UNIVERSITI SAINS MALAYSIA

First Semester Examination
2016/2017 Academic Session

December 2016 /January 2017

**MST 562 - Stochastic Processes**
***[Proses Stokastik]***

Duration : 3 hours
*[Masa : 3 jam]*

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Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions**: Answer **all seven** [7] questions.

*[**Arahan:*** *Jawab* ***semua tujuh*** *[7] soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].*

1. Let $T_1, T_2, \ldots$ denote the interarrival times of events of a nonhomogeneous Poisson process having intensity function $\lambda(t)$.

   (i) Are the $T_i$'s independent and identically distributed? Explain your answer.

   (ii) Find the distribution of $T_1$ and the distribution of $T_2$.

   (iii) If $\lambda(t) = t + 3$, what is the probability that $n$ events occur between time $t = 2$ and $t = 5$.

[ 30 marks ]

*1. Biarkan* $T_1, T_2, \ldots$ *menandakan masa antara-ketibaan bagi peristiwa-peristiwa Poisson tak homogen dengan fungsi keamatan* $\lambda(t)$.

   *(i) Adakah* $T_i$ *tak bersandar dan tertabur secaman? Jelaskan jawapan anda.*

   *(ii) Cari taburan bagi* $T_1$ *dan taburan bagi* $T_2$.

   *(iii) Jika* $\lambda(t) = t + 3$, *apakah kebarangkalian bahawa* $n$ *peristiwa berlaku antara masa* $t = 2$ *dan* $t = 5$.

*[ 30 markah ]*

2. The number of depositors who withdraw money from an ATM machine is in accordance with a Poisson process having rate 20 per hour. The amount of money withdrawn by a depositor from the machine is exponentially distributed with mean RM500.

   (i) What is the expected time until the fifth depositor arrives?

   (ii) Determine the mean and variance of the amount of money withdrawn by the depositors from the machine in 3 hours.

[ 20 marks ]

*2. Bilangan pendeposit yang mengeluarkan wang daripada mesin ATM adalah mengikut suatu proses Poisson dengan kadar 20 orang per jam. Amaun wang yang dikeluarkan oleh seseorang pendeposit daripada mesin tersebut tertabur secara eksponen dengan min RM500.*

   *(i) Apakah masa jangkaan sehingga pendeposit kelima tiba?*

   *(ii) Tentukan min dan varians bagi amaun wang yang dikeluarkan oleh pendeposit-pendeposit daripada mesin tersebut dalam 3 jam.*

*[ 20 markah ]*

3. In celebration of the 20th anniversary of its opening, a restaurant promises to give a cup of coffee free to every 20th customer to arrive. The arrivals of customers form a Poisson process with rate $\lambda$.

(i) Show that the probability density function of the times between the lucky arrivals is given by

$$f(t) = \lambda e^{-\lambda t}\left[\frac{(\lambda t)^{19}}{19!}\right].$$

(ii) Find $P[Y_t = y]$ where $Y_t$ is the number of cups of coffee given in the interval [0, $t$].

[ 25 marks ]

*3. Sempena sambutan ulangtahun ke-20 pembukaannya, sebuah restoran menjanjikan secawan kopi secara percuma kepada setiap pelanggan ke-20 yang tiba. Ketibaan pelanggan membentuk suatu proses Poisson dengan kadar* $\lambda$.

*(i) Tunjukkan bahawa fungsi ketumpatan kebarangkalian bagi masa antara ketibaan bertuah diberi oleh*

$$f(t) = \lambda e^{-\lambda t}\left[\frac{(\lambda t)^{19}}{19!}\right]$$

*(ii) Cari* $P[Y_t = y]$ *yang mana* $Y_t$ *adalah bilangan cawan kopi yang diberi dalam selang [0, t].*

*[ 25 markah ]*

4. A carwash has two workers, Syafiq and Malli. When Amena arrives to wash her car, she finds that Syafiq is washing Hassan's car and Malli is washing Khaled's car. Amena is told that her car will be washed as soon as either Hassan's car or Khaled's car is done. Suppose that the service time distributions of Syafiq and Malli are exponential with mean $1/\lambda_1$ and $1/\lambda_2$, respectively.

(i) Show that the probability that Malli who is washing Khaled's car, completes washing after Syafiq who is washing Hassan's car, is $\dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

(ii) What is the probability that, of the three customers, Amena is the last to leave the carwash?

[ 25 marks ]

*4.* *Sebuah tempat mencuci kereta mempunyai dua orang pekerja, Syafiq dan Malli. Apabila Amena tiba untuk membasuh keretanya, dia mendapati bahawa Syafiq sedang membasuh kereta Hassan dan Malli sedang membasuh kereta Khaled. Amena diberitahu yang keretanya akan dibasuh sebaik sahaja kereta Hassan atau kereta Khaled siap dibasuh. Andaikan taburan-taburan masa layan bagi Syafiq dan Malli adalah eksponen, masing-masing dengan min* $1/\lambda_1$ *dan* $1/\lambda_2$.

*(i)* *Tunjukkan bahawa kebarangkalian Malli yang sedang membasuh kereta Khaled menyiapkan cucian selepas Syafiq yang membasuh kereta Hassan adalah* $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

*(ii)* *Apakah kebarangkalian bahawa antara ketiga-tiga pelanggan, Amena adalah yang terakhir meninggalkan tempat mencuci kereta tersebut?*

*[ 25 markah ]*

5. Suppose $\{X_1, X_2, X_3, ...\}$ is a Markov chain on the state space $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-\beta & \beta \\ 1-\beta & \beta & 0 \end{pmatrix}, \quad \text{where } 0 < \beta < 1.$$

(i) If the chain is equally likely to be in each of the three states at time 1, find $P(X_0 = 3, X_1 = 2, X_2 = 1)$.

(ii) Verify that the stationary distribution of $P$ is

$$\pi^T = \begin{pmatrix} \dfrac{1-\beta}{3-\beta} & \dfrac{1}{3-\beta} & \dfrac{1}{3-\beta} \end{pmatrix}$$

( iii) Does $X_t$ converege to a stationary distribution as $t \to \infty$ ?

[ 30 marks ]

*5.* *Andaikan* $\{X_1, X_2, X_3, ...\}$ *ialah suatu rantai Markov pada ruang keadaan* $S = \{1,2,3\}$ *dengan matriks peralihan*

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-\beta & \beta \\ 1-\beta & \beta & 0 \end{pmatrix}, \quad \textit{dengan } 0 < \beta < 1.$$

*(i)* *Jika rantai tersebut sama berkemungkinan berada dalam setiap keadaan pada masa 1, dapatkan* $P(X_0 = 3, X_1 = 2, X_2 = 1)$.

*(ii)* *Tentusahkan bahawa taburan pegun bagi P ialah*

$$\pi^T = \begin{pmatrix} \dfrac{1-\beta}{3-\beta} & \dfrac{1}{3-\beta} & \dfrac{1}{3-\beta} \end{pmatrix}$$

*(iii)* *Adakah* $X_t$ *menumpu ke suatu taburan pegun apabila* $t \to \infty$ ?

*[ 30 markah ]*

6. Let $\{Z_n, n = ..., -1, 0, 1, 2, ...\}$ be a sequence of independent and identically distributed random variables with $P(Z_n = 0) = P(Z_n = 1) = \frac{1}{2}$. Let $X_n = Z_{n-1} + Z_n + Z_{n+1}$.

   (i) Write down the state space for $\{X_n\}$. By considering the possible transitions for $X_n$, explain whether $\{X_n\}$ is a reversible process.

   (ii) Determine $P(X_0 = 3, X_1 = 2, X_2 = 1)$ and $P(X_0 = 3, X_1 = 2)$. Hence, find $P(X_2 = 1 | X_0 = 3, X_1 = 2)$.

   (iii) Find $P(X_2 = 1 | X_1 = 2)$.

   (iv) Use your answers in parts (ii) and (iii) to explain whether $\{X_n\}$ is a Markov process.

[ 35 marks ]

*6. Andaikan $\{Z_n, n = ..., -1, 0, 1, 2, ...\}$ ialah suatu jujukan pembolehubah rawak yang tak bersandar dan tertabur secaman dengan $P(Z_n = 0) = P(Z_n = 1) = \frac{1}{2}$. Andaikan $X_n = Z_{n-1} + Z_n + Z_{n+1}$.*

   *(i) Tuliskan ruang keadaan bagi $\{X_n\}$. Dengan mempertimbangkan peralihan yang berkemungkinan bagi $X_n$, terangkan samada $\{X_n\}$ ialah suatu proses boleh diterbalikkan.*

   *(ii) Tentukan $P(X_0 = 3, X_1 = 2, X_2 = 1)$ dan $P(X_0 = 3, X_1 = 2)$. Oleh yang demikian, dapatkan $P(X_2 = 1 | X_0 = 3, X_1 = 2)$.*

   *(iii) Dapatkan $P(X_2 = 3 | X_1 = 2)$.*

   *(iv) Gunakan jawapan anda dalam bahagian (ii) dan (iii) untuk menerangkan samada $\{X_n\}$ ialah suatu proses Markov.*

*[ 35 markah ]*

7. Suppose that customers arrive at a 2-server service station in accordance with a Poisson process at a rate of 5 per hour. Each customer, upon arrival, goes directly into service if any of the servers are free, and if not, then the customer joins the queue (that is, he waits in line). When a server finishes serving a customer, the customer leaves the system, and the next customer in line (if there are any waiting) enters the service. The successive service times are assumed to be independent exponential random variables with rate of 2 per hour. If $X(t)$ denotes the number in the system at time *t*, then $\{X(t), t \geq 0\}$ is a birth and death process.

(i) Specify the parameters of the process $X(t)$

(ii) Determine the stationary distribution for $X(t)$.

(iii) Determine $P_0$, the proportion of time that the system is empty in the long run.

[ 35 marks ]

*7. Andaikan pelanggan tiba ke suatu stesen servis 2–pelayan menurut suatu proses Poisson pada kadar 5 setiap jam. Setiap pelanggan yang tiba terus memasuki servis jika mana-mana pelayan tidak sibuk dan jika tidak, pelanggan akan menyertai barisan (iaitu dia menunggu giliran). Apabila pelayan selesai melayan seorang pelanggan, pelanggan meninggalkan sistem dan pelanggan seterusnya yang dalam barisan (jika ada) memasuki servis. Masa layanan berturutan diandaikan sebagai pembolehubah exponen yang tak bersandar dengan kadar 2 setiap jam. Jika* $X(t)$ *mewakili bilangan dalam sistem pada masa t, maka* $\{X(t), t \geq 0\}$ *ialah suatu proses kelahiran dan kematian.*

*(i) Nyatakan parameter-parameter bagi proses* $X(t)$.

*(ii) Tentukan taburan pegun bagi* $X(t)$.

*(iii) Tentukan* $P_0$, iaitu *kadaran masa sistem tersebut kosong dalam jangka masa panjang.*

*[ 35 markah ]*

**APPENDIX**

1. If $X$ is distributed as Poisson with parameter $\lambda > 0$, then

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0,1,2,...$$

2. If $X$ distributed as geometric with parameter $p$, $0 < p < 1$, then

$$P(X = x) = p(1-p)^{x-1} \quad ; \quad x = 1,2,...$$

3. If $X$ distributed as Binomial with parameter $p$, $0 < p < 1$, then

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0,1,2,...,n$$

4. If $X$ distributed as exponential with parameter $\lambda > 0$, then

$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

5. If $X$ is distributed as Gamma with parameter $\alpha > 0$ and $\beta > 0$ then

$$f(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1} e^{-\beta x} \quad ; \quad x > 0$$

6. If $X$ is distributed as normal with parameter $\mu$ and $\alpha^2 > 0$ then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad ; \quad -\infty < x < \infty$$

7. Formula of geometric series

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad ; \quad |r| < 1$$

8. For an arbitrary event $E$ and for any random variable $Y$,

$$\Pr\{E\} = E\left[\Pr\{E|Y\}\right]$$

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