if it preserves the angular and also ratio of length. Affine transformation refers to transformation which preserves collinearity and ratio of distance. Refer Moreton (1992) for CAD system application; the curve must be invariant under similarity transformation whilst invariant under affine transformation is not prerequisite for CAD utility.

2.6.4 Local

A curve is local if, by moving a control point will only cause a small change around the neighborhood of the moved point. For curve which exhibits locality, it possessess local control. Local control is important for interactive design as the designer is allowed changing the small unsatisfied part without affecting the others well satisfied parts. The curve is considered as having global control if correcting one part would result in changing the entire shape of the curve. This will cause a lot of inconvenience. Example of the curve with local control is shown in Chapter 4.

2.6.5 Roundness

Roundness is another key aspect for fair curve. Circle is acknowledged as the smoothness or fairness curve. In curve interpolation, the resulting curve is round if it form an approximating circular arc when the data points are arranged in the co-circular form. The term “approximating circular arc” is used because it is impossible for a polynomial to form an exact circular arc. The roundness property measure how close the approximating circular arc is to the exact one. Since circular arc is the segment of circle, hence it is said that round curve is fair but not vice versa. MEC is a good example of fair curve, but it is not round.

2.6.6 Monotone curvature

Human judge how fair a curve is by its visual appearance. According to Burchard et al. (1994) the visual properties of a curve can be translated into mathematic properties by its intrinsic equation, the curvature. Human eyes are particularly sensitive to curvature extrema. Spiral is a good example of fair curve; it has monotone curvature everywhere and no curvature extrema. Unfortunately, the polynomial parametric curve used in CAD application has more complicated curvature distribution. Authors in the literature have good vibes in evaluating the fairness of complicated curve in term of pieces of monotone curvature or distribution of curvature extrema. A few of them are shown below:

- Curvature extrema of a fair curve curve should only occur where explicitly desired by designer. (Farin, Sapidis and Worsey 1987)
- A curve is fair if its curvature plot consists of relatively few monotone pieces.. (Farin and Sapidis 1989)
- A curve is fair if the curvature plot is continuous and is as close as possible to a piecewise monotone function with as few monotone pieces as possible. (Poliakoff, Wong and Thomas 1999)
- Curves with no undesirable curvature extrema are referred to as fair curves. (Walton and Meek 2010)

CHAPTER 3

ELASTICA AND NONLINEAR SPLINE

This Chapter traces the development of spline in literature, starting from the elastica, followed by the nonlinear spline. We will start the discussion on the relation between spline with elastica in Section 3.1. The brief history of elastica and the derivation from elastica to variational approach used today is given in Section 3.2. Section 3.3 shows the derivation of equation measuring strain energy through understanding the reaction of force and moment. Section 3.4 discussed the nonlinear spline by Mehlum (1974) and Horn (1983). The derivation of proposed approach in this study is mainly derived from the classic elastica, and nonlinear spline, the derivation is presented in Section 3.5.

3.1 Introduction

Spline appeared regularly in literature and has a few different meanings. This included, as the draftsman mechanical spline, as the piecewise curve interpolation and the approximating curve for a given set of data point. Herein we refer the spline as the draftsman spline. Draftsman spline is a curve used for shipbuilding in the 13th to 16th century. It was the day before computer modelling; architects draw a smooth curve through a set of points by using the wooden or metal beam. The beam is bent in a way that it smoothly passes through all the given points on the Euclidean plane. They control the shape of the beam by using metal weight or ducks. Draftsmen adjust the shape of the beam by moving the duck or add ducks onto the beam. The spline will "try" to bend as little as possible, resulting in shapes which are both aesthetically pleasing and physically optimal Farin (2002a). Elastica is the theory of bending spline. In earlier literature, mathematicians study elastica through the physical approach. We will start the discussion with the early discovery of elastica followed by its development into the variation approach in the mathematical form used today.

3.2 Brief history of elastica

3.2.1 From elastica to variational problem

In the 17^{th} to 18^{th} century, there is a group of researchers paying attention in elastica. Elastica derived from the Latin word *Elasticus*, meaning thin strip of elastic material. Elastica has long histories; here we highlighted them from studies done by Levien (2009). More detail description can be found in (Love 1906; Goss 2003; Levien 2009).

In 1638, Galileo had approached the problem of elastica by building a model of a beam with one side attached to the wall, and load the other side is with weight. The moment of force and resistant moment is the main focus of his study. Although Galileo did not discuss the deviation of the beam, he had established the concept of the moment to determine the force on an elastic material. In 1678, Hooke had approached the problem of elastica through spring. He had derived the Hooke's law which claims that the applied force is proportional to the change in length. Hooke also slightly touched the problem of elastic strip by providing an illustration of compound elasticity. However, Hooke did not discuss the curvature of the strip as curvature was not fully discovered at that time.

James Bernoulli did not accept the linear law of spring by Hooke easily and intend to verify it again himself. He had approached the problem of elastica by using the catgut model. Bernoulli had found the significant nonlinearities of elastica and posed the general elastica problem which is now known as rectangular elastic. He had solved the problem partially by working the evolution of elastica as geometric construction for curvature. Based on Bernoulli's finding, Huygens pointed out that James's solution was not the general solution for elastica but was only limited to rectangular elastica. Responding to Huygens's point, Bernoulli continued to work on the problem by giving another general solution. However, the solution did not receive a favourable response at that time.

James Bernoulli's nephew, Daniel Bernoulli, working with Euler on elastica, had brought the next breakthrough in it. In a letter to Euler, he had stated the elastica as variational problem in terms of the stored energy and also mentioned about the energy minimizing principle. Based on Bernoulli's suggestion, Euler had expressed the problem of elastica in variational form and written it in terms of first and second derivatives. However, Euler was unable to solve the equation at that time because he was unable to solve the second derivative until he discovered the Euler-Poisson equation. Pleasantly surprised, the resulting elastica solution obtained by Euler was the same as James Bernoulli's general solution.

Euler Spiral is the special case of elastica. Euler comes to this important discovery when he studies the case of "an elastica spring freely coiled on the form of a spiral". In mathematics, Euler spiral is given in integrating form. Where the integral are known as Fresnel integral, this is because Fresnel comes across the same integral in the study of light diffraction. More detail of Euler spiral will be discussed in Section 5.1.2.

Euler was aware of the simple moment approach to the elastica. He had verified the equivalence of variational approach with moment approach by manipulating the integral formulation of elastica and thus further affirmed the variational technique. Today, the variational problem, is equivalent to finding the minimal of integral of curvature square over arc length, $\int k(s)^2 ds$.

3.2.2 Elliptic integral

In the early 18th century, where finding the length of ellipse was still an open question, James Bernoulli had approached this problem by modifying the ellipse to a curve called lemniscates. Bernoulli was able to find the arc length of lemniscates by applying the rectangular elastica. The resulting equation used to find the arc length of lemniscates is known as "Lemniscate integral". Lemniscate integral is given by

$$s = \int_0^x \frac{dx}{\sqrt{1-x^4}}$$

However, the solution was not entirely accurate. Fagnano carried on the problem of finding the length of lemniscates. In fact, Fagnano had achieved the doubling of lemniscates arc. Euler reviewed Fagnano's work. Based on Fagnano's impressive investigation, Euler discovered the elliptic integrals and also its inverse function, the elliptic function. Today, the elliptic integral, is useful in the study of the mechanism.

In 1859, Kirchhoff studied the kinetic energy of a swinging pendulum. He presented another significant discovery of elastica from the elliptic integral and elliptic function. From Kirchhoff's study, the trajectory of a swinging pendulum is equivalent to the shape of elastica. The motion of the pendulum is one of the inspirations among many of Levien (2009) in the computations of the series of font called Iconsolata.

3.2.3 Spline interpolation

In 1943, Schoenberg started his study in theory of splines. He was the first to use the word “spline” in smooth connection of piecewise polynomial approximation. Birkhoff and de Boor pointed out that Schoenberg’s approximation to elastica was not ideal because it was not invariant under rigid rotation. They had suggested approximating the elastica using nonlinear spline interpolation. They also discussed few methods on this problem, but there was no high impact. More details of the related work on nonlinear spline will be further discussed in section 3.4.

3.3 The elastic strain energy

Refer Section 3.1, it is easier to understand the ealstica concept through moment approach. Moment approach focuses on determining the relation between moment and curvature of the beam. This section presents the moment approach by assuming the linear relationship of force and the change of length of an infinitesimal solid. We will show that the elastic strain energy of bending beam is proportional to the integral of curvature square over the length of the beam. Studies of this section follow closely to those in Lee and Forsythe (1971) and Jou (1989).

3.3.1 Applied force and moment

According to Hooke’s law, a material is elastic if it behaves in the same way when being loaded and unloaded. Figure 3.1A and Figure 3.1B shown a bar of material being loaded and unload respectively.

Figure 3.1 Force and loading: A. Material bar being loaded; B. unloaded bar.

For the material under loading, stress, $\sigma = \frac{F}{A}$ is the average force per cross section area (A), and strain, $\varepsilon = \frac{\Delta L}{L}$ is the rate of change in length. Referred to Hooke; within a range of magnitude of an applied force, the displacement is proportional to the applied force. Thus, the relation between stress and strain can be written as $\sigma = E\varepsilon$ where E is the Young's modulus of elasticity.

Moment is defined as the tendency of causing rotation about a point. Figure 3.2A shows a thin beam with a rotation point in the middle. Moment vector (M) is given by the cross product between applied force vector (F) with lever arm vector (R).

$$M = F \times R \tag{3.1}$$

From (3.1) we can see that if the applied force passes through the rotating point (o), the moment vector will be zero and thus there is no rotation. Meanwhile, if the applied force (F) does not pass through rotation point (o), the rotation effectiveness of the applied force (F) will tend to increase as the magnitude of the lever arm vector (R) increases. Here, the rotating effectiveness is referred to the magnitude of the associate moment vector. Couple is defined as two parallel forces (here represented by F_1 and F_2) of equal magnitude but with opposite direction. Figure 3.2B illustrates the couple.

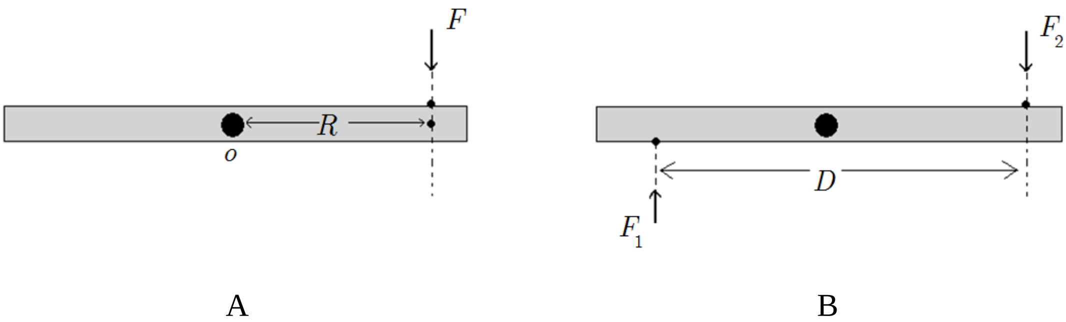


Figure 3.2 Moment and couple: A. force and moment; B. couple and moment

Couple moment, also known as torque (τ), is the result of the couple force. It is given by the cross product of either one of the applied force with distance (D) between two corresponding forces, $\tau = F_1 \times D$ (or $\tau = F_2 \times D$).

3.3.2 Equilibrium condition and bending moment

An object is in the equilibrium condition whenever there are forces acting on it, but it is not moving. Translation equilibrium exists when the sum of all the forces equals to zero. When the sum of all the moment relative to any point equals to zero, the rotational equilibrium condition exists.

Consider a simple beam satisfying the equilibrium condition; external moment is created when an external force is applied onto the beam. Internal resisting moment will be created within the cross section area of the beam to counteract the external moment. The internal moment will act in the opposite direction of the external moment, and the magnitude of internal resisting moment is equal to the magnitude of external moment. These moments together cause the beam to bend and thus it is known as bending moments.

3.3.3 Bending beam

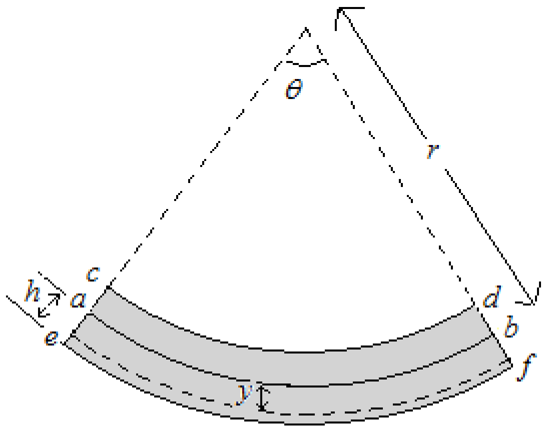


Figure 3.3 Cross section area of bending beam

Figure 3.3 shows the cross section area A (highlighted with grey) of a bending beam. Refer to the figure, arc cd is under compression, arc ef is under extension and arc ab is the neutral axis. When the beam is bent, the strain in the fibres will vary linearly with respect to the distance from the neutral axis. Let σ_h and σ_y be the stresses occur at a distance h and y from the neutral axis respectively. They can be related by the ratio:

$$\frac{\sigma_y}{y} = \frac{\sigma_h}{h} \tag{3.2}$$

The external moment created by external force is resisted by the internal bending moment (M) developed by the sum of all stress over the cross section area (A)

$$M = \int y\sigma_y dA\ . \tag{3.3}$$

By substituting (3.2) into (3.3), the internal bending moment can be written as

$$M = \frac{\sigma_h}{h}\int y^2 dA\ .$$

Moment of initial (I) is a measure of an object's resistance to acceleration. It is defined as y^2 over the cross section area.