
UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2002/2003

April/Mei 2003

JIM 414 – Pentaabiran Statistik

Masa : 3 jam

Sila pastikan bahawa kertas peperiksaan ini mengandungi DUA PULUH DUA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.

Jawab SEMUA soalan.

Setiap jawapan mesti dijawab di dalam buku jawapan yang disediakan.

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan bernilai 100 markah dan markah subsoalan diperlihatkan di penghujung subsoalan itu.

1. (a) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan $N\left(\mu,\sigma^2\right)$ dan $Z_1,\ldots,Z_n$ adalah sampel rawak daripada taburan $N(0,1)$. Takrifkan $S_z=\sqrt{\dfrac{\sum_{i=1}^{n} Z_i^2-n\bar{Z}^2}{n-1}}$ dan $\bar{Z}=\dfrac{1}{n}\sum_{i=1}^{n} Z_i$. Dapatkan taburan pembolehubah-pembolehubah berikut jika wujud:

   (i) X_1-X_2

   (ii) X_2+2X_3

   (iii) $\dfrac{X_1-X_2}{\sigma S_Z\sqrt{2}}$

   (iv) Z_1^2.

(50 markah)

   (b) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan seragam (0,1) berfungsi ketumpatan $f(x)=I_{(0,1)}(x)$. Andaikan Y_n = maks $\left(X_1,\ldots,X_n\right)$. Dapatkan taburan penghad bagi Y_n.

(20 markah)

   (c) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan Bernoulli (p) berfungsi ketumpatan $f(x)=p^x(1-p)^{1-x}I_{\{0,1\}}(x)$. Andaikan $W_n=\sum_{i=1}^{n} X_i$ dan $np=\mu$. Jika $p\to 0$ apabila $n\to\infty$, bagi $\mu>0$ yang ditetapkan, dapatkan taburan penghad bagi W_n.

(30 markah)

...3/-

2. Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan eksponen (θ) berfungsi ketumpatan $f(x;\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}}I_{(0,\infty)}(x)$, $\theta>0$.

   (a) Dapatkan penganggar kebolehjadian maksimum bagi θ.

   (b) Dapatkan penganggar kebolehjadian maksimum bagi $P(X\geq 1)$.

   (c) Dapatkan batas bawah Cramer-Rao bagi $\tau(\theta)=\theta$.

   (d) Bagaimanakah maklumat di dalam (c) dapat digunakan pada $\overline{X}$.

   (e) Nyatakan takrif am statistik cukup.

   (f) Diberikan $Y=\sum_{i=1}^{n}X_i$. Gunakan (e) untuk menunjukkan bahawa Y adalah statistik cukup.

(100 markah)

3. (a) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan $N(\mu,\sigma^2)$. Katakan $\overline{x}=19.3$ dan $n=16$. Apabila $\sigma^2=9$,

   (i) binakan selang-selang keyakinan satu sisi atas dan bawah 90% bagi μ.

   (ii) binakan selang keyakinan dua sisi 90% bagi μ.

   (iii) carikan saiz sampel yang baru supaya panjang selang keyakinan di dalam (ii) menjadi 2.

(50 markah)

(b) Andaikan $X_1,\dots,X_n$ adalah sampel rawak daripada taburan seragam $(0,\theta)$ berfungsi ketumpatan $f(x;\theta)=\frac{1}{\theta}I_{(0,\theta)}(x), \theta>0$. Andaikan $Y_n = \text{maks}(X_1,\dots,X_n)$.

(i) Dapatkan kebarangkalian θ berada di antara Y_n dan $2Y_n$.

(ii) Dapatkan pemalar c supaya $(Y_n,\ cY_n)$ adalah selang keyakinan $100(1-\alpha)\%$ bagi θ.

(50 markah)

4. (a) Pertimbangkan hipotesis ringkas $H_0:\theta=10$ lawan $H_1:\theta=11$, dengan θ adalah parameter pada taburan $N(\theta,16)$. Dua puluh lima cerapan dilakukan.

(i) Katakan rantau genting ujian ini diberikan oleh $\bar{X}\geq 11.316$, dapatkan kebarangkalian-kebarangkalian ralat jenis I dan II bagi ujian ini.

(ii) Katakan rantau genting ujian yang menjadi saingan kepada rantau genting di dalam (i) diberikan oleh $10<\bar{X}<\bar{x}_0$. Cari $\bar{x}_0$ supaya kebarangkalian ralat jenis I di dalam ujian saingan ini sama dengan kebarangkalian ralat jenis I di dalam (i).

(iii) Seterusnya dapatkan kebarangkalian ralat jenis II yang baru berdasarkan rantau genting di dalam (ii).

(iv) Apakah yang dapat disimpulkan tentang kedua-dua ujian yang berdasarkan kepada rantau-rantau genting yang berlainan tadi?

(50 markah)

(b) Diberikan $X_1,\dots,X_n$ adalah sampel daripada taburan $N(\mu,\sigma^2)$, σ^2 diketahui. Dapatkan ungkapan bagi λ di dalam ujian nisbah kebolehjadian bagi $H_0:\mu=\mu_0$ lawan $H_1:\mu\neq\mu_0$.

(20 markah)

(c) Andaikan sampel rawak daripada taburan eksponen (θ) berfungsi ketumpatan $f(x;\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}}I_{(0,\infty)}(x),\theta>0$. Bina ujian paling berkuasa secara seragam bersaiz α bagi $H_0:\theta=\theta_0$ lawan $H_1:\theta=\theta_1,\theta_1>\theta_0$.

(30 markah)

5. (a) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan Bernoulli (p) berfungsi ketumpatan $f(x)=p^x(1-p)^{1-x}I_{\{0,1\}}(x)$. Tunjukkan

$$\frac{\hat{p}-p}{\sqrt{p(1-p)/n}}\xrightarrow{d}Z\sim N(0,1).$$

(25 markah)

(b) Andaikan sampel rawak daripada taburan eksponen (θ) berfungsi ketumpatan $f(x;\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}}I_{(0,\infty)}(x),\theta>0$. Tunjukkan $T_1=1/\bar{X}$ adalah penganggar pincang bagi $\tau(\theta)=1/\theta$.

(25 marlah)

(c) Diberikan $\frac{\hat{p}_2-\hat{p}_1-(p_2-p_1)}{\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1+\hat{p}_2(1-\hat{p}_2)/n_2}}\xrightarrow{d}Z\sim N(0,1)$. Dapatkan selang keyakinan hampiran $100(1-\alpha)\%$ bagi p_2-p_1.

(25 markah)

(d) Andaikan $X_1,\ldots,X_n$ adalah sampel rawak daripada taburan $N(\mu,\sigma^2)$. Kita ingin menguji $H_0:\sigma^2=16$ lawan $H_1:\sigma^2>16$. Diberikan n = 15, $\alpha=0.1$ dan $\sigma^2=32$ apabila H_1 benar, dapatkan kuasa ujian ini.

(25 markah)

Lampiran

1. $\underset{n \to \infty}{had} F_n(z) = \Phi\,(\mathrm{z})$

2. $\mathrm{E[cX] = c\ E[X]}$

3. $\mathrm{Var\ (aX + b) = a^2\ Var\ (X)}$

4. $\mathrm{E}[\overline{X}] = \mu$

5. $\mathrm{Var}\,(\overline{X}) = \dfrac{\sigma^2}{n}$

6. $\mathrm{P}(\,|\,\mathrm{X} - \mu\,|\geq \varepsilon\,) \leq \dfrac{\sigma^2}{\varepsilon^2}$

7. $Z_n = \dfrac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$

8. $M_{\sum X_i}(t) = [M_X(t)]^n$

9. $M_{\overline{X}}(t) = [M(t/n)]^n$

10. $g_1(y) = n[1 - F(y)]^{n-1} f(y)$

11. $g_\alpha(y) = \dfrac{n!}{(\alpha - 1)!(n-\alpha)!}[F(y)]^{\alpha-1} f(y)[1-F(y)]^{n-\alpha}$

12. $g_{\alpha,\beta}(x,y) = \frac{n!}{(\alpha-1)!(\beta-\alpha-1)!(n-\beta)!}[F(x)]^{\alpha-1} f(x)[F(y)-F(x)]^{\beta-\alpha-1} f(y)[1-F(y)]^{n-\beta}$, $\alpha < \beta$

13. $g_n(y) = n[F(y)]^{n-1} f(y)$

14. $f_Y(t) = f_X[g^{-1}(t)]\,|J|$

15. $J = \frac{dg^{-1}(t)}{dt}$

16. $L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i, \theta)$

17. $f(x; \theta) = a(\theta)\, b(x) \exp[c(\theta)\, d(x)]$

18. $\text{Var}(T) \geq \frac{[\tau'(\theta)]^2}{nE[\{\frac{\partial}{\partial\theta}\log f(x;\theta)\}^2]}$

19. $E[\{\frac{\partial}{\partial\theta}\log f(x;\theta)\}^2] = -E[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)]$

Rumus-Rumus

Modul 1

Pelajaran 1

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2. $P(A) = P(A \cap \bar{B}) + P(A \cap B)$

3. $P(\bar{A}) = 1 - P(A)$

4. $^{n}P_{r} = \frac{n!}{(n-r)!}$

5. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

6. $N = \frac{n!}{n_1!\, n_2! \dots n_k!}$

Pelajaran 2

1. $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

2. $P(A \cap B) = P(A)P(B)$

3. $P(A) = P(A \mid B)\, P(B) + P(A \mid \bar{B})\, P(\bar{B})$

4. $P(B_i \mid A) = \frac{P(A \cap B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)}$

Pelajaran 3

1. $P(a \leq X \leq b) = \int_{o}^{b} f(x)\, dx$

2. $P(a < X < b) = \sum_{a<x<b} p(x)$

3. $F(t) = P(X \leq t)$

4. $P(a < X \leq b) = F(b) - F(a)$

5. $\frac{d}{dt} F(t) = f(t)$

6. $F_Y(t) = F_X(g^{-1}(t))$

7. $F_Y(t) = 1 - F_X(g^{-1}(t))$

8. $f_Y(t) = f_X(g^{-1}(t)) \mid J \mid$

9. $J = \frac{dg^{-1}(t)}{dt}$

10. $f_Y(t) = \sum_{i=1}^{k} f_X \ (g_i^{-1}(t)) \mid J_i \mid)$

11. $J_i = \frac{d}{dt} g_i^{-1} (t)$

12. $P_Y(y) = \sum_{x \in A} P_X(x)$

Modul 2

Pelajaran 1

1. $E(X) = \sum_{x \in \text{Julat } X} xp(x)$

2. $1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}, \ \mid x \mid < 1$

3. $1 + 2x + \dots + n x^{n-1} + \dots = \frac{1}{(1-x)^2}, \ \mid x \mid < 1$

4. $E(X) = \int_{-\infty}^{\infty} x f(x) \ dx$

5. $E(X) = \int_{0}^{\infty} [1 - f(x)] \ dx - \int_{-\infty}^{0} F(x) \ dx$

6. $E[G(X)] = \sum_{x \in \text{Julat } X} G(x) \ p(x)$

7. $E[G(X)] = \int_{-\infty}^{\infty} G(x) f(x)\, dx$

8. $E[c] = c$

9. $E[cX] = c\,E[X]$

10. $E[X + c] = E[X] + c$

11. $\mathrm{Var}(X) = E[X - E[X]]^2$

12. $\mathrm{Var}(X) = E[X^2] - \mu_X^2$

13. $\mathrm{Var}(X) = \sum_{x \in \text{Julat } X} x^2 p(x) - \mu_X^2$

14. $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu_X^2$

15. $\mathrm{Var}(a) = 0$

16. $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

17. $F_X(t_k) = k,\ 0 < k < 1$

Pelajaran 2

1. $m_k = E[X^k]$

2. $m_k = \sum_{x \in \text{Julat } X} x^k p(x)$

3. $m_k = \int_{-\infty}^{\infty} x^k f(x)\, dx$

4. $\mu_k = E[(X - \mu_X)^k$

5. $\gamma_1 = \mu_3 / \sigma_X^3$

6. $\gamma_2 = \frac{\mu_4}{\sigma_X^4} - 3.$

7. $\mu_{[k]} = E[X(X-1)(X-2) \ldots (X-k+1)]$

8. $m(t) = E[e^{tX}]$

9. $m(t) = \sum_{x \in \text{Julat } X} e^{tx} p(x)$

10. $m(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

11. $m_Y(t) = E[e^{tg(X)}]$

12. $m_Y(t) = \sum_{x \in \text{Julat } X} e^{tg(x)} p(x)$

13. $m_Y(t) = \int_{-\infty}^{\infty} e^{tg(x)} f(x) dx$

14. $m_Y(t) = e^{bt} m_X(at)$

15. $m^{(i)}(0) = m_i$

16. $k(t) = \ell n\, m(t)$

17. $\psi(t) = E[t^X]$

18. $f(t) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (t-a)^i$

19. $\psi^{(i)}(0) = i!\, p(i)$

20. $P(|X| \geq a) < \frac{1}{a^2} E[X^2]$

21. $P(|X - \mu| \geq a\sigma) \leq \frac{1}{a^2}$

22. $P(|X - \mu| < a\sigma) \geq 1 - \frac{1}{a^2}$

23. $P(X \geq a) \leq \frac{E[X]}{a}$

24. $E[X^n] = \int_0^{\infty} nx^{n-1} (1 - F(x)) dx$

Pelajaran 3

1. (i) $p(x) = \begin{cases} q, & x = 0 \\ p, & x=1 \\ 0, & \text{ditempat lain} \end{cases}$ X ~ Bernoulli (p)

   (ii) $E[X] = p$

   (iii) $\text{Var}(X) = pq$

   (iv) $m(t) = q + pe^t$

2. (i) $p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x=0, 1, 2,\ldots, n \\ 0, & \text{di tempat lain} \end{cases}$ X ~ Binomial (n, p)

   (ii) $E[X] = np$

   (iii) $\text{Var}(X) = npq$

   (iv) $m(t) = (q + pe^t)^n$

3. (i) $p(x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, & x=0, 1, 2,\ldots, n \\ 0, & \text{di tempat lain} \end{cases}$ X ~ hipergeometri (N, k, n)

   (ii) $E[X] = \dfrac{nK}{N}$

   (iii) $\text{Var}(X) = \dfrac{nK(N-K)(N-n)}{N^2(N-1)}$

4. $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

5. (i) $p(x) = \begin{cases} q^{x-1}p\ , & x = 1, 2, 3, \ldots \\ 0\ , & \text{di tempat lain} \end{cases}$ $X \sim$ geometri (p)

(ii) $E[X] = 1/p$

(iii) $\mathrm{Var}(X) = q/p^2$

(iv) $m(t) = \dfrac{pe^t}{1 - qe^t}$

6. (i) $p(x) = \begin{cases} \dbinom{x-1}{r-1} p^r q^{x-r}, & x=r, r+1, r+2 \\ & r=2, 3, 4, \ldots \\ 0\ , & \text{di tempat lain} \end{cases}$ $X \sim$ negatif binomial (r, p)

(ii) $E[X] = r/p$

(iii) $\mathrm{Var}(X) = rq/p^2$

(iv) $m(t) = \left[\dfrac{pe^t}{1 - qe^t}\right]^r$

7. (i) $p(x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!}, & x = 0, 1, 2, \ldots \\ 0\ , & \text{di tempat lain} \end{cases}$ $X \sim$ Poisson (λ)

(ii) $E[X] = \lambda$

(iii) $\mathrm{Var}(X) = \lambda$

(iv) $m(t) = e^{\lambda(e^t - 1)}$

8. $\underset{x \to 0}{\text{had}}\ (1 + x)^{1/x} = e$

9. $\underset{x \to \infty}{\text{had}} \left(1 + \dfrac{1}{x}\right)^x = e$

10. $\underset{x \to 0}{\text{had}}\ (1 + ax)^{1/x} = e^a$

Pelajaran 4

1. (i) $f(x)=\begin{cases}\dfrac{1}{b-a}, & a<x<b\\ 0, & \text{di tempat lain}\end{cases}$ $X \sim$ seragam (a, b)

   (ii) $E[X] = \dfrac{a+b}{2}$

   (iii) $\text{Var}(X) = \dfrac{(b-a)^2}{12}$

   (iv) $m(t) = \dfrac{e^{bt}-e^{at}}{t(b-a)}$

2. (i) $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < \infty$ $X \sim N(\mu, \sigma^2)$

   (ii) $E[X] = \mu$

   (iii) $\text{Var}(X) = \sigma^2$

   (iv) $m(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

3. $\underset{n\to\infty}{\text{had}}\, P\left[a \le \dfrac{S_n - np}{\sqrt{npq}} \le b\right] \to P(Z \ge a) - P(Z > b)$

4. $\underset{\lambda\to\infty}{\text{had}}\, P\left[a \le \dfrac{X-\lambda}{\sqrt{\lambda}} < b\right] \to P(Z > a) - P(Z \ge b)$

5. (i) $f(x)=\begin{cases}\lambda e^{-\lambda}, & x \ge 0\\ 0, & \text{di tempat lain}\end{cases}$ $X \sim$ eksponen (λ)

   (ii) $E[X] = 1/\lambda$

   (iii) $\text{Var}(X) = 1/\lambda^2$

   (iv) $m(t) = \dfrac{\lambda}{\lambda - t}$

6. $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

7. $\Gamma(n) = (n-1)\Gamma(n-1)$

8. $\Gamma(n) = (n-1)!$

9. (i) $f(x) = \begin{cases} \dfrac{\lambda^n x^{n-1}}{\Gamma(n)} e^{-\lambda x}, & x > 0 \\ 0 & , \text{ di tempat lain} \end{cases}$ $\qquad X \sim \text{Gamma}(n, \lambda)$

   (ii) $E[X] = n/\lambda$

   (iii) $\text{Var}(X) = n/\lambda^2$

   (iv) $m(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^n$

10. (i) $f(x) = \begin{cases} \dfrac{1}{2^{\upsilon/2}\Gamma\left(\dfrac{\upsilon}{2}\right)} x^{\upsilon/2-1} e^{-x/2} & , x > 0 \\ 0 \ , & \text{di tempat lain} \end{cases}$ $\qquad X \sim \chi^2_\upsilon$

    (ii) $E[X] = \upsilon$

    (iii) $\text{Var}(X) = 2\upsilon$

    (iv) $m(t) = \left(\dfrac{1}{1-2t}\right)^{\upsilon/2}$

11. $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$

12. $B(x, y) = \int_0^\infty \dfrac{t^{x-1}}{(1+t)^{x+y}} dt$

13. $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

14. (i) $f(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & , 0 < x < 1 \\ 0 \ , & \text{di tempat lain} \end{cases}$ $\quad X \sim \text{Beta}(a, b)$

(ii) $F_X(p) = \sum_{x=a}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

(iii) $E[X] = \dfrac{a}{a+b}$

(iv) $\text{Var}(X) = \dfrac{ab}{(a+b+1)(a+b)^2}$

Modul 3

Pelajaran 1

1. $P(X \leq x, \ Y \leq y) = \sum_{t_1 \leq x} \sum_{t_2 \leq y} p(t_1, t_2)$

2. $P(X \leq x, \ Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(t_1, t_2) \, dt_1 \, dt_2$

3. $F(x, y) = P(X \leq x, \ Y \leq y)$

4. $f(x, y) = \dfrac{\partial^2 F(x, y)}{dxdy}$

Pelajaran 2

1. $p(x) = \sum_{y} p(x, y)$

2. $p(y) = \sum_{x} p(x, y)$

3. $f(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$

4. $f(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$

5. $F(x) = F(x, \infty)$

6. $F(y) = F(\infty, y)$

7. $f(x) = \dfrac{\partial F(x, \infty)}{\partial x}$

8. $f(y) = \dfrac{\partial F(\infty, y)}{\partial y}$

9. $p(x \mid y) = \dfrac{p(x, y)}{p(y)}$

10. $f(x \mid y) = \dfrac{f(x, y)}{f(y)}$

11. $p(x, y) = p(x)\, p(y)$

12. $f(x, y) = f(x)\, f(y)$

Pelajaran 3

1. $E[g(X, Y)] = \sum_{x} \sum_{y} g(x, y)\, p(x, y)$

2. $E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\,dx\,dy$

3. $E[g_1(X, Y) + g_2(X, Y)] = E[g_1(X, Y)] + E[g_2(X, Y)]$

4. $E[h_1(X)\, h_2(Y)] = E[h_1(X)]\, E[h_2(Y)]$

5. (i) $\text{Cov}(X, Y) = E[X - \mu_X)(Y - \mu_Y)]$

   (ii) $\text{Cov}(X, Y) = E[XY] - \mu_X \mu_Y$

6. $\text{Cov}(aX, bY) = ab\, \text{Cov}(X, Y)$

7. $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\, \text{Cov}(X, Y)$

8. $\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X, Y)$

9. $\rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

10. $E[g(X, Y) \mid Y = y] = \sum_{x} g(x, y)\, p(x \mid y)$

11. $E[g(X, Y) \mid Y = y] = \int_{-\infty}^{\infty} g(x,y) f(x|y)\, dx$

12. $E[E[X \mid Y = y]] = E[X]$

13. $E[E[Y \mid X = x]] = E[Y]$

14. $E[E[g(X) \mid Y = y]] = E[g(X)]$

15. $E[E[g(Y) \mid X = x]] = E[g(Y)]$

16. $\text{Var}(X \mid Y = y) = E[X^2 \mid Y = y] - (E[X \mid Y = y)^2$

17. $m(t_1, t_2) = E\left[e^{t_1 X_1 + t_2 X_2}\right]$

18. $m(t_1, t_2, ..., t_n) = E\left[e^{\sum_{i=1}^{n} t_i X_i}\right]$

19. $m(t_1) = \lim_{t_2 \to 0} m(t_1, t_2)$

20. $m(t_1, t_2, ..., t_n) = m(t_1)\, m(t_2) \ldots m(t_n)$

Pelajaran 4

1. (i) $p(x_1, x_2, ..., x_k) = \dfrac{n!}{x_1!\, x_2! \ldots x_k!}\, p_1^{x_1}\, p_2^{x_2} \ldots p_k^{x_k}$

   (ii) $p(x_i) = \dbinom{n}{x_i} p_i^{x_i}(1-p_i)^{n-x_i}$

   (iii) $p(x_i, x_j) = \dfrac{n!}{x_i!\,x_j!\,(n - x_i - x_j)!}\, p_i^{x_i}\, p_j^{x_j}\,(1 - p_i - p_j)^{n-x_i-x_j}$

   (iv) $E[X_i X_j] = n(n-1)\, p_i p_j$

   (v) $\text{Cov}(X_i, X_j) = -np_i p_j$

2\. (i) $$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\},$$

$$-\infty < x < \infty, -\infty < y < \infty$$

(ii) $$f(x|y) = \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_X^2}\left[x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right]^2\right\}$$

$$-\infty < x < \infty$$

(iii) $$m(t_1,t_2) = \exp\left[t_1\mu_X + t_2\mu_Y + \frac{1}{2}\left(t_1^2\sigma_X^2 + 2\rho t_1t_2\sigma_X\sigma_Y + t_2^2\sigma_Y^2\right)\right]$$

(iv) $E[XY] = \mu_X\mu_Y + \rho\,\sigma_X\sigma_Y$

(v) $\mathrm{Cov}(X, Y) = \rho\,\sigma_X\sigma_Y$

Modul 4

Pelajaran 1

1\. $M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$

2\. $E[M_k] = m_k$

3\. $\mathrm{Var}(M_k) = \frac{1}{n}[m_{2k} - m_k)^2]$

4\. $E[\bar{X}] = \mu$

5\. $\mathrm{Var}(\bar{X}) = \frac{1}{n}\sigma^2$

6\. $S^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2$

7. $E[S^2] = \sigma^2$

8. $\mathrm{Var}(S^2) = \frac{1}{n}\left(\mu_4 - \frac{(n-3)}{(n-1)}\sigma^4\right)$

9. $\sum_{i=1}^{n}(X_i - \mu)^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$

10. $\bar{X} - \mu = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$

Pelajaran 2

1. $p(u, v) = p_{X,Y}\left(g_1^{-1}(u, v),\ g_2^{-1}(u, v)\right)$

2. $f(u, v) = f_{X,Y}\left(g_1^{-1}(u, v), g_2^{-1}(u, v)\right) \mid J \mid$

3. $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

4. $f(u,v) = \sum_{i=1}^{m} |J_i| f_{X,Y}\left(g_i^{-1}(u,v), h_i^{-1}(u,v)\right)$

5. $J_i = \begin{vmatrix} \frac{\partial g_i^{-1}(u,v)}{\partial u} & \frac{\partial g_i^{-1}(u,v)}{\partial v} \\ \frac{\partial h_i^{-1}(u,v)}{\partial u} & \frac{\partial h_i^{-1}(u,v)}{\partial v} \end{vmatrix}$

6. $m_{u,v}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 g(x,y) + t_2 h(x,y)} f(x,y)\,dx\,dy$

7. $m_u(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tg(x,y)} f(x,y)\,dx\,dy$

8. (i) $f_{u=X+Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(x, u-x)\,dx$

(ii) $f_{u=X+Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(u-y, y)\,dy$

9. (i) $f_{u=X-Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-u)\,dx$

(ii) $f_{u=X-Y}(u) = \int_{-\infty}^{\infty} f_{X,Y}(u+y, y)\,dy$

10. (i) $f_{u=XY}(u) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}(x, u/x)\,dx$

(ii) $f_{u=XY}(u) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}(u/y, y)\,dy$

11. $f_{u=X/Y}(u) = \int_{-\infty}^{\infty} |y|\, f_{X,Y}(uy, y)\,dy$

Pelajaran 3

1. (i) $f(x) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{\pi n}}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}, -\infty < x < \infty$ $X \sim t_n$

(ii) $T = \frac{Z}{\sqrt{V/n}}$

(iii) $E[X] = 0$

(iv) $\text{Var}[X] = \frac{n}{n-2}$

2. (i) $$f(x)=\begin{cases}\dfrac{\Gamma[m+n)/2}{\Gamma(m/2)\Gamma(n/2)}\left(\dfrac{m}{n}\right)^{m/2}\dfrac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}}, x>0 \\ 0 \quad \text{, di tempat lain}\end{cases} \qquad X \sim F_{m,n}$$

(ii) $F = \dfrac{U/m}{V/m}$

(iii) $E[X] = \dfrac{n}{n-2}$

(iv) $\text{Var}(X) = \dfrac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$

Senarai Rumus Tambahan

1. $$\sum_{x=1}^{N} x = \frac{N(N+1)}{2}$$

2. $$\sum_{x=1}^{N} x^2 = \frac{N(N+1)(2N+1)}{6}$$

3. Diberikan $S^2 = \dfrac{1}{(n-1)}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$. Jika $X_1, X_2, \ldots, X_n$ adalah sampel rawak daripada taburan sebarang normal, maka $\dfrac{(n-1)S^2}{\sigma^2}$ tertabur secara χ^2_{n-1}.

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