UNIVERSITI SAINS MALAYSIA

Peperiksaan Semester Pertama
Sidang Akademik 2006/2007

Oktober/November 2006

EEE 453 – REKABENTUK SISTEM KAWALAN

Masa: 3 jam

Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS** muka surat dan **ENAM** muka surat LAMPIRAN bercetak sebelum anda memulakan peperiksaan ini.

Kertas soalan ini mengandungi ENAM soalan.

Jawab **LIMA** soalan.

Mulakan jawapan anda untuk setiap soalan pada muka surat yang baru.

Agihan markah bagi setiap soalan diberikan di sudut sebelah kanan soalan berkenaan.

Jawab semua soalan dalam Bahasa Malaysia.

...2/-

1. (a) (i) Merujuk kepada sistem kawalan, terangkan konsep bagi
With reference to control systems, explain the concept of

(i) Keadaan
State

(ii) Pembolehubah keadaan
State variable

(iii) Keadaan Ruang
State space

(15%)

(ii) Tuliskan persamaan keadaan dan persamaan keluaran bagi suatu sistem lelurus tak berubah dengan masa. Terangkan setiap elemen dalam persamaan tersebut.

Write the state equation and output equation for a linear time-invariant system. Explain each element in the equations

(15%)

(b) Pertimbangkan sistem berikut:
Consider the following system:

$$\dddot{y} + 5\ddot{y} + \dot{y} + 2y = u$$

Dapatkan perwakilan keadaan ruang bagi sistem tersebut dalam bentuk
Obtain a state-space representation of this system in the form of

(i) kanonikal bolehkawal
controllable canonical

(ii) kanonikal bolehperhati
observable canonical

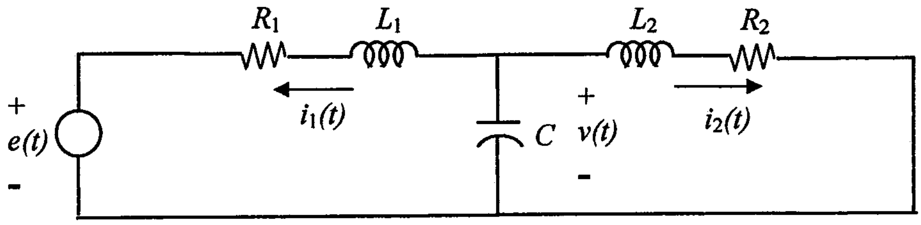
(30%)

...3/-

(c) Rajah 1 menunjukkan suatu rangkaian RLC. Input kepada sistem adalah sumber voltan, $e(t)$, dan output bagi sistem adalah voltan merintangi R_2 and arus melalui R_2. Diberi pembolehubah keadaan $x_1(t) = v(t)$, $x_2(t) = i_1(t)$, $x_3(t) = i_2(t)$, cari perwakilan keadaan ruang bagi sistem tersebut.

Figure 1 shows an RLC network. The input to the system is the voltage source, $e(t)$, and the outputs of the system are voltage across R_2 and current through R_2. Given the state variables $x_1(t) = v(t)$, $x_2(t) = i_1(t)$, $x_3(t) = i_2(t)$, find the state-space representation of the system.

(40%)



Rajah 1
Figure 1

2. (a) (i) Merujuk kepada sistem kawalan, terangkan konsep bagi
With reference to control systems, explain the concept of

(i) Kebolehkawalan
Controllability

(ii) Kebolehperhatian
Observability

(iii) Kebolehstabilan
Stabilizability

(15%)

...4/-

(ii) Tuliskan matriks kebolehkawalan dan kebolehperhatian bagi suatu sistem kawalan. Terangkan setiap elemen dalam kedua-dua matriks tersebut.

Write the controllability and observability matrices for a control system. Explain each element in the two matrices.

Apakah ciri-ciri kedua-dua matriks tersebut bagi suatu sistem yang mempunyai kebolehkawalan dan kebolehperhatian selengkapnya?

What are the characteristics of the two matrices for a system with complete controllability and observability?

(15%)

(b) (i) Diberi bahawa pekali bagi persamaan keadaan sesuatu sistem adalah seperti berikut:
Given that the coefficients of the state equation of a system are as follows:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Dapatkan matriks kebolehkawalan bagi sistem tersebut.
Obtain the controllability matrix of the system.

(20%)

(ii) Pertimbangkan suatu sistem yang ditakrifkan sebagai

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Adakah sistem tersebut bolehkawal keadaan sepenuhnya dan bolehperhati sepenuhnya? Terangkan.

Is the system completely state controllable and completely observable? Explain.

(20%)

(c) Pertimbangkan persamaan keadaan

Consider the state equation

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Tentukan matriks keadaan transisi, $G(t)$, dan vektor keadaan, $x(t)$, bagi $t \geq 0$ apabila input adalah $u(t) = 1$ baig $t \geq 0$.

Determine the state-transition matrix, $G(t)$, and the state vector $x(t)$ for $t \geq 0$ when the input is $u(t) = 1$ for $t \geq 0$.

(30%)

3. (a) (i) Dalam konteks rekabentuk sistem kawalan keadaan ruang, apakah keadaan yang diperlukan dan mencukupi untuk perletakan kutub yang sebarangan?

In terms of state-space control system design, what is the necessary and sufficient condition for arbitrary pole placement?

(10%)

(ii) Suatu sistem regulator mempunyai model berikut

A regulator system has the following model

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)}$$

Takrifkan pembolehubah keadaan seperti $x_1 = y$, $x_2 = \dot{x}_1$, $x_3 = \dot{x}_2$. Dengan menggunakan kawalan suapbalik-keadaan, kutub gelung tertutup ingin diletakkan pada $s_{1,2} = -2 \pm j2\sqrt{3}$, $s_3 = -10$. Tentukan matriks gandaan suapbalik-keadaan, ***K***.

Define state variables as $x_1 = y$, $x_2 = \dot{x}_1$, $x_3 = \dot{x}_2$. By using the state-feedback control, it is desired to place the closed-loop poles at $s_{1,2} = -2 \pm j2\sqrt{3}$, $s_3 = -10$. Determine the necessary state-feedback gain matrix, ***K***.

(30%)

(b) Pertimbangkan sistem
Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Rekabentuk suatu sistem regulator dengan kaedah perletakan-kutub-dengan-pemerhati. Anggapkan kutub gelung tertutup yang diingini diletakkan pada $s_{1,2}=-1\pm j$, $s_3=-5$, dan kutub pemerhati yang diingini diletakkan pada $s_1=s_2=s_3=-6$

Design a regulator system by the pole-placement-with-observer-approach. Assume that the desired closed-loop poles for pole placement are located at $s_{1,2}=-1\pm j$, $s_3=-5$, *and the desired observer poles are located at* $s_1=s_2=s_3=-6$.

Lukiskan suatu gambarajah blok bagi sistem dengan keadaan-suapbalik yang diperhatikan.
Draw a block-diagram for the system with observed-state-feedback.

(60%)

4. (a) (i) Dengan menggunakan suatu gambarajah blok, terangkan bagaimana masalah identifikasi sistem diwakilkan

By using a block diagram, explain how the problem of system identification is represented.

(15%)

(ii) Terangkan dengan jelasnya prosidur yang boleh digunakan untuk menyelesaikan sesuatu masalah identifikasi sistem.

Explain in detail what is the procedure that can be used to solve a system identification problem.

(20%)

(b) Terangkan kaedah "least-squares" yang digunakan dalam anggaran parameter dalam data eksperimen.
Explain the least-squares method used in parameter estimation of experimental data.

Apakah ciri-ciri statistik bagi kaedah "least-squares"?
What are the statistical properties of the least-squares method?

(20%)

(c) Merujuk kepada Rajah 4, suatu siri *m* pemerhatian untuk kedua-dua *y* dan **x** telah dibuat pada masa $t_1, t_2, \cdots, t_m$. Sampel data yang diukur diwakili oleh $y(i)$ dan $x_1(i), x_2(i), \cdots, x_n(i),\ i = 1,2,\cdots,m$, dan sampel data adalah dihubungkan oleh persamaan $y(i) = \theta_i x_1(i) + \theta_2 x_2(i) + \cdots + \theta_n x_n(i),\ i = 1,2,\cdots,m$.

*With reference to Figure 4, a sequence of m observations on both y and **x** has been made at times* $t_1, t_2, \cdots, t_m$. *The measured data samples are represented by* $y(i)$ *and* $x_1(i), x_2(i), \cdots, x_n(i),\ i = 1,2,\cdots,m$, *and the data samples are related by the equation* $y(i) = \theta_i x_1(i) + \theta_2 x_2(i) + \cdots + \theta_n x_n(i),\ i = 1,2,\cdots,m$

(i) Apakah keadaan yang diperlukan untuk membolehkan anggaran *n* parameter θ_i?

What is the necessary condition to be able to estimate the n parameters θ_i*?*

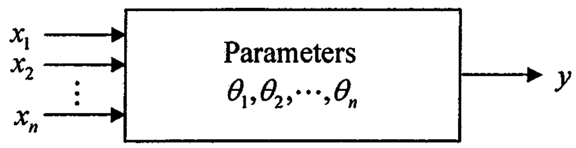
(ii) Terbitkan penganggar "ordinary least-squares" bagi θ.

Derive the ordinary least-squares estimator of θ.

(iii) Jika setiap ralat diberi pemberat yang berlainan oleh matriks pemberat ***W***, terbitkan penganggar "weighted least-squares" bagi θ_W.

If each error term is weighted differently by the weighting matrix ***W****, derive the weighted least-squares estimator of* θ_W

(45%)



Rajah 4 : Suatu sistem lelurus dengan *n* parameters
Figure 4 : An n-parameter linear system

5. (a) Terangkan dua kaedah yang boleh digunakan untuk menentukan kesesuaian tertib bagi sesuatu model semasa identifikasi sistem.

Explain two methods that can be used to determine a suitable order for a model during system identification.

(30%)

(b) Diberi model suatu proses adalah dalam bentuk $y(k)=\dfrac{bz^{-1}}{1+az^{-1}}u(k)$, dan data input/output adalah seperti dalam Jadual 5.
Anggarkan parameter *a* and *b* dengan menggunakan kaedah "least-squares".

Given that the process model is of the form $y(k)=\dfrac{bz^{-1}}{1+az^{-1}}u(k)$ *with the input/output data as in Table 5.*

Estimate parameters a and b using the least squares method.

k	1	2	3	4
u(k)	0.05	0.9725	0.455	1
y(k)	-32.78	18.356	-1.28	3.72

Jadual 5
Table 5

(40%)

(c) Terangkan dengan detil langkah-langkah yang terlibat dalam algoritma "recursive least-squares".

Explain in detail the steps involved in the recursive least-squares algorithm.

(30%)

6. (a) Merujuk kepada sistem kawalan, apa itu masalah servo-mekanisma? Apakah objektif rekabentuk yang biasa digunakan dalam suatu masalah servo-mekanisma? Cadangkan suatu indeks prestasi yang boleh digunakan untuk mencapai objektif rekabentuk tersebut?

With reference to control systems, what is a servo-mechanism problem?
What normally is the design objective in a servo-mechanism problem? Suggest a performance index that could be used to achieve the stated design objective.

(30%)

...11/-

(b) Pertimbangkan suatu sistem kawalan digital yang ditunjukkan dalam Rajah 6. Anggapkan kitaran sampel adalah $T = 1.0$ saat dan rangkap pindah adalah

Consider a digital control system as shown in Figure 6. Assume that the sampling period $T = 1.0$ sec and the transfer functions of

$$G_h(s) = \frac{1-e^{-Ts}}{s} \quad \text{dan } \textit{and} \quad G_p(s) = \frac{1}{s+1}$$

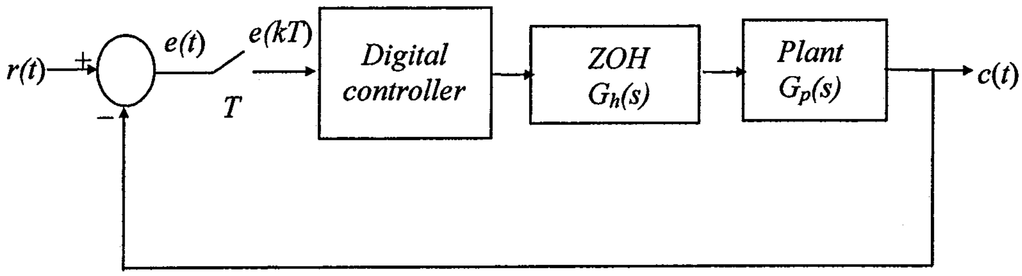
Pengawal digital adalah suatu penguat dengan gandaan *K*, dan input adalah step unit.
The digital controller is an amplifier of gain K, and the input is unit step.

Dapatkan nilai optimum *K* supaya
Find the optimal value of K so that

(i) $J = \sum_{k=0}^{\infty} e^2(kT)$ adalah minimum
is minimised

(ii) $J = \sum_{k=0}^{\infty} [e^2(kT) + u^2(kT)]$ adalah minimum
is minimised.

(70%)



Rajah 6
Figure 6

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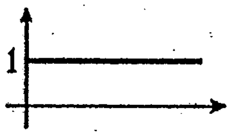
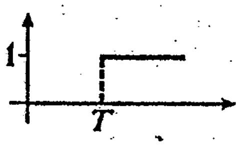
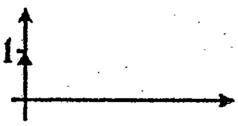
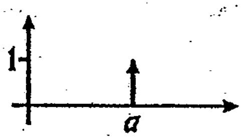
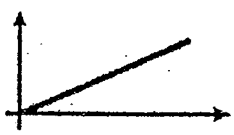
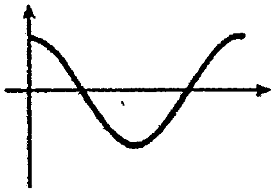
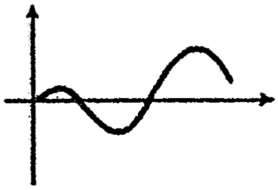
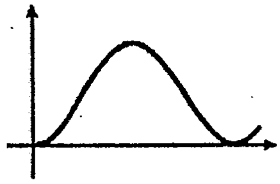
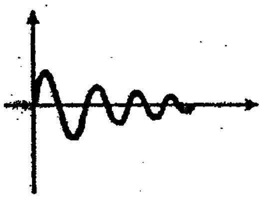
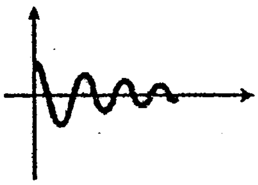
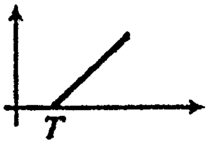
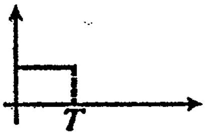
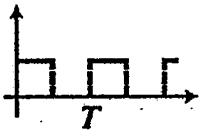
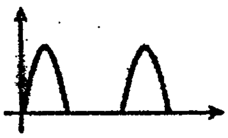
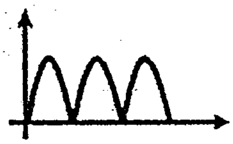
A Table of Laplace Transforms

		$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$
1. Sum		$af(t)+bg(t)$	$aF(s)+bG(s)$ or $a\bar{f}(s)+b\bar{g}(s)$
2. First derivative		$\frac{d}{dt}f(t)$ or $f'(t)$	$sF(s)-f(0)$ or $s\bar{f}(s)-f_0$
3. Second derivative		$\frac{d^2}{dt^2}f(t)$ or $f''(t)$	$s^2F(s)-sf(0)-f'(0)$ or $s_2\bar{f}(s)-sf_0-f_1$
4. Third derivative		$\frac{d^3}{dt^3}f(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
5. Fourth derivative		$\frac{d^4}{dt^4}f(t)$	$s^4F(s)-s^3f(0)-s^2f'(0)-sf''(0)-f'''(0)$
6. Definite integral		$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$ or $\frac{1}{s}\bar{f}(s)$
		$\int_{-\infty}^t f(t)dt$	$\frac{1}{s}F(s)+\frac{1}{s}\int_{-\infty}^0 f(t)dt$
7. Exponential multiplier		$e^{-\alpha t}f(t)$	$F(s+\alpha)$ or $\bar{f}(s+\alpha)$
8. Time shift		$f(t-T)u(t-T)$	$e^{-sT}F(s)$ or $e^{-sT}\bar{f}(s)$
9. Periodic function		$f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$
10. Convolution		$f(t)\times g(t)=\int_0^t f(t-u)g(u)du$	$F(s)G(s)$ or $\bar{f}(s)\bar{g}(s)$
11. Unit step		$u(t)$ or $H(t)$	$\frac{1}{s}$
12. Delayed step		$u(t-T)$	$\frac{1}{s}e^{-sT}$
13. Unit impulse		$\delta(t)$	1
14. Delayed unit impulse		$\delta(t-a)$	e^{-as}
15. Linear ramp		t	$\frac{1}{s^2}$

		$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
28. Cosine wave		$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
29.		$\cos(\omega t \pm \phi)$	$\frac{s\cos(\phi) \pm \omega\sin(\phi)}{s^2+\omega^2}$
30.		$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
31.		$1-\cos(\omega t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$
32.		$\sin(\omega t) - t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
33. Exponentially damped		$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
34.		$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
35.		$e^{-\alpha t}[\sin(\omega t) - \omega t\cos(\omega t)]$	$\frac{2\omega^3}{[(s+\alpha)^2+\omega^2]^2}$
36. Hyperbolic function		$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
37.		$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
38. Damped Hyperbolic		$e^{-\alpha t}\sinh(\omega t)$	$\frac{\omega}{(s+\alpha)^2-\omega^2}$
39.		$e^{-\alpha t}\cosh(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2-\omega^2}$
40.		$e^{-\alpha t}[\sinh(\omega t) - \omega t\cosh(\omega t)]$	$\frac{-2\omega^3}{[(s+\alpha)^2-\omega^2]^2}$

			$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
41. (a)	$\zeta < 1$ and $\omega_d = \omega_0(1-\zeta^2)^{1/2}$ where ω_d is the frequency of free damped oscillation.		$u(t) - e^{-\zeta\omega_0 t} \times \left[\cos(\omega_d t) + \frac{\zeta\omega_0}{\omega_d}\sin(\omega_d t)\right]$	$\left[s\left(\frac{s^2}{\omega_0^2} + \frac{2\zeta s}{\omega_0} + 1\right)\right]^{-1}$ where ω_0 is the frequency of undamped oscillations, i.e. if $\zeta = 0$
(b)	$\zeta = 1$		$u(t) - e^{-\omega_0 t}[1 + \omega_0 t]$	
(c)	$\zeta = 1$ and $\beta = \omega_0(\zeta^2 - 1)^{1/2}$		$u(t) - e^{-\zeta\omega_0 t} \times \left[\cosh(\beta t) + \frac{\zeta\omega_0}{\beta}\sinh(\beta t)\right]$	
42.	Delayed ramp		$(t-T)u(t-T)$	$\frac{1}{s^2}e^{-sT}$
43.	Rectangular pulse		$u(t) - u(t-T)$	$\frac{1}{s}(1 - e^{-sT})$
44.	Rectangular periodic		$f(t) = \begin{cases} 1 & 0 < t < T/2 \\ 0 & T/2 < t < T \end{cases}$	$\frac{1}{s(1 + e^{-sT/2})}$
45.	Half-wave-rectified sine, period $T = 2\pi/\omega$		$f(t) = \begin{cases} \sin(\omega t) & 0 < t < t/2 \\ 0 & T/2 < t < T \end{cases}$	$\frac{\omega}{(s^2+\omega^2)(1 - e^{-\pi s/\omega})}$
46.	Full-wave-rectified sine, period $T = 2\pi/\omega$		$f(t) = \lvert\sin(\omega t)\rvert$	$\frac{\omega}{s^2+\omega^2}\,\frac{(1 + e^{-\pi s/\omega})}{(1 - e^{-\pi s/\omega})}$

Initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ Final value theorem $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Z-Transform Table

	$x(t)$ or $x(k)$	$\mathscr{Z}[x(t)]$ or $\mathscr{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\cdots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\cdots-zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to1} [(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

Table 1 Tabulation of Definite Integral for Continuous-Time Systems

$$J_n = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{B(s)B(-s)}{A(s)A(-s)}ds$$

$$B(s) = \sum_{k=0}^{n-1} b_k s^k$$

$A(s) = \sum_{k=0}^{n} a_k s^k$; $A(s)$ has zeros in left half plane only.

$$J_1 = \frac{b_0^2}{2a_0a_1}$$

$$J_2 = \frac{b_1^2a_0 + b_0^2a_2}{2a_0a_1a_2}$$

$$J_3 = \frac{b_2^2a_0a_1 + (b_1^2 - 2b_0b_2)a_0a_3 + b_0^2a_2a_3}{2a_0a_3(-a_0a_3 + a_1a_2)}$$

$$J_4 = \frac{b_3^2(-a_0^2a_3 + a_0a_1a_2) + (b_2^2 - 2b_1b_3)a_0a_1a_4 + (b_1^2 - 2b_0b_2)a_0a_3a_4 + b_0^2(-a_1a_4^2 + a_2a_3a_4)}{2a_0a_4(-a_0a_3^2 - a_1^2a_4 + a_1a_2a_3)}$$

Table 2 Tabulation of Definite Integral for Sampled-Data Systems

$$J_n = \frac{1}{2\pi j}\oint_{\substack{unit\\circle}} X(z)X(z^{-1})\frac{dz}{z}$$

$$X(z) = \frac{b_0z^n + b_1z^{n-1} + \cdots + b_n}{a_0z^n + a_1z^{n-1} + \cdots + a_n}$$

$$J_1 = \frac{(b_0^2 + b_1^2)a_0 - 2b_0b_1a_1}{a_0(a_0^2 - a_1^2)}$$

$$J_2 = \frac{B_0a_0e_1 - B_1a_0a_1 + B_2(a_1^2 - a_2e_1)}{a_0[(a_0^2 - a_2^2)e_1 - (a_0a_1 - a_1a_2)a_1]}$$

where

$$B_0 = b_0^2 + b_1^2 + b_2^2$$
$$B_1 = 2(b_0b_1 + b_1b_2)$$
$$B_2 = 2b_0b_2$$
$$e_1 = a_0 + a_2$$

$$J_3 = \frac{a_0B_0Q_0 - a_0B_1Q_1 + a_0B_2Q_2 - B_3Q_3}{[(a_0^2 - a_3^2)Q_0 - (a_0a_1 - a_2a_3)Q_1 + (a_0a_2 - a_1a_3)Q_2]a_0}$$

$$B_0^2 = b_0^2 + b_1^2 + b_2^2 + b_3^2$$
$$B_1 = 2(b_0b_1 + b_1b_2 + b_2b_3)$$
$$B_2 = 2(b_0b_2 + b_1b_3)$$
$$B_3 = 2b_0b_3$$
$$Q_0 = (a_0e_1 - a_3a_2)$$
$$Q_1 = (a_0a_1 - a_1a_3)$$
$$Q_2 = (a_1e_2 - a_2e_1)$$
$$Q_3 = (a_1 - a_3)(e_2^2 - e_1^2) + a_0(a_0e_2 - a_3e_1)$$
$$e_1 = a_0 + a_2$$
$$e_2 = a_1 + a_3$$