SAID-BALL CUBIC TRANSITION CURVE AND ITS APPLICATION TO SPUR GEAR DESIGN

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by

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DEDICATION

One of the joys of this completion is to look over the journey past and remember the most important people in my life, my family, to whom this thesis is dedicated to, who have helped and supported me along this long but fulfilling road.

To my wife, Hafizah, who has been a source of motivation and strengths during moments of despair and discouragement. Thank you for making the time to read, comment and proof-read the thesis. For all that, thank you!

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LENGKUNG PERALIHAN KUBIK SAID-BALL DAN PENGGUNAANNYA DALAM REKA BENTUK GEAR BINTANG

ABSTRAK

Tidak dinafikan, gear adalah merupakan antara elemen yang paling banyak digunakan dalam pemesinan dan industri. Kajian lepas telah menunjukkan bahawa lengkung berbentuk involut adalah profil yang sering digunakan di dalam mereka bentuk gigi gear bintang, yang dibangunkan berdasarkan teori-teori penghampiran seperti penghampiran Chebyshev dan kaedah menyurih titik. Walau bagaimanapun, kaedah yang digunakan ini tidak jitu dan hanya menumpukan kepada konsep-konsep penghampiran sahaja. Pengurangan bunyi gear dan kekuatan gigi gear sentiasa menjadi tumpuan kajian dan eksperimen terutamanya dengan pengubahsuaian bentuk gigi atau profilnya. Oleh itu, kajian ini adalah untuk mereka bentuk lengkung peralihan S dan C dengan menggunakan lengkung kubik Said-Ball berdasarkan templet bulatan bagi kes ketiga dan kelima dengan beberapa pembuktian matematik. Salah satu objektif kajian ini adalah untuk menyiasat keupayaan model lengkung berbentuk S dan C dalam mengurangkan tahap bunyi atau bising melalui eksperimen akustik. Dalam kajian ini, gear bintang dipilih sebagai kajian kes kerana ia adalah gear asas dan hakikatnya ia mudah untuk dibina dan dibuat. Berdasarkan eksperimen dan simulasi yang dijalankan, keputusan menunjukkan bahawa dengan menggunakan lengkung kubik Said-Ball, teori-teori bagi lengkung peralihan S dan C telah berjaya dibangunkan. Lengkung-lengkung ini telah terbukti secara matematik, dengan menggunakan ujian terbitan kedua, ujian kelekukan dan teorem Kneser. Ia juga mendedahkan bahawa lengkung peralihan S dan C telah berjaya digunakan dalam

mereka bentuk gigi gear bintang. Ini membuktikan juga bahawa model pepejal bagi gear bintang dapat dibangunkan melalui penggunaan berintegrasi antara perisian matematik dan CAD. Apabila diukur melalui analisis statik linear, analisis kelesuan dan DE, kebolehan reka bentuk yang dicadangkan beserta bahan, AISI 304, menunjukkan bahawa polinomial Newton interpolasi peringkat pertama boleh digunakan sebagai peramal kelesuan bagi semua model reka bentuk. Kaedah reka bentuk gigi baru iaitu lengkung-lengkung berbentuk C dan S adalah kaedah yang boleh diterima pakai di dalam mereka bentuk gigi gear bintang di mana kedua-dua kaedah ini telah membentangkan DE yang lebih besar daripada 85% keberkesanan reka bentuk. Semua model juga telah berjaya diukur melalui analisis dinamik dan akustik. Model berbentuk C telah terbukti mempunyai sesaran yang paling rendah berbanding model peralihan S dan EM. Dengan menggunakan model ini, kebisingan gear atau bunyinya terbukti boleh dikurangkan secara konsisten. Model berbentuk C juga lebih dipercayai daripada model-model lain yang menepati PS. Pengubahsuaian profil gigi terbukti sebagai faktor utama dalam mengurangkan kebisingan gear atau bunyi secara signifikan dan konsisten. Sumbangan kajian ini akan memberi manfaat kepada pereka atau pembuat dalam mereka bentuk profil gear bintang di mana lengkung yang disebutkan di atas boleh digunakan sebagai kaedah alternatif bagi profil gear ini. Untuk penyelidikan masa depan, keupayaan lengkung peralihan C boleh lagi diterokai dalam mereka bentuk model-model aerodinamik contohnya, kereta, kereta api berkelajuan tinggi, peluru dan lain-lain. Kajian mengenai reka bentuk gear juga boleh diteruskan lagi dengan jenis gear yang lain seperti gear heliks herringbone.

SAID-BALL CUBIC TRANSITION CURVE AND ITS APPLICATION TO SPUR GEAR DESIGN

ABSTRACT

Undoubtedly, gears are some of the most widely used elements in consumer and industrial machineries. Past studies have shown that an involute curve is the most common profile used in designing the spur gear tooth, developed based on the approximation theories such as Chebyshev approximation and the tracing points method. However, these employed methods are not accurate (or inexact) and only focusing on the approximation concepts. Gear noise reduction and tooth strength are continually being the focus of exploration and experimentation particularly the modification of the tooth shape or tooth profile. Therefore, this study is to design the S and C-shaped transition curves using Said-Ball cubic curve based on the third and fifth cases of circle to circle templates with some mathematical proofs. One of the objectives is to investigate the capability of this proposed S and C-shaped model in reducing sound or noise level through an acoustic experiment. In this study, spur gear is chosen as a case model due to its fundamental gear and the fact that it is simple to construct and manufacture. Based on the conducted experiment and simulation, results show that by using Said-Ball cubic curve, the theories of S and C-shaped transition curves have successfully developed. These curves have been mathematically proven, by using the concavity and second derivative tests and also Kneser's theorem. It is also revealed that S and C-shaped transition curves can be applied successfully in designing spur gear tooth. This proves that the solid model of spur gear can also be developed through the integrated use of mathematical and CAD

software. When measured through linear static analysis, fatigue analysis and DE, the applicability of the proposed design and the material, AISI 304 shows that First-order Newton interpolating polynomial can be employed as a fatigue predictor for all design models. The new teeth design methods, S and C-shaped curves are the acceptable methods in designing the spur gear teeth where both methods have presented DE greater than 85% of the design effectiveness. All models have also been successfully measured via dynamic and acoustic response analyses. C-shaped model has been proven to have the lowest displacement when compared to S-shaped (transition) and EM models. By utilizing this model, it is proven that gear noise or sound can be reduced consistently. C-shaped model is more reliable than other models in accordance to PS. It is proven that tooth profile modification is the main factor in reducing sound or noise in a very significant and consistent way. The contribution of this study will be beneficial to the designers or manufacturers in designing the spur gear profiles where the above-mentioned curves can be applied as an alternate method of these profiles. For future research, the capability of C transition curve can be further explored in designing aerodynamic models for example, car, high-speed train, bullet etc. Study on gear design can also be further explored on other type of gears such as helical herringbone gear.

CHAPTER 1

INTRODUCTION

1.1 History of CAGD

Lately, Computer-Aided Geometric Design (CAGD) plays a major role in the world of design. CAGD or in another term called, Geometric Modelling is a research field geared toward the development and representation of freeform curves, surfaces or volumes (Làvička, 2011). CAGD is a new field, originally created to bring some great benefits of computers to industries such as automotive, aerospace, shipbuilding or in various applications. Historically, CAGD emerged in the middle of 1970s. Barnhill and Riesenfeld (1974) can be claimed as the early pioneers in this field as they have organized a conference on this field which was held at the University of Utah, USA, in 1974. The main objective of this conference was primarily to discuss the aspect of CAGD that has attracted a great number of international researchers around Europe and USA to participate, for the first time, in this conference.

Today, CAGD becomes a sovereign discipline in its own right (Làvička, 2011). After the success of this first conference, research findings from the conference have been produced in various forms including the first textbook on CAGD entitled "Computational Geometry for Design and Manufacture" by Faux and Pratt (1988) and also first journal published focusing on the field of CAGD (Figure 1.1). Both publications have been used as the major references for many students and young researchers who want to understand further or to study CAGD. Earlier, de Casteljau (1959) and Coons (1964) have constructed the fundamental aspects of CAGD before a conference which was related to the use of CAGD in automotive industries was organized by a French man, Bézier, in 1971 (Farin, 2002).

(a) (b)

Figure 1.1: (a) The cover of the first journal on CAGD published by Barnhill and Boehm in 1984 (Farin, 1992); (b) is an illustration of "Uccello's Chalice" used for the cover (Talbot, 2006)

CAGD deals with mathematical expressions to control the shapes when designing curves and surfaces. Several essential mathematical concepts are fully utilized in this control such as geometry, vector, coordinate system and some basic knowledge of calculus. Shapes or profiles are typically produced by related parametric equations (functions). Abstractly, a parametric equation can be defined as a method to determine the relationship amongst equations or functions using independent variables (parameters) (Thomas et al., 1988). One of the most common functions that is always used in this field is Bézier function, normally in cubic but can be represented either in quadratic or in any degree. As for an intention of smoothness or visually pleasant curves and surfaces, the idea of continuity is then applied.

Hence, control points (in coordinate form) are highly needed to design curves or surfaces completely. Rockwood and Chambers (1996) explained that control points are points in two or more dimensions, which can define the behaviour of the resulting curve. Figure 1.2 shows a generated curve using four control points with the incorporation of cubic Bézier function and GC^1 continuity (Sarfraz, 2008). Hazewinkel (1997) concludes that GC^1 continuity can be classified as tangent (G^1) continuity. This GC^1 continuity has been utilized in designing an airplane wing (Brakhage and Lamby, 2005).

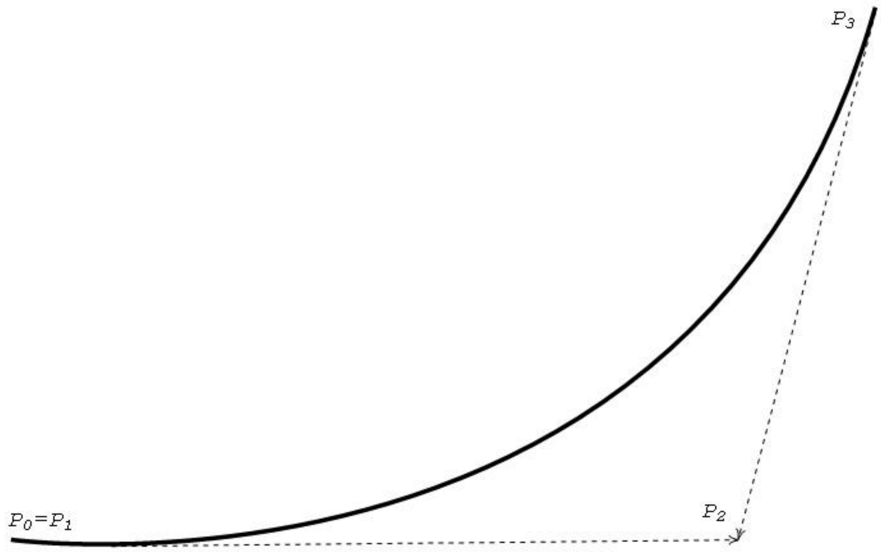
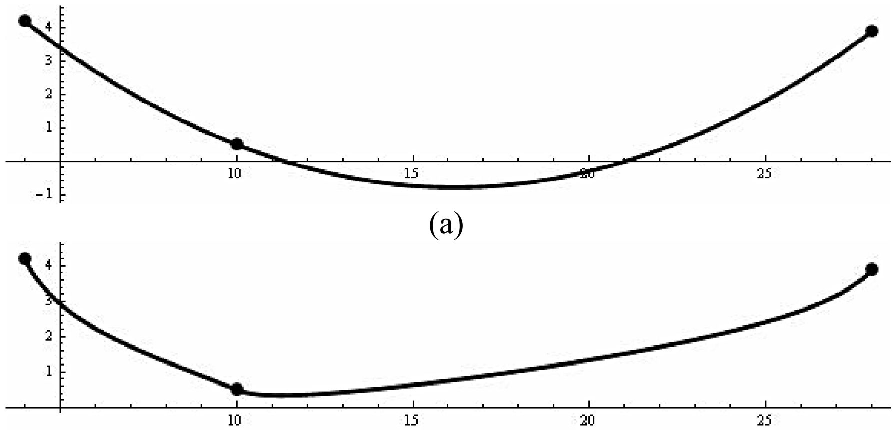


Figure 1.2: An example of curve design with GC^1 continuity

Nth-order parametric continuity (C^n) with $n = 0, 1, 2, 3, ..., k$ are the well-known smoothness properties in shape preserving or in interpolation problem. Figure 1.3 depicts the use of C^1 continuity in preserving a shape between curves. These concepts of CAGD will be further discussed in chapter 2.



(a)

(b)

Figure 1.3: Both curves apply C^1 continuity, (a) non-positivity and (b) positivity preserving

1.2 Early Applications

Over the past decades, CAGD has been expanded rapidly in the fields of automobile, aircraft, aerospace or in ship industries. Nowacki (2000) found that the ship's ribs, introduced during Roman Empire in 13^{th} century, were the earliest geometry application in free-form shapes. Several curves, such as splines have been recorded in these ribs. The revolution is then continued by Liming (1944) and Coon (1947) who proposed conic construction in aeronautics manufacture (aircraft) design. Farin (1992) defined the conic as a perspective projection of a parabola in Euclidean three space into a plane. Figure 1.4 depicts these conics in aircraft.

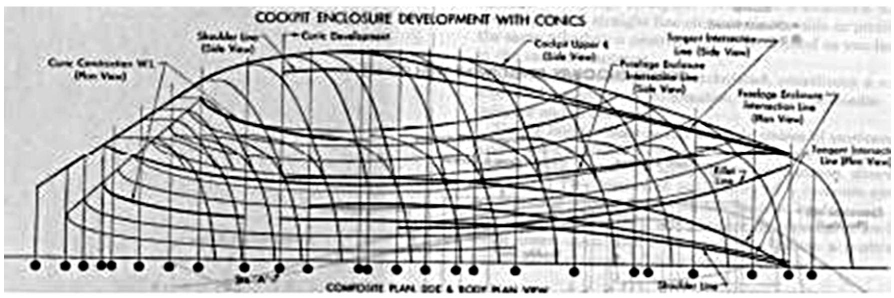


Figure 1.4: Conics as represented in cockpit development (Liming, 1944)

The industrial companies such as General Motors has hired de Boor and Gordon to develop its first CAD/CAM system called DAC-1 (Farin, 2002). The fundamental of curves and surfaces is used in this system. Meanwhile, Citroen and Renault, two famous French car companies, are mainly utilizing the CAGD knowledge discovered by de Casteljau in 1959 and Bézier in 1962, respectively. The CAGD knowledge has been utilized in designing car body shape profiles for both companies. The applications of CAGD continue to be invented in many commercial modelers (software) such as CATIA, EUCLID, STRIM, ANVIL, and GEOMOD (Rockwood and Chambers, 1996). Figure 1.5 shows CATIA's logo.

Figure 1.5: An example of CATIA's logo (www.ccam.de/edgecam.html, accessed 20 March 2013)

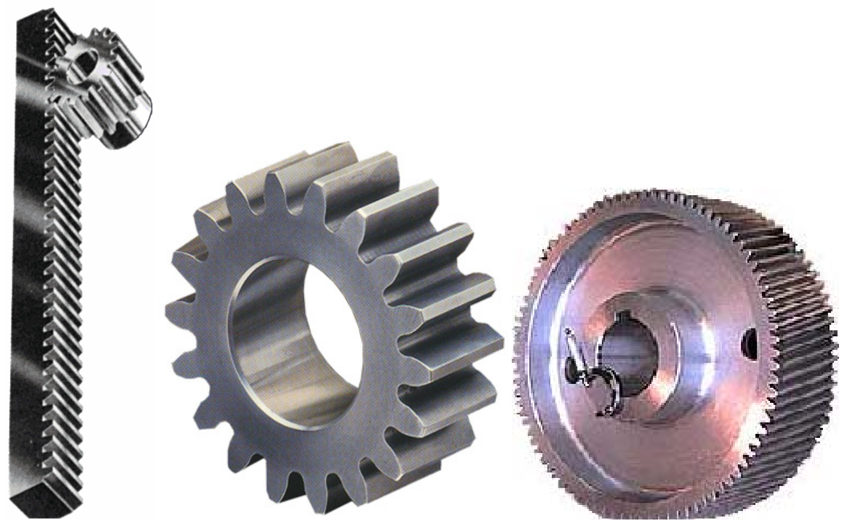
Geoscience is one of the related areas that apply CAGD methods to demonstrate the seismic horizons (Amorim et al., 2012). For example, image processing algorithms are normally applied to interpret and visualize the characteristic changes in the seismic horizons (Bondar, 1992; Aurnhammer and Tonnies, 2002; Parks, 2010). Similarly, designers of computer graphics use surface properties to model graphic objects. CAGD is also useful for word processing (font design), drafting program of the interface protocols using free-form curves or in moviemakers. TRON, Jurassic Park and Terminator 2 are some of the films that apply CAGD techniques in their animation processes and graphical effects

(Rockwood and Chambers, 1996; Prince, 1996). In 2012, Life of Pi is the latest film that applies this legacy technique successfully (Figure 1.6). Several concepts have been applied in the film for instance; geometric shapes and computational method which are strongly connected to this field. The applicability of CAGD will be continuously used throughout this study such as in spur gear design.

Figure 1.6: The making of tiger in the film of 'Life of Pi'
(http://99designs.com/designer-blog/2013/02/15/oscars-best-visual-effects/, accessed 20 March 2013)

1.3 Problem Statement

Gears are some of the most widely used elements in both applications such as in consumer and industrial machineries. The family of gears also includes spur, helical, rack and pinion, worm and bevel (Figure 1.7). In this study, spur gear is chosen as the case model due to its fundamental gear and the fact that it is simple to construct and manufacture. Babu and Tsegaw (2009); Yoon (1993); Bradford and Guillet (1943) and Higuchi et al. (2007) have identified an involute curve (Figure 1.8) as the most common profiles used in designing the spur gear tooth. This curve is developed based on the approximation theories such as Chebyshev approximation (Higuchi and Gofuku, 2007) and the tracing points method (Margalit, 2005; Reyes et al., 2008). However, these employed methods are not accurate (or inexact) and only focus on the approximation concepts (Babu and Tsegaw, 2009). Furthermore, the gear noise reduction and tooth strength are continually viewed as the main issues for consideration, with emphasis on the tooth shape (profile) modification (Yoon, 1993; Sweeney, 1995; Sankar et al., 2010; Åkerblom, 2001; Beghini et al., 2006).



(a) (b) (c)

Figure 1.7: The illustration of the family of gears: (a) rack and pinion, (b) spur and (c) helical (www.gearsandstuff.com, accessed 20 March 2013)

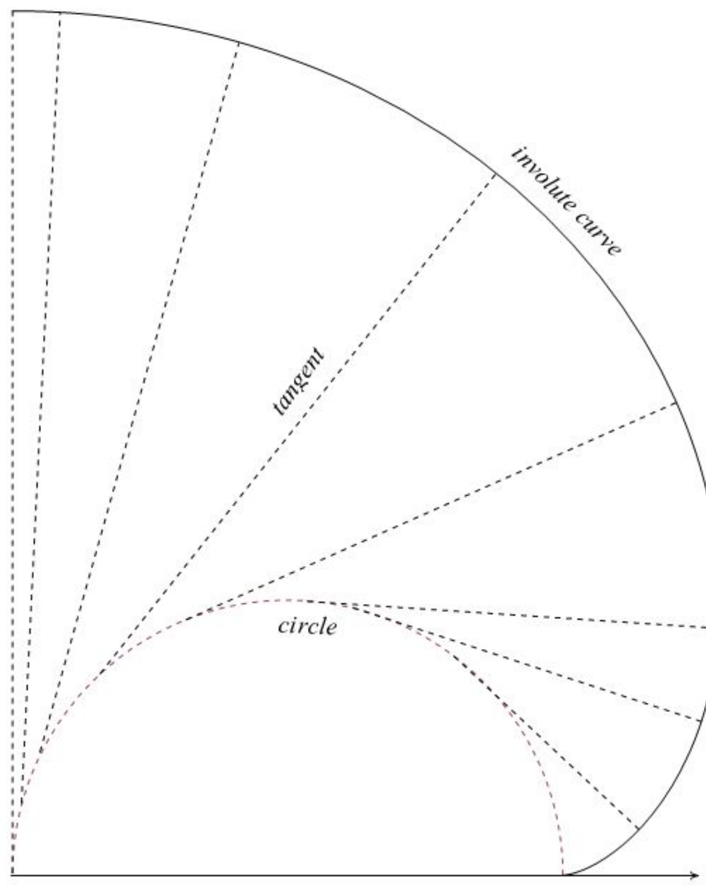


Figure 1.8: Generation of an involute curve

1.4 Significance of the Study

This study will be a significant endeavour on the construction of the parametric (or known as an exact) curve theories namely, S and C-shaped transition curves with some related mathematical proofs. This study will also be beneficial to the designers or manufacturers in designing the spur gear profiles where the above-mentioned curves can be applied as an alternative method of these profiles. In addition, the shape of these profiles is preserved exactly over the curvature continuity (G^2 continuity).

The applicability of the proposed designs is now measured using linear static, fatigue, normal modes, frequency and transient analyses with the material selected is Stainless Steel Grade 304 (AISI 304). These analyses covered all static and dynamic behaviour. At present, first-order Newton interpolating polynomial is used as a fatigue predictor to predict the fatigue mode in the design. Continuously, acoustic analysis is also carried out through the related experiment to perceive the sound or noise level in the proposed design with the material (AISI 304) remained. The comparison is made between the proposed and existing designs in all analyses. This study uses design efficiency (DE), probabilistic simulation (PS) and coefficient of variation (CV) as the lens of either the proposed designs are acceptable or otherwise by setting and computing the benchmark improvements and the design consistency amongst all models.

This study provides future recommendations on the function used in designing the curves, the use of integrated software as a significant technique and also to explore new applications.

1.5 Research Objectives

The objectives of this thesis are:

a. To study the characteristics of the circle to circle templates together with the applied function, Said-Ball cubic curve.
b. To design the S and C-shaped transition curves in accordance to the third and fifth cases of circle to circle templates and Said-Ball cubic curve with some mathematical proofs.
c. To apply the S and C-shaped transition curves in spur gear design.
d. To analyze the proposed and existing models using appropriate engineering analyses such as linear static, fatigue and frequency analyses.
e. To find out the sound or noise level of the proposed and existing models throughout an acoustic experiment.
f. To validate between the proposed and existing models using DE, PS and CV.

1.6 Thesis Organization

This thesis begins by introducing the history of CAGD, its first ideas and involvements, contribution of this field, the early applications of CAGD technique and concept in various areas or industries. Chapter 1 also focuses on the explanation on the problem statement, significance of the study, research objectives and thesis organization.

Chapter 2 reviews the differential geometry of the curves, which will be extensively used throughout this study. The present review deals with parameterized and plane curves, degree of smoothness (continuity) and also some notations and convections. Cubic Bézier curve in Bernstein form representation will be discussed followed by the discussion on the introduction of transition and spiral curves. Methods of designing these curves will be touched such as straight line to circle, circle to circle with an S-shaped transition and circle to circle with C-shaped spiral. The application of this curve design in highway and railway designs or in path planning will also be described in details. Chapter 2 also focuses on the investigation of gears by exploring its history, terminology and classification. The general description of spur gears is then discussed together with current curve that has been applied in design process. Finally, a brief overview of fabrication tools such as turning and wire-cut machines will be done.

Theory development will be the focus of discussion in chapter 3. It consists of a review of Said-Ball cubic curve and its curve characteristics and the designing of S and C-transition curves. A method of designing these curves will be dictated by the circle to circle templates (as described in chapter 2). Relatively, S and C-shaped curves will be analyzed by examining their curvature profiles and will be concluded with mathematical proofs.

Chapter 4 further elaborates the use of S and C-transition curves in designing spur gear tooth profiles. These tooth profiles will be converted into spur gear solid models using the integrated software, Wolfram Mathematica 7.0 and CATIA V5. Meanwhile, the existing model is also explained in this chapter.

By using structural response analysis, chapter 5 focuses on the evaluation of outcomes resulting from chapter 4. One of common schemes used is static strength analysis with the tool, FEA. FEA includes CAD model, meshing process with several conditions such as displacement, boundary and loading needed in this tool. This scheme will determine the stress distributions and safety factor amongst the models. In addition, fatigue analysis will be highlighted in this chapter and finally, the computation of DE in all spur gear models will be discussed and shown.

Chapter 6 discusses the measurement of spur gear models by using the dynamic and acoustic response analyses. Dynamic response comprises of the schemes of normal modes, frequency and transient response analyses with the influence of the damping factor. These schemes will to find out for instance; natural frequency, displacement and stress distributions in real-time computing or in frequency domain amongst the models. Simultaneously, this analysis will be completed by using a simulation method. Conversely, the noise or sound levels of the models are evaluated experimentally. An experiment on the gear acoustics is initially intended to study the effect of tooth shape (profile) amongst the proposed and existing models. This experiment involves fabrication of spur gear (model) and experimental setup as well as fundamental features which are mostly applied in this analysis. This chapter ends by briefly discussing the results obtained throughout the presentation of PS and CV. An approximate normal distribution is the method of representing PS.

Chapter 7 discusses and summarizes the findings and highlights suggestions for further study.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction to Differential Geometry of Curve

Differential Geometry (DG) or known as the mathematical disciplines are the most fundamental properties in CAGD with its concentration on the shape of the objects (curves and surfaces). The disciplines for instance, calculus and linear algebra have been identified as a main contributor to DG. Since the 18^{th} and 19^{th} centuries, DG has been developed using such a theory of curves and surfaces in Euclidean (real vector) space (Schlichtkrull, 2011). Euler (1707-1783), Monge (1746-1818) and Gauss (1777-1855) are the early mathematicians involved in expanding this theory. For example, modern theory of plane curves is developed by Euler while in the year 1825, Gauss contributed his work on DG of surfaces (Schoen, 2011). Schoen (2011) also explained that DG always begins with the plane curves.

Plane curve is basically a special curve or profile (two-dimensional) situated along the plane (Lawrence, 1972). It can also be defined parametrically, explicitly or implicitly such as in (2.2)-(2.4) below. In general, a standard notation of this curve is depicted as

$$z:(a,b)\rightarrow \Re^{2} \tag{2.1}$$

with (a,b) an open interval (Mare, 2012). Plane curves may visualize in closed or open region (Figure 2.1). Closed region means the curve is without endpoints (enclosed area), whereas vice versa in open curve (Berger and Prior, 2006). On top of that, equation (2.2) below shows that the parametric form in a set of Cartesian coordinates with the relationships between dependent and independent variables, as mentioned previously in chapter 1. Hence, the form expressed as

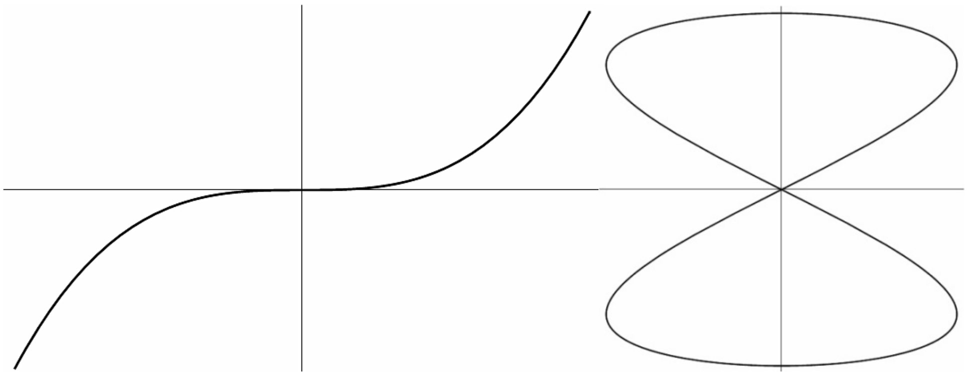
$$z(t) = (x(t), y(t)), \tag{2.2}$$

where $x = x(t)$ and $y = y(t)$ with t is in the real interval (non-negative) either open or closed range for instance, $t \in [0,1]$ while an explicit form is represented by

$$y = g(x). \tag{2.3}$$

It can be seen clearly that both forms, parametric and explicit have the same structures regarding their representation types. Accordingly, the following property (2.4) controls an implicit appearance. In addition, explicit and implicit functions are also identified as the non-parametric forms:

$$f(x, y) = 0. \tag{2.4}$$



(a) (b)

Figure 2.1: The example of (a) open and (b) closed curves

Every form has its own advantage and strength which depends on the application used (Du and Qin, 2007). However, the degree of freedom (DOF) (or a set of independent parameters) can be increased once the parametric form (curve) is employed (Martinsson et al., 2007). This advantage is highly desired in controlling such shapes. Agarwal (2013) in his work clarified that the smooth curves can be generated using this form. Thus, it is mostly preferred in representing the plane curves such as in CAD or in geometric modelling. Alpers (2006) also described that the flexible CAD model will be constructed together with the model modifications can be changed easily and rapidly after applying this parametric form. Presently, one of the parametric forms is

$$z(t) = (r\cos t, r\sin t), \tag{2.5}$$

where $x(t) = r\cos t$ and $y(t) = r\sin t$ with r as the radius. Equation (2.5) will produce a closed curve as a circle. Conversely, this curve can also be generated implicitly using

$$f(x, y) = x^2 + y^2 - r^2 = 0. \tag{2.6}$$

The simple closed curve (circle) using (2.5) or (2.6) is displayed with r equals to 2 as shown in Figure 2.2.

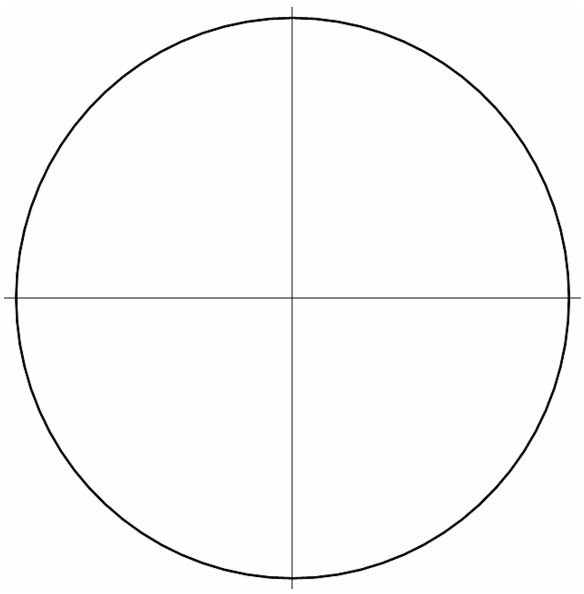


Figure 2.2: A circle defined parametrically and implicitly

Relatively, the represented forms (definition) are always delivered along with some general notations and convections. These rules (calculus and linear algebra) are useful in DG and to generate the smooth plane curve. Consider the Euclidean system consisting of the vectors, $\mathbf{A} =< A_x, A_y >$ and $\mathbf{B} =< B_x, B_y >$. The dot and cross products of these vectors are symbolized as, $\mathbf{A} \bullet \mathbf{B}$ and $\mathbf{A} \wedge \mathbf{B}$, respectively. Hence, these products can be expanded to (Juhász, 1998; Artin, 1957)

$$\begin{aligned} \mathbf{A} \bullet \mathbf{B} &= \|\mathbf{A}\|\|\mathbf{B}\| \cos(\theta) = A_x B_x + A_y B_y, \\ \mathbf{A} \wedge \mathbf{B} &= \|\mathbf{A}\|\|\mathbf{B}\| \sin(\theta) = A_x B_y - A_y B_x, \end{aligned} \tag{2.7}$$

and where, θ (or angle) is normally measured in anti-clockwise direction. Let $z(t)$ be as defined in (2.2), thus its velocity (tangent) will be denoted by $z'(t)$ and followed by the norm (speed) equivalents to

$$\|z'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}, \tag{2.8}$$

Equation (2.8) is essentially associated to compute the arc length of a curve as depicted by

$$S(t) = \int_a^b \|z'(t)\| dt. \tag{2.9}$$

Jia (2014) and Hagen et al. (1995) claimed that the curve is regular (smooth) if the parametric form is employed and $z'(t) \neq 0.$ In addition, these velocities and speeds are fully dependent on this form. Due to $z'(t) \neq 0$ consequently, the studies such as by Hoschek and Lasser (1993) and Faux and Pratt (1988) have discovered the existence of curvature along the curve (the regular characteristic will begin to form). Curvature can be prescribed as a local measure (set of measurement) of the curve shape (Sullivan, 2008). This is the best approach of describing the curves which are said to be entirely beautiful (Margalit, 2005). It is also agreed by Struik (1931) who said that the curvature is the major property in DG.

Initially, this curvature theorized through mathematicians such as Aristotle (384-322 BC) and Proclus (412-485 C.E.) are the Greeks developed the notations of curvature through classical Geek curves while Pergaeus (ca. 262 BC-ca. 190 BC) in his significant works about the proposed method to identify the radius of curvature and the integration between conic section and normal line (Margalit, 2005). Hence, the study of curvature is then indispensable since this exploration. Fermat (1601-1665) and Descartes (1596-1650) expanded theory of curvature with some algebraic equations whereas Newton (1642-1727) in his remarkable contribution to conclude that the curvature is inversely proportional to the radius in all circles:

$$\kappa(t) = \frac{1}{r}. \tag{2.10}$$

This relationship is mainly used as the basis form in constructing smoothness of the curve namely, second order geometric (G^2) continuity. The shapes in CAGD or in CAD are the current studies which apply this continuity. Euler (1707-1783) has mentioned that the parameterized curves should be lens to DG. He was responsible to modify the definition of curvature by including the tangent concept (Kline, 1972). These mathematicians are known as the father of DG. Hence, curvature, $\kappa(t)$ and its derivative have been referred to as

$$\kappa(t) = \frac{z'(t) \wedge z''(t)}{\|z'(t)\|^3}. \tag{2.11}$$

$$\kappa'(t) = \frac{\phi(t)}{\|z'(t)\|^5}, \tag{2.12}$$

where

$$\phi(t) = \|z'(t)\|^2 \frac{d}{dt}\{z'(t) \wedge z''(t)\} - 3\{z'(t) \wedge z''(t)\}\{z'(t) \bullet z''(t)\}.$$

Equation (2.12) is engaged as an indicator to recognize the plane curve either spiral or transition feature if certain conditions have been fulfilled (will be further discussed in chapter 3). An aesthetic (the concept of beauty and interactive) appearance amongst the curves can be found once this recognition is truly finished (Jacobsen et al., 2006). Due to the order derivatives which are applied in (2.12), as explained by Costa (2002) that the curve will be produced smoothly. Besides, Lin (2009) has confirmed that the order derivatives give an influence to the shape of curves becomes smoother and visually pleasing. After knowing these several theories, the descriptions of this chapter will be continued on cubic Bézier curve.

2.2 Cubic Bézier Curve

Cubic Bézier Curve (CBC) is a well-known form in the fields of CAGD, CAD and Geometric Modelling. Walton and Meek (1999 and 2001) described that CBC is frequently chosen as a function due to the properties such as one of the parametric curves (classification); the lowest degree polynomial to permit the inflection points (related to curvature extrema and stability reason); have the geometric and numerical properties that satisfy CBC suitable for use in CAGD or in CAD (flexibility) and ease to handle and implement when compare to other degree.

Historically, Bézier representation is used as the basis form in CBC. This representation has been introduced to the world by Bézier (1910-1999) and de Casteljau (1930-1999), the French engineers to overcome the problems in representing and preserving smooth curves and surfaces in automobile company (Farin, 2002). For example, Citroën and Renault use this curve completely.

Throughout their life, Bézier (1910-1999) and de Casteljau (1930-1999) are also known as the pioneers in many areas such as solid, geometric and physical modelling. In general, Bézier curve of degree n can be depicted as

$$Z(t) = \sum_{i=0}^{n} P_i B_{i,n}(t),\ t \in [0,1] \tag{2.13}$$

where P_i are defined as the control points and;

$$B_{i,n}(t) = \begin{cases} \dfrac{n!}{(n-i)!i!}(1-t)^{n-i}t^{i}, & 0 \le i \le n \\ 0, & \text{otherwise} \end{cases} \tag{2.14}$$

as the Bernstein polynomials or the Bernstein basis functions of degree n (Qian et al., 2011). These polynomials are indispensable as the core of Bézier curves and have different form when compared with the rational Bézier curves. Statistically, $\dfrac{n!}{(n-i)!i!}$ or $\begin{pmatrix} n \\ i \end{pmatrix}$ is also classified as the binomial coefficients (Sury et al., 2004). Farouki (2012) in his review describes several properties of Bernstein polynomials:

a. Symmetry

   The basis functions, $B_{n-i,n}(t) = B_{i,n}(1-t)$ for $i = 0,\dots,n$ (mirroring).

b. Non-Negativity (Positivity)

   The basis functions, $B_{i,n}(t) \ge 0$ in all $t \in [0,1]$.

c. Partition of Unity

   The total of binomial expansion, $\sum_{i=0}^{n} B_{i,n}(t) = 1$ for $t \in [0,1]$.

d. Recurrence Relation (de Casteljau's algorithm)

   For example, the basis polynomial of degree $n+1$ can be produced using the basis polynomial of degree n where

   $B_{i,n+1}(t) = (1-t)B_{i,n}(t) + tB_{i-1,n}(t)$ since $t \in [0,1]$.

Sánchez-Reyes and Chacón (2005) state the basic properties of Bézier curves as follows:

a. Endpoint interpolation expressed as

$$Z(0) = P_0 \ \& \ Z(1) = P_n$$

b. Geometric continuity (Tangent) for instance,

$$Z'(0) = n(P_1 - P_0) \ \& \ Z'(1) = n(P_n - P_{n-1})$$

c. Convex hull (Polygon)

This property always exists in the control points of Bézier curve. It is also crucial for numerical stability.

d. Invariant under affine transformations (Geometric mappings)

This property engages with any blending of translations, reflections, stretches or rotations (original form remains) such that

$$\gamma\left(\sum_{i=0}^{n} P_i B_{i,n}(t)\right) \equiv \sum_{i=0}^{n} \gamma P_i B_{i,n}(t)$$

e. Variation diminishing (VD)

This property has verified that a Bézier curve alternates less than its control polygon (point) due to the influence of the segments. Moreover, VD is widely applied in the algorithms for example, intersection and fairness.

These properties are well-suited for interactive design environments and are especially useful in path planning (Ho and Liu, 2009). Figure 2.3 shows the terminology of Bézier curve.

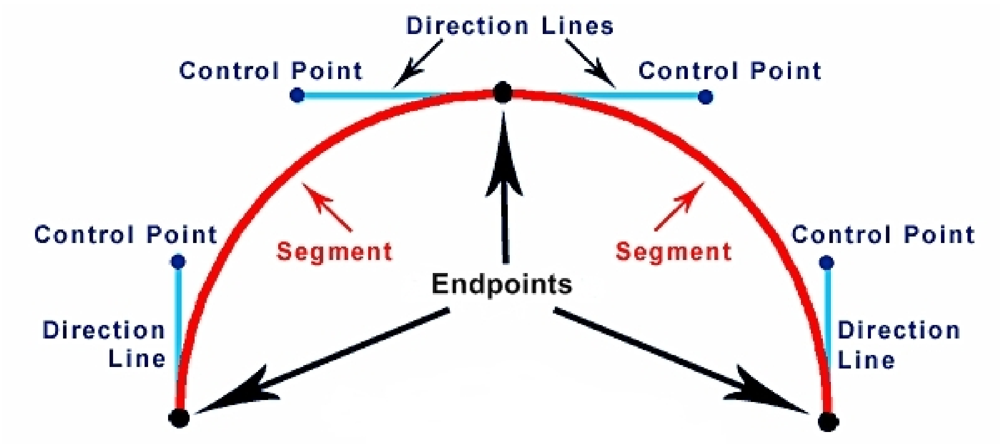


Figure 2.3: The art of Bézier curve (red colour) (Novin, 2007)

If $n = 3$, (2.13) and (2.14) become

$$Z(t) = P_0 B_{0,3}(t) + P_1 B_{1,3}(t) + P_2 B_{2,3}(t) + P_3 B_{3,3}(t) \tag{2.15}$$

where

$$\begin{aligned} B_{0,3}(t) &= (1-t)^3, \quad B_{1,3}(t) = 3t(1-t)^2, \\ B_{2,3}(t) &= 3t^2(1-t), \quad B_{3,3}(t) = t^3. \end{aligned} \tag{2.16}$$

Both equations are characterized as CBC. CBC consists of four control points symbolized by P_0, P_1, P_2 and P_3 while the visualization of Bernstein functions (2.16) is displayed in Figure 2.4. The exploration of this chapter will be continued by introducing the most common types of parametric curves, namely transition and spiral curves.

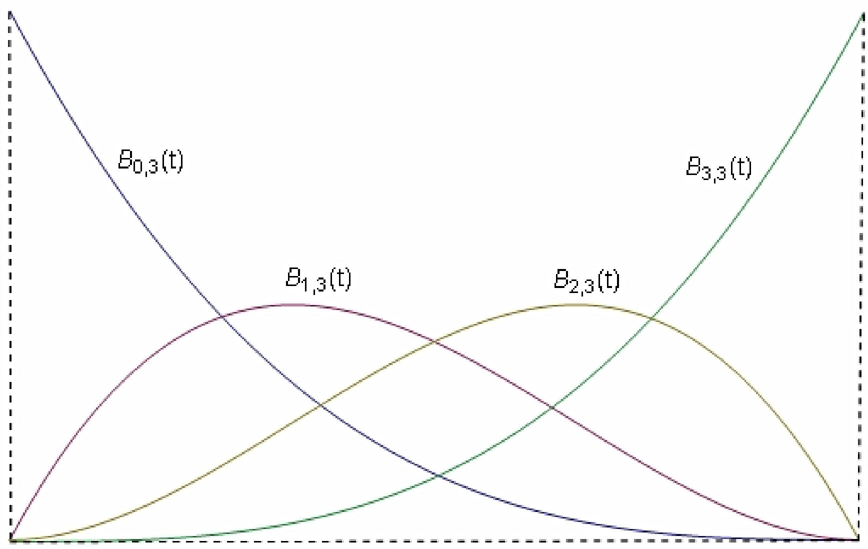


Figure 2.4: Cubic Bernstein basis functions in 2D plot

2.3 Transition and Spiral Curves

Shen et al. (2013) explicate that transition curve is a segment with varied radius, gradually increasing or decreasing. This increment happens during the connection between two curves with different radius for instance, circular arc (curve) and tangent track (straight line). The idea of connecting curves is utilized to enable the gradual change (smooth) amongst the curvature and its acceleration or speed (Lindahl, 2001). Therefore, this idea becomes crucial since it has been widely used in civil and transportation engineering particularly; in highway or in railway design (Figure 2.5).

Figure 2.5: The example of road design using transition curves (Myers, 2001)

In contrast, spiral curve defines as a plane curve with the curvature varies monotonically either increasing or decreasing (only in one sign) (Kurnosenko, 2009; Leyton, 1987). Since the denominator in (2.12) is always positive, thus monotonic curvature (MC) must satisfy the following spiral condition (SC) such as

$$\mathrm{SC} = \begin{cases} \text{If } \kappa'(t) > 0 \text{ means MC is increasing} \\ \text{If } \kappa'(t) < 0 \text{ means MC is decreasing} \end{cases} \tag{2.17}$$

Clothoid or also known as Euler or cornu spiral is one of the basic spirals which has the curvature changes linearly with its arc-length (Figure 2.6) (Yates, 1974). This review shows that the curvature and arc length have the same identity (identical) since the clothoid has its own function.

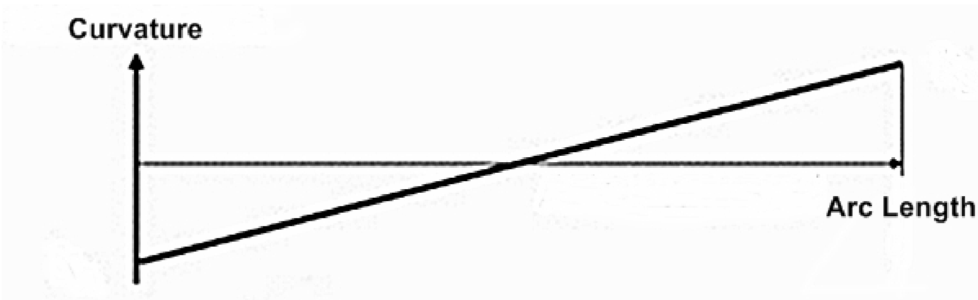


Figure 2.6: Relationship between curvature and arc length in clothoid (Séquin, 2005)

However, in real situation, this curvature plot (Figure 2.6) might be difficult to achieve when using Bézier curve. The clothoid function is formulated with the use of parametric form and Fresnel integral (Abramowitz and Stegun, 1964; Meek and Walton, 2004) where

$$H(t) = \beta \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad t \geq 0 \tag{2.18}$$

and β is the scaling factor, the Fresnel integrals are

$x(t) = \int_0^t \sin[\frac{\pi u^2}{2}]du$ and $y(t) = \int_0^t \cos[\frac{\pi u^2}{2}]du$.

Some scholars believed that spiral curves can be recognized as aesthetic curves (Ziatdinov et al., 2013; Harary and Tal, 2012; Yoshida and Saito, 2006). Besides, several researchers also figured out that transition and spiral curves contain an equivalence relation for example, geometric smoothness has been used effectively to design these curves and both are highly useful for the same engineering fields (Levien, 2008; Perco, 2006; Kimia et al., 2003). Nevertheless, spiral curve requires some extensions on its curvature profile (to ensure either MC or not).

2.4 Introduction to Clothoid Templates

Clothoid or circle to circle templates have been introduced to the world by Baass (1984). These templates are firstly utilized in highway design to obtain two major outcomes for instance, to enhance the quality, comfortable and safe driving to the users as well as to design more natural alignments such as in highways that are suitable for its surrounding area while traditional approaches consist of straight line and circular arc (also known as horizontal alignment) (Figure 2.7) are difficult to achieve these outcomes (Baass, 1984).