UNIVERSITI SAINS MALAYSIA

Peperiksaan Semester Kedua
Sidang Akademik 2002/2003

Februari/Mac 2003

JIM 414/4 – Pentaabiran Statistik

Masa : 3 jam

Sila pastikan bahawa kertas peperiksaan ini mengandungi **DUA PULUH EMPAT** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.

Jawab SEMUA soalan yang disediakan.

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

1. (a) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan $N(0,1)$. Takrifkan

$$\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i \text{ dan } \overline{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i .$$

Dapatkan taburan

(i) $\frac{1}{2}\left(\overline{X}_k + \overline{X}_{n-k}\right)$.

(ii) $k\overline{X}_k^2 + (n-k)\overline{X}_{n-k}^2$.

(iii) X_1^2 / X_2^2.

(iv) X_1/X_2.

(50 markah)

(b) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada $f(x;\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$, $\theta > 0$. Bandingkan taburan asimptot bagi $\overline{X}_n$ dengan taburan asimptot bagi median sampel.

(20 markah)

(c) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan $U(0, 1)$. Andaikan $Y_1 \leq \ldots \leq Y_n$ menandakan statistik tertib yang sepadan. Dapatkan min dan varians bagi Y_{k+1} jika $n = 2k+1$, $k = 0, 1, \ldots$

(30 markah)

2. (a) Andaikan $X_{11}, \ldots, X_{1n}$ adalah sampel rawak daripada taburan $N(a + b + c, \sigma^2)$. $X_{21}, \ldots, X_{2n}$ adalah sampel rawak daripada taburan $N(a + b - c, \sigma^2)$. $X_{31}, \ldots, X_{3n}$ adalah sampel rawak daripada taburan $N(a - b + c, \sigma^2)$ manakala $X_{41}, \ldots, X_{4n}$ adalah sampel rawak daripada taburan $N(a - b - c, \sigma^2)$. Dapatkan penganggar-penganggar kebolehjadian maksimum bagi a, b, c dan σ^2.

(50 markah)

(b) Diberikan $f(x;\theta) = \dfrac{1}{\pi\left[1+(x-\theta)^2\right]}$, $-\infty < \theta < \infty$. Tentukan sama ada θ adalah parameter lokasi ataupun parameter skala.

(20 markah)

(c) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan yang berfungsi ketumpatan $f(x;\theta) = \dfrac{2x}{\theta^2} I_{(0,\infty)}(x)$, $\theta > 0$.

(i) Adakah $Y_n = \text{maks}[X_1, \ldots, X_n]$ suatu statistik cukup?

(ii) Adakah Y_n lengkap?

(30 markah)

3. (a) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan $N(\mu, \sigma^2)$.

(i) Binakan selang keyakinan 95% bagi μ, apabila σ^2 tak diketahui.

(ii) Binakan selang keyakinan 95% bagi μ, apabila σ^2 diketahui.

(iii) Jika $n = 9$, bandingkan jangkaan panjang selang di dalam (i) dengan panjang selang di dalam (ii).

(50 markah)

(b) $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan yang berfungsi ketumpatan $f(x;\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$, $\theta > 0$. Dapatkan penganggar selang keyakinan bagi $e^{-\theta} = P[X > 1]$.

(20 markah)

(c) Satu kepala dan dua bunga muncul daripada tiga lambungan sekeping syiling. Dapatkan selang keyakinan 90% bagi kebarangkalian munculnya kepala.

(30 markah)

4. (a) Pertimbangkan hipotesis ringkas H_0: $\theta = 2$ lawan H_1: $\theta = 1$, θ adalah parameter pada taburan $f(x;\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$, $\theta > 0$. Satu cerapan diperoleh.

(i) Katakan rantau genting ujian ini diberikan oleh $X \geq 1$, dapatkan kebarangkalian-kebarangkalian ralat jenis I dan II bagi ujian ini.

(ii) Katakan rantau genting ujian yang menjadi saingan pada rantau genting di dalam (i) diberikan oleh $X \leq x_0$. Cari x_0 supaya kebarangkalian ralat jenis I di dalam ujian saingan ini sama dengan kebarangkalian ralat jenis I di dalam (i).

(iii) Seterusnya dapat kebarangkalian ralat jenis II yang baru berdasarkan rantau genting di dalam (ii).

(iv) Apakah yang dapat disimpulkan tentang kedua-dua ujian yang berdasarkan pada rantau-rantau genting yang berlainan tadi?

(50 markah)

(b) Diberikan X tertabur secara $N(0, \sigma^2)$. Dapatkan ungkapan bagi λ di dalam ujian nisbah kebolehjadian bagi H_0: $\sigma^2 = 1$.

(20 markah)

(c) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan Poisson(θ) yang berfungsi ketumpatan $f(x;\theta) = e^{-\theta}\dfrac{\theta^x}{x!}$, $x = 0, 1, \ldots$. Binakan ujian paling berkuasa secara seragam bersaiz α bagi H_0: $\theta = \theta_0$ lawan H_1: $\theta < \theta_0$.

(30 markah)

5. (a) Andaikan $X_1, \ldots, X_n$ adalah sampel rawak daripada taburan $N(\mu, \sigma^2)$.

Tunjukkan

$$\widetilde{S}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \xrightarrow{P} \sigma^2 .$$

(25 markah)

(b) Andaikan X sebagai cerapan tunggal bagi taburan Bernoulli yang berfungsi ketumpatan $f(x;\ \theta) = \theta^x(1-\theta)^{1-x}I_{(0,\ 1)}(x)$, $0 < \theta < 1$. Diberikan $t_1(X) = X$ dan $t_2(X) = ½$.

(i) Yang mana satu antara $t_1(X)$ dan $t_2(X)$ saksama?

(ii) Bandingkan min ralat kuasa dua $t_1(X)$ dan $t_2(X)$.

(25 markah)

(c) Tunjukkan panjang selang keyakinan bagi σ daripada taburan normal menuju ke 0 apabila saiz sampel dinaikkan.

(25 markah)

(d) Diberikan $f(x;\theta) = e^{-\theta}\dfrac{\theta^x}{x!}$, $x = 0, 1, \ldots$, H_0: $\theta = 1$ lawan H_1: $\theta < 1$.

Andaikan $W = \sum_{i=1}^{10} X_i \leq 4$ adalah rantau genting ujian ini berdasarkan sampel rawak bersaiz 10. Dapatkan ungkapan bagi fungsi kuasa ujian ini.

(25 markah)

Lampiran

Bab 5

1. $f_{X_1,X_2,\ldots,X_n}^{(x_1,x_2,\ldots,x_n)} = \prod_{i=1}^{n} f(x_i)$

2. $M_r = \frac{1}{n} \sum_{i=1}^{n} X_i^r$, $r = 1, 2, 3, \ldots$

3. $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

4. $M_r' = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^r$, $r = 1, 2, 3, \ldots$

5. $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

6. $\tilde{S}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$

7. $E[M_r] = E[X^r]$

8. $E[\bar{X}_n] = \mu$

9. $\mathrm{Var}(\bar{X}_n) = \sigma^2/n$

10. $\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2.$

11. $E\left[S_n^2\right] = \sigma^2.$

12. $M_{\bar{X}}(t) = [M(t/n)]^n$

13. $\underset{n\to\infty}{\mathrm{had}} P(|\bar{X}_n - \mu| < \varepsilon) = 1$

14. $\underset{n\to\infty}{\mathrm{had}} F_n = F$

15. $Z_n = \dfrac{\sqrt{n}\,(\overline{X}_n - \mu)}{\sigma}$

16. $\underset{n\to\infty}{\text{had}}\ F_n(z) = \Phi(z)$

17. $(X_1 - \overline{X})^2 = \left[\sum_{i=2}^{n}(X_i - \overline{X})\right]^2$

18. $\sum_{i=1}^{n} \dfrac{(X_i - \overline{X})^2}{\sigma^2} = (n-1)\dfrac{S_n^2}{\sigma^2}$

19. $G_1(y) = 1 - [1 - F(y)]^n$

20. $G_n(y) = [F(y)]^n$

21. $g_1(y) = n[1-F(y)]^{n-1} f(y)$

22. $g_n(y) = n\,[F(y)]^{n-1} f(y)$

23. $G_\alpha(y) = \sum_{j=\alpha}^{n} \binom{n}{j} [F(y)]^j [1 - F(y)]^{n-j}$

24. $g_\alpha(y) = \dfrac{n!}{(\alpha-1)!(n-\alpha)!}\ [F(y)]^{\alpha-1} f(y)\ [1-F(y)]^{n-\alpha}$

25. $g_{\alpha,\beta}(x,y) = \dfrac{n!}{(\alpha-1)!(\beta-\alpha-1)!(n-\beta)!}\ [F(x)]^{\alpha-1} f(x)$
$[F(y) - F(x)]^{\beta-\alpha-1} f(y)[1-F(y)]^{n-\beta},\ \alpha < \beta$

26. $g(y_1, y_2, \ldots, y_n) = n!\ \prod_{i=1}^{n} f(y_i)$

27. $\text{Median sampel} = \begin{cases} \frac{1}{2}(Y_{n/2}+Y_{(n+2)/2}), & \text{jika n genap} \\ Y_{(n+1)/2}, & \text{jika n ganjil} \end{cases}$

28. Julat sampel $= Y_n - Y_1$

29. Tengah julat sampel $= \frac{1}{2}(Y_1 + Y_n)$

30. $\underset{n\to\infty}{\text{had}}\, F_n(x) = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$

31. $P(|X_n - c| < \varepsilon) = 1,\ \varepsilon > 0$

32. $\underset{n\to\infty}{\text{had}}\, M_n(t) = M(t)$

33. $e^x = 1 + x + \frac{x^2}{2!} + \frac{e^z x^3}{3!},\ 0 < z < x$

34. $\underset{n\to\infty}{\text{had}}\left[1 + \frac{a}{n} + \frac{\psi(n)}{n}\right]^n = \underset{n\to\infty}{\text{had}}\left(1 + \frac{a}{n}\right)^n = e^a$, jika $\underset{n\to\infty}{\text{had}}\, \psi(n) = 0$

Bab 6

1. $L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i, \theta)$

2. $L(\theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2, \ldots, \theta_k)$

3. $\chi^2 = \sum_{i=1}^{k} \frac{[N_i - np_i(\theta)]^2}{np_i(\theta)}$

4. $E[T] = \tau(\theta)$

5. $E_\theta[\{T - \tau(\theta)\}^2] = \text{Var}(T) + \{E[T] - \tau(\theta)\}^2$

6. $\text{Var}(T) \geq \dfrac{[\tau'(\theta)]^2}{nE\left[\left\{\frac{\partial}{\partial\theta}\log f(X;\theta)\right\}^2\right]}$

7. $E\left[\left\{\frac{\partial}{\partial\theta}\log f(X;\theta)\right\}^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]$

8. $\underset{n\to\infty}{\text{had}}\ P_\theta\,[|T_n - \tau(\theta)| < \varepsilon] = 1,\ \varepsilon > 0$

9. $\underset{n\to\infty}{\text{had}}\ E_\theta\left[\{T_n - \tau(\theta)\}^2\right] = 0$

10. $f(x_1, x_2, ..., x_n; \theta) = g(t\,;\,\theta)\,h(x_1, x_2, ..., x_n)$

11. $f(x_1, x_2, ..., x_n; \theta) = g(t_1, t_2, ..., t_r; \theta)\,h(x_1, x_2, ..., x_n)$

12. $L(\theta;\ x_1,\ ..., x_n) = g(t; \theta)\,h(x_1, x_2, ..., x_n)$

13. $E[X] = E[E[X \mid Y = y]] = E[E[X \mid Y]]$

14. $\text{Var}\,(X \mid Y = y) = E[(X - E[X \mid y])^2 \mid y\,]$

15. $\text{Var}\,(X) = \text{Var}\,(E[X \mid Y]) + E[\text{Var}\,(X \mid Y)]$

16. $E[z(T)] = 0 \Rightarrow P[z(T) = 0] = 1$

17. $f(x; \theta) = a(\theta)\,b(x)\exp\,[c(\theta)\,d(x)]$

18. $f(x; \theta_1, ..., \theta_k) = a(\theta_1, ...\ \theta_k)\,b(x)\exp\,[c_1(\theta_1, ..., \theta_k)\,d_1(x) + ... + c_k(\theta_1, ..., \theta_k)\,d_k(x)]$

19. $f(x; \theta) = h(x - \theta)$

20. $f(x; \theta) = \frac{1}{\theta}h(x/\theta)$

21. $u(x_1 + c, x_2 + c, ..., x_n + c) = u(x_1, x_2, ..., x_n) + c.$

22. $u(cx_1, cx_2, ..., cx_n) = cu(x_1, x_2, ..., x_n)$

23. $u(X_1, X_2, ..., X_n) = \dfrac{\int \theta \prod_{i=1}^{n} f(X_i;\theta)\, d\theta}{\int \prod_{i=1}^{n} f(X_i;\theta)\, d\theta}$

Rumus-Rumus
JIM 312 - Teori Kebarangkalian

Modul 1

Pelajaran 1

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2. $P(A) = P(A \cap \bar{B}) + P(A \cap B)$

3. $P(\bar{A}) = 1 - P(A)$

4. $^{n}p_{r} = \dfrac{n!}{(n-r)!}$

5. $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

6. $N = \dfrac{n!}{n_1!\, n_2! \dots n_k!}$

Pelajaran 2

1. $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

2. $P(A \cap B) = P(A)P(B)$

3. $P(A) = P(A \mid B)\, P(B) + P(A \mid \bar{B})\, P(\bar{B})$

4. $P(B_i \mid A) = \dfrac{P(A \cap B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)}$

Pelajaran 3

1. $P(a \le X \le b) = \int_{o}^{b} f(x)\ dx$

2. $P(a < X < b) = \sum_{a<x<b} p(x)$

3. $F(t) = P(X \le t)$

4. $P(a < X \le b) = F(b) - F(a)$

5. $\frac{d}{dt} F(t) = f(t)$

6. $F_Y(t) = F_X(g^{-1}(t))$

7. $F_Y(t) = 1 - F_X(g^{-1}(t))$

8. $f_Y(t) = f_X(g^{-1}(t))\ |J|$

9. $J = \frac{dg^{-1}(t)}{dt}$

10. $f_Y(t) = \sum_{i=1}^{k} f_X\ (g_i^{-1}(t))\ |J_i|)$

11. $J_i = \frac{d}{dt}\ g_i^{-1}\ (t)$

12. $P_Y(y) = \sum_{x \in A} P_X(x)$

Modul 2

Pelajaran 1

1. $E(X) = \sum_{x \in \text{Julat } X} xp(x)$

2. $1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}\ ,\ |x| < 1$

3. $1 + 2x + \ldots + n\,x^{n-1} + \ldots = \frac{1}{(1-x)^2}\ ,\ |x| < 1$

4. $E(X) = \int_{-\infty}^{\infty} x\,f(x)\ dx$

5. $E(X) = \int_{0}^{\infty} [1 - f(x)]\ dx - \int_{-\infty}^{0} F(x)\ dx$

6. $E[G(X)] = \sum_{x \in \text{Julat } X} G(x)\ p(x)$

7. $E[G(X)] = \int_{-\infty}^{\infty} G(x)\, f(x)\ dx$

8. $E[c] = c$

9. $E[cX] = c\, E[X]$

10. $E[X + c] = E[X] + c$

11. $Var(X) = E[X - E[X]]^2$

12. $Var(X) = E[X^2] - \mu_X^2$

13. $Var(X) = \sum_{x \in \text{Julat } X} x^2 p(x) - \mu_X^2$

14. $Var(X) = \int_{-\infty}^{\infty} x^2 f(x)\ dx - \mu_X^2$

15. $Var(a) = 0$

16. $Var(aX + b) = a^2\, Var(X)$

17. $F_X(t_k) = k,\ 0 < k < 1$

Pelajaran 2

1. $m_k = E[X^k]$

2. $m_k = \sum_{x \in \text{Julat } X} x^k\, p(x)$

3. $m_k = \int_{-\infty}^{\infty} x^k f(x)\ dx$

4. $\mu_k = E[(X - \mu_X)^k]$

5. $\gamma_1 = \mu_3 / \sigma_X^3$

6. $\gamma_2 = \dfrac{\mu_4}{\sigma_X^4} - 3.$

7. $\mu_{[k]} = E[X(X-1)(X-2) \ldots (X-k+1)]$

8. $m(t) = E[e^{tX}]$

9. $m(t) = \sum_{x \in \text{Julat } X} e^{tx}\, p(x)$

10. $m(t) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx$

11. $m_Y(t) = E[e^{tg(X)}]$

12. $m_Y(t) = \sum_{x \in \text{Julat } X} e^{tg(x)}\, p(x)$

13. $m_Y(t) = \int_{-\infty}^{\infty} e^{tg(x)}\, f(x)\, dx$

14. $m_Y(t) = e^{bt}\, m_X(at)$

15. $m^{(i)}(0) = m_i$

16. $k(t) = \ell n\, m(t)$

17. $\psi(t) = E[t^X]$

18. $f(t) = \sum_{i=0}^{\infty} \dfrac{f^{(i)}(a)}{i!}(t-a)^i$

19. $\psi^{(i)}(0) = i!\, p(i)$

20. $P(|X| \geq a) < \dfrac{1}{a^2} E[X^2]$

21. $P(|X-\mu| \geq a\sigma) \leq \dfrac{1}{a^2}$

22. $P(|X-\mu| < a\sigma) \geq 1 - \dfrac{1}{a^2}$

23. $P(X \geq a) \leq \frac{E[X]}{a}$

24. $E[X^n] = \int_0^\infty nx^{n-1}(1 - F(x))\,dx$

Pelajaran 3

1. (i) $p(x) = \begin{cases} q, & x = 0 \\ p, & x = 1 \\ 0, & \text{ditempat lain} \end{cases}$ $\qquad X \sim \text{Bernoulli}(p)$

   (ii) $E[X] = p$

   (iii) $\text{Var}(X) = pq$

   (iv) $m(t) = q + pe^t$

2. (i) $p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{di tempat lain} \end{cases}$ $\qquad X \sim \text{Binomial}(n, p)$

   (ii) $E[X] = np$

   (iii) $\text{Var}(X) = npq$

   (iv) $m(t) = (q + pe^t)^n$

3. (i) $p(x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{di tempat lain} \end{cases}$ $\qquad X \sim \text{hipergeometri}(N, k, n)$

   (ii) $E[X] = \frac{nK}{N}$

   (iii) $\text{Var}(X) = \frac{nK(N-K)(N-n)}{N^2(N-1)}$

4. $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

5. (i) $p(x) = \begin{cases} q^{x-1}p \ , x = 1, 2, 3, \ldots \\ 0 \quad , \text{ di tempat lain} \end{cases}$ $X \sim$ geometri (p)

(ii) $E[X] = 1/p$

(iii) $Var(X) = q/p^2$

(iv) $m(t) = \dfrac{pe^t}{1 - qe^t}$

6. (i) $p(x) = \begin{cases} \binom{x-1}{r-1} p^r q^{x-r}, \ x=r, r+1, r+2 \\ \qquad r=2, 3, 4, \ldots \\ 0 \quad , \text{ di tempat lain} \end{cases}$ $X \sim$ negatif binomial (r, p)

(ii) $E[X] = r/p$

(iii) $Var(X) = rq/p^2$

(iv) $m(t) = \left[\dfrac{pe^t}{1-qe^t}\right]^r$

7. (i) $p(x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!}, \ x = 0, 1, 2, \ldots \\ 0 \quad , \text{ di tempat lain} \end{cases}$ $X \sim$ Poisson (λ)

(ii) $E[X] = \lambda$

(iii) $Var(X) = \lambda$

(iv) $m(t) = e^{\lambda(e^t - 1)}$

8. $\underset{x \to 0}{\text{had}} \ (1+x)^{1/x} = e$

9. $\underset{x \to \infty}{\text{had}} \left(1 + \dfrac{1}{x}\right)^x = e$

10. $\underset{x \to 0}{\text{had}} \ (1+ax)^{1/x} = e^a$

PELAJARAN 4

1. (i) $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{di tempat lain} \end{cases}$ $X \sim$ seragam (a, b)

   (ii) $E[X] = \frac{a+b}{2}$

   (iii) $Var(X) = \frac{(b-a)^2}{12}$

   (iv) $m(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$

2. (i) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < \infty$ $X \sim N(\mu, \sigma^2)$

   (ii) $E[X] = \mu$

   (iii) $Var(X) = \sigma^2$

   (iv) $m(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

3. $\underset{n\to\infty}{\text{had}}\ P\left[a \le \frac{S_n - np}{\sqrt{npq}} \le b\right] \to P(Z \ge a) - P(Z > b)$

4. $\underset{\lambda\to\infty}{\text{had}}\ P\left[a \le \frac{X-\lambda}{\sqrt{\lambda}} < b\right] \to P(Z > a) - P(Z \ge b)$

5. (i) $f(x) = \begin{cases} \lambda e^{-\lambda}, & x \ge 0 \\ 0, & \text{di tempat lain} \end{cases}$ $X \sim$ eksponen (λ)

   (ii) $E[X] = 1/\lambda$

   (iii) $Var(X) = 1/\lambda^2$

   (iv) $m(t) = \frac{\lambda}{\lambda - t}$

6. $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$

7. $\Gamma(n) = (n-1)\,\Gamma(n-1)$

8. $\Gamma(n) = (n-1)!$

9. (i) $f(x) = \begin{cases} \dfrac{\lambda^n x^{n-1}}{\Gamma(n)} e^{-\lambda x}, & x > 0 \\ 0, & \text{di tempat lain} \end{cases}$ $\quad X \sim \text{Gamma}(n, \lambda)$

(ii) $E[X] = n/\lambda$

(iii) $Var(X) = n/\lambda^2$

(iv) $m(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^n$

10. (i) $f(x) = \begin{cases} \dfrac{1}{2^{\upsilon/2}\Gamma\left(\dfrac{\upsilon}{2}\right)} x^{\upsilon/2-1} e^{-x/2}, & x > 0 \\ 0, & \text{di tempat lain} \end{cases}$ $\quad X \sim \chi^2_\upsilon$

(ii) $E[X] = \upsilon$

(iii) $Var(X) = 2\upsilon$

(iv) $m(t) = \left(\dfrac{1}{1-2t}\right)^{\upsilon/2}$

11. $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$

12. $B(x, y) = \int_0^\infty \dfrac{t^{x-1}}{(1+t)^{x+y}} \, dt$

13. $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

14. (i) $f(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & , 0 < x < 1 \\ 0 \ , & \text{di tempat lain} \end{cases}$ $\quad X \sim \text{Beta}(a, b)$

(ii) $F_X(p) = \sum_{x=a}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

(iii) $E[X] = \dfrac{a}{a+b}$

(iv) $\text{Var}(X) = \dfrac{ab}{(a+b+1)(a+b)^2}$

Modul 3

Pelajaran 1

1. $P(X \le x, \ Y \le y) = \sum_{t_1 \le x} \sum_{t_2 \le y} p(t_1, t_2)$

2. $P(X \le x, \ Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(t_1, t_2) \, dt_1 \, dt_2$

3. $F(x, y) = P(X \le x, \ Y \le y)$

4. $f(x, y) = \dfrac{\partial^2 F(x, y)}{dxdy}$

Pelajaran 2

1. $p(x) = \sum_{y} p(x, y)$

2. $p(y) = \sum_{x} p(x, y)$

3. $f(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$

4. $f(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$

5. $F(x) = F(x, \infty)$

6. $F(y) = F(\infty, y)$

7. $f(x) = \dfrac{\partial F(x, \infty)}{\partial x}$

8. $f(y) = \dfrac{\partial F(\infty, y)}{\partial y}$

9. $p(x \mid y) = \dfrac{p(x, y)}{p(y)}$

10. $f(x \mid y) = \dfrac{f(x, y)}{f(y)}$

11. $p(x, y) = p(x)\, p(y)$

12. $f(x, y) = f(x)\, f(y)$

Pelajaran 3

1. $E[g(X, Y)] = \sum_x \sum_y g(x, y)\, p(x, y)$

2. $E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\,dx\,dy$

3. $E[g_1(X, Y) + g_2(X, Y)] = E[g_1(X, Y)] + E[g_2(X, Y)]$

4. $E[h_1(X)\, h_2(Y)] = E[h_1(X)]\, E[h_2(Y)]$

5. (i) $\text{Cov}(X, Y) = E[X - \mu_X)(Y - \mu_Y)]$

   (ii) $\text{Cov}(X, Y) = E[XY] - \mu_X \mu_Y$

6. $\text{Cov}(aX, bY) = ab\, \text{Cov}(X, Y)$

7. $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y)$

8. $\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X, Y)$

9. $\rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

10. $E[g(X, Y) \mid Y = y] = \sum_{x} g(x, y)\, p(x \mid y)$

11. $E[g(X, Y) \mid Y = y] = \int_{-\infty}^{\infty} g(x, y) f(x \mid y)\, dx$

12. $E[E[X \mid Y = y]] = E[X]$

13. $E[E[Y \mid X = x]] = E[Y]$

14. $E[E[g(X) \mid Y = y]] = E[g(X)]$

15. $E[E[g(Y) \mid X = x]] = E[g(Y)]$

16. $\text{Var}(X \mid Y = y) = E[X^2 \mid Y = y] - (E[X \mid Y = y)^2$

17. $m(t_1, t_2) = E\left[e^{t_1 X_1 + t_2 X_2}\right]$

18. $m(t_1, t_2, \ldots, t_n) = E\left[e^{\sum_{i=1}^{n} t_i X_i}\right]$

19. $m(t_1) = \lim_{t_2 \to 0} m(t_1, t_2)$

20. $m(t_1, t_2, \ldots, t_n) = m(t_1)\, m(t_2) \ldots m(t_n)$

Pelajaran 4

1. (i) $p(x_1, x_2, \ldots, x_k) = \dfrac{n!}{x_1!\, x_2! \ldots x_k!}\, p_1^{x_1}\, p_2^{x_2} \ldots p_k^{x_k}$

   (ii) $p(x_i) = \dbinom{n}{x_i} p_i^{x_i} (1 - p_i)^{n - x_i}$

   (iii) $p(x_i, x_j) = \dfrac{n!}{x_i!\, x_j!\, (n - x_i - x_j)!}\, p_i^{x_i}\, p_j^{x_j} (1 - p_i - p_j)^{n - x_i - x_j}$

   (iv) $E[X_i X_j] = n(n-1)\, p_i p_j$

   (v) $\text{Cov}(X_i, X_j) = -np_i p_j$

2. (i) $$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\},$$

$$-\infty<x<\infty,\ -\infty<y<\infty$$

(ii) $$f(x\,|\,y)=\frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)\sigma_X^2}\left[x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right]^2\right\}$$

$$-\infty<x<\infty$$

(iii) $$m(t_1,t_2)=\exp\left[t_1\mu_X+t_2\mu_Y+\frac{1}{2}\left(t_1^2\sigma_X^2+2\rho t_1t_2\sigma_X\sigma_Y+t_2^2\sigma_Y^2\right)\right]$$

(iv) $E[XY] = \mu_X\mu_Y + \rho\,\sigma_X\sigma_Y$

(v) $\text{Cov}(X, Y) = \rho\,\sigma_X\sigma_Y$

Modul 4

Pelajaran 1

1. $M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$

2. $E[M_k] = m_k$

3. $\text{Var}(M_k) = \frac{1}{n}[m_{2k} - m_k)^2]$

4. $E[\bar{X}] = \mu$

5. $\text{Var}(\bar{X}) = \frac{1}{n}\sigma^2$

6. $S^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2$

7. $E[S^2] = \sigma^2$

8. $\mathrm{Var}(S^2) = \frac{1}{n}\left(\mu_4 - \frac{(n-3)}{(n-1)}\sigma^4\right)$

9. $\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2$

10. $\bar{X}-\mu = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)$

Pelajaran 2

1. $p(u,v) = p_{X,Y}\left(g_1^{-1}(u,v),\ g_2^{-1}(u,v)\right)$

2. $f(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right)\ |J|$

3. $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

4. $f(u,v) = \sum_{i=1}^{m} |J_i| f_{X,Y}\left(g_i^{-1}(u,v), h_i^{-1}(u,v)\right)$

5. $J_i = \begin{vmatrix} \frac{\partial g_i^{-1}(u,v)}{\partial u} & \frac{\partial g_i^{-1}(u,v)}{\partial v} \\ \frac{\partial h_i^{-1}(u,v)}{\partial u} & \frac{\partial h_i^{-1}(u,v)}{\partial v} \end{vmatrix}$

6. $m_{u,v}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 g(x,y)+t_2 h(x,y)} f(x,y)\,dx\,dy$

7. $m_u(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tg(x,y)} f(x,y)\,dx\,dy$

8. (i) $f_{u=X+Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(x, u-x)\, dx$

(ii) $f_{u=X+Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(u-y, y)\, dy$

9. (i) $f_{u=X-Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-u)\, dx$

(ii) $f_{u=X-Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(u+y, y)\, dy$

10. (i) $f_{u=XY}(u) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}(x, u/x)\, dx$

(ii) $f_{u=XY}(u) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}(u/y, y)\, dy$

11. $f_{u=X/Y}(u) = \int_{-\infty}^{\infty} |y|\, f_{X,Y}(uy, y)\, dy$

Pelajaran 3

1. (i) $f(x) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{\pi n}}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}, -\infty < x < \infty$ $\qquad X \sim t_n$

(ii) $T = \frac{Z}{\sqrt{V/n}}$

(iii) $E[X] = 0$

(iv) $\mathrm{Var}[X] = \frac{n}{n-2}$

2. (i) $$f(x) = \begin{cases} \dfrac{\Gamma[m+n)/2}{\Gamma(m/2)\Gamma(n/2)}\left(\dfrac{m}{n}\right)^{m/2}\dfrac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}}, x>0 \\ 0 \quad , \text{di tempat lain} \end{cases} \qquad X \sim F_{m,n}$$

(ii) $F = \dfrac{U/m}{V/m}$

(iii) $E[X] = \dfrac{n}{n-2}$

(iv) $\text{Var}(X) = \dfrac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$

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