UNIVERSITI SAINS MALAYSIA

First Semester Examination
2016/2017 Academic Session

December 2016 / January 2017

MST561 - Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **FIVE** (5) questions.

*[**Arahan:** Jawab **LIMA** (5) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Assume that a continuous random variable X has a probability density function given by $f(x) = 2e^{-2x}, x \geq 0,$ and zero elsewhere

(i) Show that the mean is ½ and the variance is ¼.

(ii) Hence, find the value of $P(\left|X - \frac{1}{2}\right| < 1)$

(iii) Use Chebyshev inequality to find the lower bound for $P(\left|X - \frac{1}{2}\right| < 1)$

[40 marks]

(b) If the moment generating function for random variable X is $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$, find $P(X = 2 \text{ or } 3)$

[30 marks]

(c) Given the joint density function for random variables X and Y is $f(x, y) = e^{-x-y}, 0 < x < \infty, 0 < y < \infty.$ Are X and Y stochastically independent? Explain. Find $E(e^{t(X+Y)})$ for $t < 1$.

[30 marks]

1. *(a)* *Andaikan pembolehubah rawak selanjar X mempunyai fungsi ketumpatan kebarangkalian yang diberikan oleh* $f(x) = 2e^{-2x}, x \geq 0,$ dan sifar selainnya

(i) *Tunjukkan bahawa min adalah ½ dan varians adalah ¼.*

(ii) *Seterusnya, cari nilai* $P(\left|X - \frac{1}{2}\right| < 1)$

(iii) *Gunakan ketaksamaan Chebyshev untuk mendapatkan nilai had bawah bagi* $P(\left|X - \frac{1}{2}\right| < 1)$

[40 markah]

(b) *Jika fungsi penjana momen bagi pembolehubah rawak X ialah* $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$, *cari P(X = 2 atau 3).*

[30 markah]

(c) *Diberi fungsi ketumpatan tercantum bagi pembolehubah rawak X dan Y ialah* $f(x, y) = e^{-x-y}, 0 < x < \infty, 0 < y < \infty.$ *Adakah X dan Y tak bersandar secara stokastik. Jelaskan. Cari* $E(e^{t(X+Y)})$ *untuk t < 1.*

[30 markah]

2. (a) Joint density function for X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}, \quad -\infty < x_1, x_2 < \infty.$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint density function for Y_1 and Y_2.

[40 marks]

(b) Let $Y_1 < Y_2 < Y_3 < Y_4$ be order statistics for a random sample of size 4 from a distribution with density function.

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

What is $P\left(\frac{1}{2} < Y_3\right)$?

[30 marks]

(c) Assume that X_1, X_2 is a random sample of size 2 from the distribution $N(0,1)$ and Y_1, Y_2 be random sample of size 2 from the distribution $N(0,2)$. If both samples are independent, find the distribution for each of the following statistics:

(i) $\overline{X} - \overline{Y}$

(ii) $\frac{1}{4}(Y_1 - Y_2)^2$

(iii) $\dfrac{2(\overline{X} - \overline{Y})}{\sqrt{\frac{3}{2}}(Y_1 - Y_2)}$

[30 marks]

2. a) Fungsi ketumpatan tercantum X_1 dan X_2 ialah

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}, \quad -\infty < x_1, x_2 < \infty.$$

Biarkan $Y_1 = X_1 + X_2$ dan $Y_2 = X_1 - X_2$. Cari fungsi ketumpatan tercantum Y_1 dan Y_2.

[40 markah]

(b) Biarkan $Y_1 < Y_2 < Y_3 < Y_4$ mewakili statistik tertib untuk suatu sampel rawak bersaiz 4 daripada taburan dengan fungsi ketumpatan

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{selainnya} \end{cases}.$$

Apakah $P\left(\frac{1}{2} < Y_3\right)$?

[30 markah]

(c) *Andaikan X_1, X_2 menandakan sampel rawak saiz 2 daripada taburan N(0,1) dan Y_1, Y_2 menandakan sampel rawak saiz 2 daripada taburan N(0,2). Jika kedua-dua sampel tak bersandar, cari taburan untuk setiap statistik berikut*:

(i) $\bar{X} - \bar{Y}$

(ii) $\frac{1}{4}(Y_1 - Y_2)^2$

(iii) $\dfrac{2(\bar{X} - \bar{Y})}{\sqrt{\frac{3}{2}}(Y_1 - Y_2)}$

[30 markah]

3. (a) If X_n follows a gamma distribution with parameters *n* and λ , find the limiting distribution of the random variable $Y_n = \dfrac{X_n}{n}$

[30 marks]

(b) Assume $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution with parameter α, $0 \le \alpha \le 1$.

(i) Find the maximum likelihood estimate for α.

(ii) Show that $\dfrac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \dfrac{n}{n-1}\bar{X}^2$ is an unbiased estimator for $\alpha(1-\alpha)$.

(iii) Find the Cramer-Rao lower bound for the variance of the unbiased estimator of $\alpha(1-\alpha)$.

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ represent a random sample from the Be(θ) distribution. If the prior distribution of Θ is given by $g_\Theta(\theta) = I_{(0,1)}(\theta)$, find the posterior Bayes estimator of $\theta(1-\theta)$ with respect to the prior pdf $g_\Theta(\theta)$.

[30 marks]

3. *(a)* *Jika X_n mengikuti taburan gama dengan parameter n dan λ , cari taburan penghad bagi pembolehubah rawak $Y_n = \frac{X_n}{n}$*

[30 markah]

(b) *Andaikan $X_1, X_2, ..., X_n$ menandai suatu sampel rawak daripada taburan Bernoulli dengan parameter α, $0 \le \alpha \le 1$.*

(i) *Cari penganggar kebolehjadian maksimum bagi α.*

(ii) *Tunjukkan bahawa $\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\overline{X}^2$ ialah penganggar saksama bagi $\alpha(1-\alpha)$.*

(iii) *Cari batas bawah Cramer-Rao bagi varians penganggar saksama $\alpha(1-\alpha)$.*

[40 markah]

(c) *Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan Be(θ). Jika taburan prior bagi Θ diberi oleh $g_\Theta(\theta) = I_{(0,1)}(\theta)$, cari penganggar Bayes posterior bagi $\theta(1-\theta)$ terhadap fungsi ketumpatan kebarangkalian prior $g_\Theta(\theta)$.*

[30 markah]

4. (a) Assume that $X_1, X_2, ..., X_n$ is a random sample of size *n* with density function

$$f(x,\alpha) = \begin{cases} (\alpha+1)x^\alpha, 0 < x < 1 \\ 0, \quad \text{elsewhere} \end{cases}$$

(i) Is the distribution a member of the exponential family? Explain.

(ii) Find the uniformly minimum variance unbiased estimator (UMVUE) for $\alpha+1$.

[40 marks]

(b) Let $X_1, X_2, ..., X_n$ be random sample from distribution with density function

$f(x;\beta) = (\beta+1)x^\beta I_{(0,1)}(x)$, $\beta > 0$.

Find the 100γ percent confidence interval for β.

[40 marks]

(c) If $X_1, X_2, ..., X_n$ are random samples from normal distribution, $N(\mu,3)$, and $\overline{X}_n = \sum_{i=1}^{n} X_i \Big/ n$, is $\overline{X}_n + \mu$ a pivotal quantity? Explain.

[20 marks]

4. *(a)* *Andaikan $X_1, X_2, ..., X_n$ ialah satu sampel rawak bersaiz n yang mempunyai taburan dengan fungsi ketumpatan*

$$f(x;\alpha) = \begin{cases} (\alpha+1)x^{\alpha}, & 0 < x < 1 \\ 0 \quad , \text{ dtl} \end{cases}$$

(i) *Adakah taburan ini ahli bagi famili eksponen? Jelaskan.*

(ii) *Cari penganggar saksama bervarians minimum secara seragam (PSVMS) bagi α+1.*

[40 markah]

(b) *Biarkan $X_1, X_2, ..., X_n$ sampel rawak daripada taburan dengan fungsi ketumpatan*

$f(x;\beta) = (\beta+1)x^{\beta} I_{(0,1)}(x)$, *$\beta > 0$.*

Cari selang keyakinan 100γ peratus bagi β.

[40 markah]

(c) *Jika $X_1, X_2, ..., X_n$ sampel rawak daripada taburan normal, N(μ,3), dan $\overline{X}_n = \sum_{i=1}^{n} X_i \Big/ n$, adakah $\overline{X}_n + \mu$ suatu kuantiti pangsian? Jelaskan.*

[20 markah]

5. (a) Based on a random sample of size 20 from the distribution $N(0,\sigma^2)$, find the uniformly most powerful test (UMP) of size α = 0.10 to test $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 > 1$.

[30 marks]

(b) Assume that X is a single observation from distribution with density function $f(x;\beta) = (1+\beta)x^{\beta} I_{(0,1)}(x)$, with $\beta > -1$.

(i) Find the most powerful test of size β to test $H_0 : \beta = 0$ versus $H_1 : \beta = 1$.

(ii) To test $H_0 : \beta \le 0$ versus $H_1 : \beta > 0$, the following test is used: Reject H_0 if $X \ge 3/4$. Find the power function and size of the test.

[40 marks]

(c) Assume *X* be a single observation from distribution with density function $f(x;\theta) = \theta x^{\theta-1} I_{(0,1)}(x); \quad \theta > 0.$

To test $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$, the test used is reject H_0 if and only if $X \ge \frac{1}{2}$. Find the power function and size of the test.

[30 marks]

5. *(a)* *Berdasarkan sampel rawak saiz 20 daripada taburan* $N(0,\sigma^2)$*, cari ujian paling berkuasa secara seragam (UPBS) bersaiz* α *= 0.10 untuk menguji* $H_0 : \sigma^2 = 1$ *lawan* $H_1 : \sigma^2 > 1.$

[30 markah]

(b) *Andaikan X suatu cerapan tunggal daripada taburan dengan fungsi ketumpatan* $f(x;\beta) = (1+\beta)x^{\beta} I_{(0,1)}(x)$*, dengan* β *> -1.*

(i) *Cari ujian paling berkuasa bersaiz* β *untuk menguji* $H_0 : \beta = 0$ *lawan* $H_1 : \beta = 1.$

(ii) *Untuk menguji* $H_0 : \beta \le 0$ *lawan* $H_1 : \beta > 0$ *, ujian berikut digunakan: Tolak* H_0 *jika* $X \ge 3/4$*. Cari fungsi kuasa dan saiz bagi ujian tersebut.*

[40 markah]

(c) *Andaikan X cerapan tunggal daripada taburan yang mempunyai fungsi ketumpatan*

$f(x;\theta) = \theta x^{\theta-1} I_{(0,1)}(x); \quad \theta > 0.$

Untuk menguji $H_0 : \theta \le 1$ *lawan* $H_1 : \theta > 1$*, ujian yang digunakan ialah tolak* H_0 *jika dan hanya jika* $X \ge \frac{1}{2}$*. Cari fungsi kuasa dan saiz ujian tersebut.*

[30 markah]

APPENDIX/***LAMPIRAN***

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x} p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x} x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2} x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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