DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

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DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

by

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LIST OF SYMBOLS

Symbol	Description	Page
$\mathbf{A}[f]$	Alexander operator	24
$\mathcal{A}_n$	Class of normalized analytic functions f of the form	3
	$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (z \in \mathbb{U})$	
$\mathcal{A} := \mathcal{A}_1$	Class of normalized analytic functions f of the form	2
	$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U})$	
$\mathcal{A}_{n,b}$	Class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$	20
	where b is a fixed non-negative real number.	
$\mathcal{A}_b := \mathcal{A}_{1,b}$	Class of all functions $f(z) = z + bz^2 + a_3z^3 + \cdots$	20
	where b is a fixed non-negative real number.	
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	12
$\mathcal{CV}_b$	Class of convex functions in $\mathcal{A}_b$	20
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	14
$\mathcal{CV}_b(\alpha)$	Class of convex functions of order α in $\mathcal{A}_b$	20
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	25
D	Domain	1
$\mathcal{D}_n$	$\{\varphi \in \mathcal{H}[1,n] \,:\, \varphi(z) \neq 0,\ z \in \mathbb{U}\}$	54
$\mathcal{D}_{n,-\beta}$	$\{\varphi \in \mathcal{H}_{-\beta}[1,n] \,:\, \varphi(z) \neq 0,\ z \in \mathbb{U}\}$	55
$\mathcal{D} := \mathcal{D}_1$	$\{\varphi \in \mathcal{H}[1,1] \,:\, \varphi(z) \neq 0,\ z \in \mathbb{U}\}$	55
$\mathcal{H}(\mathbb{U})$	Class of analytic functions in $\mathbb{U}$	2
$\mathcal{H}[a,n]$	Class of analytic functions f of the form	2
	$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ (z \in \mathbb{U})$	
$\mathcal{H}_\beta[a,n]$	Class of analytic functions f with fixed initial coefficient	19
	of the form $p(z) = a + \beta z^n + a_{n+1} z^{n+1} + \cdots,\ (\beta \geq 0,\ z \in \mathbb{U})$	
$\mathcal{HB}(M)[0,1]$	$\{f \in \mathcal{H}[0,1] \,:\, \lvert f(z) \rvert < M,\ M > 0,\ z \in \mathbb{U}\}$	59
$\mathcal{HB}_\beta(M)[0,1]$	$\{f \in \mathcal{H}_\beta[0,1] \,:\, \lvert f(z) \rvert < M,\ M > 0,\ z \in \mathbb{U}\}$	59

$\mathcal{H}_C[0,1]$	$\{f \in \mathcal{H}[0,1] \,:\, f \text{ is convex}, \, z \in \mathbb{U}\}$	63
$\mathcal{H}_{C\beta}[0,1]$	$\{f \in \mathcal{H}_\beta[0,1] \,:\, f \text{ is convex}, \, z \in \mathbb{U}\}$	63
$\mathbf{I}[f]$	Integral operator	24
Im	Imaginary part of a complex number	35
$k(z)$	Koebe function	6
$\mathbf{L}[f]$	Libera operator	25
$\mathbf{L}_\gamma[f]$	Bernardi-Libera-Livingston operator	25
$m(z)$	Möbius function	10
$\mathbb{N}$	Set of all natural numbers	2
$\mathcal{P}_n$	$\{f \in \mathcal{H}[1,n] \,:\, \text{Re } f(z) > 0, \, z \in \mathbb{U}\}$	55
$\mathcal{P}_{n,\beta}$	$\{f \in \mathcal{H}_\beta[1,n] \,:\, \text{Re } f(z) > 0, \, z \in \mathbb{U}\}$	55
Q	Set of analytic and univalent functions on $\overline{\mathbb{U}}\backslash E(q)$	21
$\mathbb{R}$	Set of all real numbers	6
Re	Real part of a complex number	10
$\mathcal{S}$	Class of normalized univalent functions in $\mathcal{A}$	6
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	13
$\mathcal{ST}_b$	Class of starlike functions in $\mathcal{A}_b$	20
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	14
$\mathcal{ST}_b(\alpha)$	Class of starlike functions of order α in $\mathcal{A}_b$	20
$\mathbb{U}$	Open unit disk, $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$	2
$\overline{\mathbb{U}}$	Close unit disk, $\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$	21
$\partial\mathbb{U}$	Boundary of unit disk, $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$	21
$\psi(r,s,t;z)$	Admissible function	22
$\Psi_n(\Omega,q)$	Class of admissible functions	22
$\Psi_{n,\beta}(\Omega,q)$	Class of β-admissible functions	30
$\Phi_{C,\beta}(\Omega,q)$	Class of β-admissible functions for convexity	79
$\Phi_{C,\beta}(\triangle)$	Class of β-admissible functions for convexity	85
$\Phi_{S,\beta}(\Omega,q)$	Class of β-admissible functions for starlikeness	71
$\Phi_{S,\beta}(\triangle)$	Class of β-admissible functions for starlikeness	78

$\{f,z\}$	Schwarzian derivative of f	8
$\prec$	Subordinate to	20

LIST OF PUBLICATIONS

[1] N. Salleh, R. M. Ali and V. Ravichandran. (2014). Admissible second-order differential subordinations for analytic functions with fixed initial coefficient, The 21st National Symposium on Mathematical Sciences (SKSM21): Germination of Mathematical Sciences Education and Research towards Global Sustainability, AIP Conference Proceedings 1605(1): 655-660.

SUBORDINASI PEMBEZA FUNGSI ANALISIS DENGAN PEKALI AWAL TETAP

ABSTRAK

Tesis ini mengkaji fungsi analisis bernilai kompleks dalam cakera unit dengan pekali awal tetap atau dengan pekali kedua tetap dalam pengembangan sirinya. Kaedah subordinasi pembeza disesuai dan dipertingkatkan untuk membolehkan penggunaannya, yang diperlukan bagi mendapatkan kelas-kelas fungsi teraku yang sesuai. Tiga masalah penyelidikan dibincangkan di dalam tesis ini. Pertama, subordinasi pembeza linear peringkat kedua

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z),$$

dipertimbangkan. Syarat-syarat pada fungsi bernilai kompleks A,B,C,D dan h diterbitkan untuk memastikan implikasi pembeza yang bersesuaian diperoleh yang melibatkan penyelesaian p. Untuk pilihan tertentu bagi fungsi h, implikasi-implikasi ini ditafsirkan secara geometri. Hubungkait akan dibuat dengan penemuan-penemuan terdahulu. Hasil subordinasi-subordinasi tersebut seterusnya digunakan untuk mengkaji sifat-sifat rangkuman untuk pengoperasi kamiran linear pada subkelas fungsi analisis dengan pekali awal tetap tertentu. Kepentingannya akan menjadi pengoperasi kamiran linear berbentuk

$$\mathbf{I}[f](z)=\frac{\rho+\gamma}{z^{\gamma}\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt,$$

dengan ρ dan γ adalah nombor kompleks, dan fungsi ϕ,φ dan f tergolong dalam be-

berapa kelas fungsi analisis. Pengoperasi kamiran linear ditunjukkan memeta subkelas fungsi analisis dengan pekali awal tetap tertentu ke dalam dirinya sendiri. Masalah terakhir yang dipertimbangkan adalah untuk mendapatkan syarat-syarat cukup untuk fungsi analisis dengan pekali awal tetap untuk menjadi bak-bintang atau cembung. Syarat-syarat ini dirangka menggunakan terbitan Schwarzian.

DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

ABSTRACT

This thesis investigates complex-valued analytic functions in the unit disk with fixed initial coefficient or with fixed second coefficient in its series expansion. The methodology of differential subordination is adapted and enhanced to enable its use, which requires obtaining appropriate classes of admissible functions. Three research problems are discussed in this thesis. First, the linear second-order differential subordination

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z) \prec h(z),$$

is considered. Conditions on the complex-valued functions A, B, C, D and h are derived to ensure an appropriate differential implication is obtained involving the solutions p. For particular choices of h, these implications are interpreted geometrically. Connections are made with earlier known results. These subordination results are next used to study inclusion properties for linear integral operators on certain subclasses of analytic functions with fixed initial coefficient. Of interest would be the linear integral operator of the form

$$\mathbf{I}[f](z)=\frac{\rho+\gamma}{z^{\gamma}\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt,$$

where ρ and γ are complex numbers, and ϕ, φ and f belong to some classes of analytic functions. The linear integral operator is shown to map certain subclasses of analytic functions with fixed initial coefficient into itself. The final problem considered is in obtaining sufficient conditions for analytic functions with fixed initial coefficient to be starlike or convex. These conditions are framed in terms of the Schwarzian derivative.

CHAPTER 1

INTRODUCTION

The theory of differential subordination is one of the active research topics in the theory of univalent functions. Research on the theory of differential subordination was pioneered by Miller and Mocanu and their monograph [35] compiled a very comprehensive discussion and many applications of the theory. In the last few decades, hundreds of articles related to the subject have been published and many interesting results obtained. By employing the methodology of differential subordination, this thesis investigates the analytic function in unit disk having the fixed initial coefficient in their series expansion.

In the following, a brief introduction of elementary concepts from the theory of univalent functions as well as the theory of differential subordination will be given which will be very useful in later chapters. The relevant definitions, known results and proofs of most of the results can be found in the standard text books by [2, 20, 22, 24, 35].

1.1 Analytic Univalent Functions

Let $\mathbb{C}$ be the complex plane. Let $z_0 \in \mathbb{C}$ and $r > 0$. Denote by

$$\mathrm{D}(z_0, r) := \{z : z \in \mathbb{C}, |z - z_0| < r\}$$

to be the *neighbourhood of* z_0. A set D of $\mathbb{C}$ is called an open set if for every point z_0 in D, there is a neighborhood of z_0 contained in D. An open set D is connected if there is a polygonal path in D joining any pair of points in D.

A *domain* is an open connected set and it is said to be simply connected if the interior domain to every simple closed curve in D lies completely within D. Geometrically, a simply connected domain is a domain without any holes.

A continuous complex-valued function f is *differentiable* at a point $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. Such a function f is said to be *analytic* at z_0 if it is differentiable at z_0 and at every point in some neighbourhood of z_0. It is analytic on D if it is analytic at every point in D. It is known in [2, Corollary 3.3.2, p. 179] that an analytic function f has derivatives of all orders. Thus f has a Taylor series expansion given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!},$$

convergent in some open disk centered at z_0.

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions f of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}).$$

Let $\mathcal{A}$ denote the class of all analytic functions f defined on $\mathbb{U}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each such function f has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}). \tag{1.1}$$

Generally, let $\mathcal{A}_n$ denote the class of all normalized analytic functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \qquad (z \in \mathbb{U},\ n \in \mathbb{N}).$$

where $\mathcal{A}_1 \equiv \mathcal{A}$.

A function f is *univalent* in D if it is one-to-one in D. In other words, the function f does not take the same value twice, that is, $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D with $z_1 \neq z_2$. Thus, a function f is called *locally univalent* at z_0 if it is one-to-one in some neighbourhood of z_0. For an analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

Theorem 1.1. *Let f be analytic in a domain* D*. Then f is locally univalent in a neighbourhood of z_0 in* D *if and only if $f'(z_0) \neq 0$.*

Proof. Let f be locally univalent in a neighbourhood of z_0 in D and suppose that $f'(z_0) = 0$. Then

$$g(z) := f(z) - f(z_0)$$

has a zero of order n, $n \geq 2$, at z_0. Since zeroes of a non-constant analytic function are isolated, there exists an $r > 0$ so that both g and f' have no zeroes in the punctured disk $0 < |z - z_0| \leq r$. Let

$$m = \min_{z \in C} |g(z)|$$

where $C = \{z : |z - z_0| = r\}$, and $h(z) := f(z_0) - a$, where $a \in \mathbb{C}$ satisfies $0 < |a - f(z_0)| < m$. Then $|h(z)| < |g(z)|$ on C. It follows from Rouche's theorem [20, p. 4] that g and $g + h$ have the same numbers of zeroes inside C. Thus $f(z) - a$ has at least two zeroes inside C. Observe that none of these zeros can be at z_0. Since $f'(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq r$, these zeros must be simple. Thus $f(z) = a$ at two or more points inside C. This contradicts the assumption that f is locally univalent in a neighbourhood of z_0 in D.

Now, assume that $f'(z_0) \neq 0$ and f is not locally univalent in any neighbourhood of z_0 in D. For each positive integer n, there are points α_n and β_n in $\mathrm{D}(z_0, \rho/n)$ such that $\alpha_n \neq \beta_n$ but $f(\alpha_n) = f(\beta_n)$. Since $\alpha_n, \beta_n \in \mathrm{D}(z_0, \rho/n)$, it follows that

$$\lim_{n\to\infty} \alpha_n = z_0 \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n = z_0.$$

Since $f(\alpha_n) = f(\beta_n)$, by Cauchy's integral formula, it is evident that

$$\begin{aligned} 0 &= \frac{f(\alpha_n) - f(\beta_n)}{\alpha_n - \beta_n} \\ &= \frac{1}{\alpha_n - \beta_n} \left\{ \frac{1}{2\pi i} \int_C \left[\frac{f(z)}{z - \alpha_n} - \frac{f(z)}{z - \beta_n} \right] dz \right\} \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha_n)(z - \beta_n)}\, dz. \end{aligned}$$

Since $\alpha_n \to z_0$ and $\beta_n \to z_0$ as $n \to \infty$, it follows that $f(z)/[(z - \alpha_n)(z - \beta_n)]$ converges uniformly to $f(z)/(z - z_0)^2$. Thus

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha_n)(z - \beta_n)}\, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz = f'(z_0),$$

which contradicts the assumption that $f'(z) \neq 0$. Therefore f must be locally univalent in a neighbourhood of z_0 in D. □

Let γ be a smooth arc represented parametrically by $z = z(t),\ a \leq t \leq b$, and let f be a function defined at all points z on γ. Suppose that γ passes through a point $z_0 = z(t_0),\ a \leq t_0 \leq b$, at which f is analytic and that $f'(z_0) \neq 0$. If $w(t) = f[z(t)]$, then $w'(t_0) = f'[z(t_0)]z'(t_0)$, and this means that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0).$$

Let $\psi_0 = \arg w'(t_0)$, $\phi_0 = \arg f'[z(t_0)]$ and $\theta_0 = \arg z'(t_0)$, then $\psi_0 = \phi_0 + \theta_0$. Thus $\phi_0 = \psi_0 - \theta_0$, and the angles ψ_0 and θ_0 differs by the angle of rotation $\phi_0 = \arg f'(z_0)$.

Let γ_1 and γ_2 be two smooth arcs passing through z_0, and let θ_1 and θ_2 be angles of inclination of directed lines tangent to γ_1 and γ_2, respectively, at z_0. Then the quantities $\psi_1 = \phi_0 + \theta_1$ and $\psi_2 = \phi_0 + \theta_2$ are angles of inclination of directed lines tangent to the images curves Γ_1 and Γ_2, respectively, at $w_0 = f(z_0)$. Thus $\psi_2 - \psi_1 = \theta_2 - \theta_1$, that is, the angle $\psi_2 - \psi_1$ from Γ_1 to Γ_2 is the same as the angle $\theta_2 - \theta_1$ from γ_1 to γ_2.

This angle-preserving property leads to the notion of conformal maps. A function that preserves both the magnitude and orientation of angles is said to be *conformal*. The transformation $w = f(z)$ is conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$. It follows from Theorem 1.1 that the locally univalent functions are also conformal. A function which is both analytic and univalent on D is called a *conformal mapping of* D because of its angle-preserving property.

A *Möbius transformation* is a linear fractional transformation of the form

$$M(z) = \frac{az+b}{cz+d}, \qquad (z \in \overline{\mathbb{C}}),$$

where the coefficients a, b, c, d are complex constants satisfying $ad - bc \neq 0$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. The Möbius transformation M provides a conformal mapping of $\overline{\mathbb{C}}$ onto itself.

The famous Riemann mapping theorem states that any simply connected domain which is not the whole complex plane $\mathbb{C}$, can be mapped conformally onto $\mathbb{U}$.

Theorem 1.2. (Riemann Mapping Theorem) [20, p. 11] *Let* D *be a simply connected domain which is a proper subset of the complex plane* $\mathbb{C}$*. If* ζ *be a given point in* D*, then there is a unique function* f*, analytic and univalent in* D*, which maps* D *conformally onto the unit disk* $\mathbb{U}$ *satisfying* $f(\zeta) = 0$ *and* $f'(\zeta) > 0$*.*

In view of this theorem, the study of conformal mappings on simply connected domains may be restricted to study of analytic univalent functions in $\mathbb{U}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which are univalent and of the form (1.1). Thus $\mathcal{S}$ is the class of all normalized univalent functions in $\mathbb{U}$. An important member of the class $\mathcal{S}$ is the *Koebe function* given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n, \qquad (z \in \mathbb{U}), \tag{1.2}$$

which maps $\mathbb{U}$ conformally onto $\mathbb{C} \setminus \{w \in \mathbb{R} : w \leq -1/4\}$. The Koebe function and its rotations $e^{-i\alpha}k(e^{i\alpha}z), \alpha \in \mathbb{R}$, play a very important role in the study of the class $\mathcal{S}$. These functions are extremal for various problems in the class $\mathcal{S}$.

In 1916, Bieberbach [12] conjectured that $|a_n| \leq n, (n \geq 2)$ for f in $\mathcal{S}$. This conjecture is known as Bieberbach conjecture. But he only proved for the case when $n = 2$ and this result was called Bieberbach theorem.

Theorem 1.3. (Bieberbach Theorem) [22, p. 33] *If* $f \in \mathcal{S}$*, then*

$$|a_2| \leq 2.$$

Equality occurs if and only if f *is a rotation of the Koebe function* k*.*

This theorem will be proved later in Section 1.3.

The Bieberbach conjecture was a difficult open problem as many mathematicians have investigated it only for certain values of n. However, in 1985, De Branges [19] successfully proved this conjecture for all coefficients n with the help of the hypergeometric functions.

Theorem 1.4. (Bieberbach Conjecture or de Branges Theorem) [19] *The coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \le n$ for $n = 2, 3, \ldots$. Equality occurs if and only if f is the Koebe function k or one of its rotations.*

The coefficient inequality $|a_2| \le 2$ from the Bieberbach theorem yields many important properties of univalent functions in the class $\mathcal{S}$. One of the important consequences is the well-known covering theorem due to Koebe.

Theorem 1.5. (Koebe One-Quarter Theorem) [20, p. 31] *The range of every function of the class $\mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$.*

This theorem will be proved later in Section 1.3.

Another important consequence of the Bieberbach theorem is the distortion theorem which provides sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.6. (Distortion Theorem) [20, p. 32] *If $f \in \mathcal{S}$, then*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, \qquad (|z| = r < 1).$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

This theorem will be proved later in Section 1.3. The distortion theorem can be applied to obtain sharp upper and lower bounds for $|f(z)|$ and that result is known as the growth theorem.

Theorem 1.7. (Growth Theorem) [20, p. 33] *If $f \in \mathcal{S}$, then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad (|z| = r < 1).$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

There are many criteria for functions to be univalent. In 1915, Alexander proved an interesting result for the univalence of analytic functions. He showed that, if f is analytic in $\mathbb{U}$ satisfying $\operatorname{Re} f'(z) > 0$ for each $z \in \mathbb{U}$, then f is univalent in $\mathbb{U}$ [22, Theorem 12, p. 88]. Furthermore, in 1935, Noshiro [42] and Warschawski [57] independently proved the following well-known Noshiro-Warschawski theorem.

Theorem 1.8. (Noshiro-Warschawski Theorem) [42, 57] *If an analytic function f satisfies* $\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$ *for some real* α *and for all* z *in a convex domain* D*, then f is univalent in* D.

Another criterion for functions to be univalent involved the Schwarzian derivative. The *Schwarzian derivative* of a locally univalent analytic function f in $\mathbb{U}$ is given by

$$\{f, z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Here f' and f'' denote the first and second derivatives of f, respectively. The Schwarzian derivative of any Möbius transformation M is identically zero. Let $\mathscr{S}$ denote the mapping from f to its Schwarzian derivative. It has the property

$$\mathscr{S}(M \circ f) = (\mathscr{S}(M) \circ f) \cdot \left(f'\right)^2 + \mathscr{S}(f) = \mathscr{S}(f),$$

because $\mathscr{S}(M) = 0$ for every Möbius transformation M. This shows that the Schwarzian derivative is invariant under Möbius transformation M [20, p. 259].

In 1949, Nehari [39] discovered that certain estimates on the Schwarzian derivative imply global univalence.

Theorem 1.9. [39, Theorem I, p. 545] *If $f \in \mathcal{S}$, then*

$$|\{f,z\}| \leq \frac{6}{(1-|z|^2)^2}. \tag{1.3}$$

Conversely, if $f \in \mathcal{A}$ satisfies

$$|\{f,z\}| \leq \frac{2}{(1-|z|^2)^2}, \tag{1.4}$$

then f is univalent in $\mathbb{U}$.

The constant 6 and 2 are the best possible. In the same paper, Nehari [39] also obtained the sufficient condition $|\{f,z\}| \leq \pi^2/2$ that implies the univalence of f in $\mathbb{U}$. The constant $\pi^2/2$ is the best possible.

In a similar vein, Pokornyi [50] in 1951 obtained

$$|\{f,z\}| \leq \frac{4}{(1-|z|^2)} \tag{1.5}$$

is sufficient to ensure the univalence of f and the constant 4 is again best possible. Later, Nehari [40] unified all criterion (1.3), (1.4) and (1.5) by establishing the following general criterion of univalence

$$|\{f,z\}| \leq 2p(|z|),$$

where p is a positive continuous even function defined on the interval $(-1,1)$, with the properties that $p(-x) = p(x)$, $(1-x^2)^2 p(x)$ is nonincreasing on the interval $[0,1)$ and the differential equation $y''(x) + p(x)y(x) = 0$ has a solution which does not vanish for $(-1,1)$. The function p is referred as *Nehari function*.

The problem of finding similar bounds on the Schwarzian derivative that would imply univalence was investigated by other authors including Chuaqui et al. [17], Chuaqui et al. [18], Nunokawa et al. [43], Opoolaa and Fadipe-Josepha [44], Ovesea-Tudor and Shigeyoshi [45] and Ozaki and Nunokawa [47].

1.2 Subclasses of Analytic Univalent Functions

This section begins by discussing an important class of functions, so called the functions with positive real part. The class $\mathcal{P}$, consisting of all the functions which have positive real part in $\mathbb{U}$ will be introduced and some of their basic properties will be given as the following.

Definition 1.1. (Functions with Positive Real Part) [22, p. 78] *A normalized analytic function h in $\mathbb{U}$ of the form*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad (z \in \mathbb{U}), \tag{1.6}$$

with

$$\text{Re } h(z) > 0$$

is called a function with positive real part in $\mathbb{U}$.

A function with positive real part is also known as a Carathéodory function. An important example of a function of the class $\mathcal{P}$ is the *Möbius function* defined by

$$m(z) \equiv \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n \tag{1.7}$$

which maps $\mathbb{U}$ onto the half-plane $\{\text{Re } w > 0\}$. The role of this Möbius function m is the same as that of Koebe function in the class $\mathcal{S}$. But the function m is not the only extremal functions in the class $\mathcal{P}$, there are many other functions of the form (1.6), which are extremal for the class $\mathcal{P}$.

The following lemma gives the coefficient bound for functions in the class $\mathcal{P}$.

Lemma 1.1. (Carathéodory's Lemma) [20, p. 41] *If $h \in \mathcal{P}$ is of the form (1.6), then the following sharp estimate holds:*

$$|c_n| \leq 2, \quad (n = 1, 2, 3, \ldots).$$

Equality occurs for the Möbius function m.

The following theorem gives the growth and distortion results for the class $\mathcal{P}$.

Theorem 1.10. [24, p. 31] *If $h \in \mathcal{P}$ and $|z| = r < 1$, then*

$$\frac{1-r}{1+r} \leq |h(z)| \leq \frac{1+r}{1-r},$$
$$\frac{1-r}{1+r} \leq \text{Re } h(z) \leq \frac{1+r}{1-r},$$
$$|h'(z)| \leq \left(\frac{2}{1-r^2}\right) \text{Re } h(z) \leq \frac{2}{(1-r)^2}.$$

Equalities occurs if and only if h is a suitable rotation of the Möbius function m.

The class $\mathcal{P}$ is directly related to a number of important and basic subclasses of univalent functions. These subclasses include the well-known classes of convex and starlike functions. The geometric properties of these classes along with their relationships with each other will be given.

A set D in $\mathbb{C}$ is called *convex* if for every pair of points w_1 and w_2 lying in D, the line segment joining w_1 and w_2 also lies entirely in D, that is,

$$w_1, w_2 \in \text{D}, \ 0 \leq t \leq 1 \qquad \Longrightarrow \qquad t w_1 + (1-t) w_2 \in \text{D}.$$

Definition 1.2. (Convex Functions) [22, p. 107] *If a function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a convex domain, then f is called a convex function.*

The subclass of $\mathcal{S}$ consisting of all convex functions on $\mathbb{U}$ is denoted by $\mathcal{CV}$. An analytic description of the class $\mathcal{CV}$ is given by the following result.

Theorem 1.11. (Analytical Characterization of Convex Functions) [24, p. 38] *Let $f \in \mathcal{A}$. Then f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \qquad (z \in \mathbb{U}).$$

For instance, the Möbius function m in (1.7) and the function

$$L(z) = \frac{z}{1-z} \tag{1.8}$$

which maps $\mathbb{U}$ onto the half-plane $\{\operatorname{Re}\ z > -1/2\}$ are convex functions in $\mathbb{U}$. The following theorem gives the coefficient bound for $f \in \mathcal{CV}$ and this result was proved by Loewner [32] in 1917.

Theorem 1.12. [32] *If $f \in \mathcal{CV}$, then*

$$|a_n| \leq 1, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the function L defined in (1.8).

Let w_0 be an interior point of a set D in $\mathbb{C}$. Then D is said to be *starlike with respect to w_0* if the line segment joining w_0 to every other point w in D lies in D, that is,

$$w \in \mathrm{D},\ 0 \leq t \leq 1 \qquad \Longrightarrow \qquad (1-t)w_0 + tw \in \mathrm{D}.$$

For $w_0 = 0$, the set D is called starlike with respect to the origin or simply a starlike domain.

Definition 1.3. (Starlike Functions) [22, p. 108] *If a function f maps $\mathbb{U}$ onto a domain that is starlike with respect to w_0, then f is called a starlike function with respect to w_0. In the special case that $w_0 = 0$, f is simply called a starlike function.*

The subclass of $\mathcal{S}$ consisting of all starlike functions on $\mathbb{U}$ is denoted by $\mathcal{ST}$. An analytic description of the class $\mathcal{ST}$ is given by the following result.

Theorem 1.13. (Analytical Characterization of Starlike Functions) [24, p. 36] *Let $f \in \mathcal{A}$ with $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad (z \in \mathbb{U}).$$

The Koebe function in (1.2) is an example of starlike function in $\mathbb{U}$. The following theorem gives the coefficient bound for $f \in \mathcal{ST}$ and this result was proved by Nevanlinna [41] in 1921.

Theorem 1.14. [41] *If $f \in \mathcal{ST}$, then*

$$|a_n| \leq n, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the Koebe function k.

Every convex function is evidently starlike. Thus the subclasses of $\mathcal{S}$ consisting of convex and starlike functions satisfy the following inclusion relation:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{S}.$$

Observe that the classes $\mathcal{CV}$ and $\mathcal{ST}$ are closely related to each other. It is given by the following important relationship:

$$f \in \mathcal{CV} \qquad \Longleftrightarrow \qquad zf'(z) \in \mathcal{ST}, \qquad (z \in \mathbb{U}),$$

due to Alexander [1] in 1915. This result is known as Alexander's theorem.

In 1936, Robertson [53] introduced the classes of convex and starlike functions of order α for $0 \leq \alpha < 1$, which are defined by

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\ ;\ z \in \mathbb{U} \right\},$$

and

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\ ;\ z \in \mathbb{U} \right\},$$

respectively. In particular, $\mathcal{CV}(0) = \mathcal{CV}$ and $\mathcal{ST}(0) = \mathcal{ST}$. It is clear that

$$\mathcal{CV}(\alpha) \subseteq \mathcal{CV} \qquad \text{and} \qquad \mathcal{ST}(\alpha) \subseteq \mathcal{ST}.$$

Another important relationship between the classes $\mathcal{CV}$ and $\mathcal{ST}$ is given by the classical result of Strohhäcker [55] in 1933. He proved that if $f \in \mathcal{CV}$, then $f \in \mathcal{ST}(1/2)$, where $\mathcal{ST}(1/2)$ is the class of starlike functions of order 1/2. The following theorem is an extension of the result.

Theorem 1.15. [35, p. 115] *If* $0 \le \alpha < 1$, *then the order of starlikeness of convex functions of order* α *is given by*

$$\tau(\alpha) := \tau(\alpha; 1, 0) = \begin{cases} \frac{2\alpha - 1}{2 - 2^{2(1-\alpha)}}, & if \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\ln 2}, & if \quad \alpha = \frac{1}{2}. \end{cases}$$

The following result gives the growth and distortion theorem for convex functions of order α due to Robertson [53].

Theorem 1.16. [53] *Let* $f \in \mathcal{CV}(\alpha)$, $0 \le \alpha < 1$, *and* $|z| = r < 1$. *Then*

$$\frac{1}{(1+r)^{2(1-\alpha)}} \le |f'(z)| \le \frac{1}{(1-r)^{2(1-\alpha)}}.$$

If $\alpha \neq 1/2$, *then*

$$\frac{(1+r)^{2\alpha-1} - 1}{2\alpha - 1} \le |f(z)| \le \frac{1 - (1-r)^{2\alpha-1}}{2\alpha - 1},$$

and if $\alpha = 1/2$, *then*

$$Log\,(1+r) \le |f(z)| \le -Log\,(1-r).$$

All of these inequalities are sharp. The extremal functions are rotations of

$$f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \frac{1}{2}, \\ -Log\,(1-z), & \alpha = \frac{1}{2}. \end{cases}$$

1.3 Analytic Univalent Functions with Fixed Intial Coefficient

Closely related to the class $\mathcal{S}$ is the class Σ consisting of functions g which are analytic and univalent on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$ except for a simple pole at infinity with residue 1. The Laurent series expansion of such functions is of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \qquad (z \in \Delta). \tag{1.9}$$

This function g maps Δ onto the complement of a connected compact set E. The subclass of Σ that omits $z = 0$ in E is denoted by Σ_0.

Observe that if $f \in \mathcal{S}$ is given by (1.1), then

$$g(z) = \frac{1}{f(1/z)} = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \cdots \qquad (z \in \Delta),$$

in Σ_0. Conversely, if $g \in \Sigma_0$ is given by (1.9), then

$$f(z) = \frac{1}{g(1/z)} = z - b_0 z^2 + (b_0^2 - b_1)z^3 + \cdots \qquad (z \in \mathbb{U}),$$

in $\mathcal{S}$. In fact, the univalence of f implies the univalence of g as well.

In 1914, Gronwall [26] proved a theorem about the Laurent series coefficients of functions in the class Σ which is known as the area theorem.

Theorem 1.17. (Area Theorem) [26] *If $g \in \Sigma$ is given by (1.9), then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

with equality if and only if $g \in \widetilde{\Sigma}$.

The direct application of the area theorem can be seen clearly in the proof of Bieberbach theorem. Bieberbach theorem states that every function f in the class $\mathcal{S}$ has the property $|a_2| \leq 2$. The following proof can be found in [22, p. 34].

Proof of Theorem 1.3 (Bieberbach Theorem). Suppose that $f \in \mathcal{S}$. A square root transformation yields the function

$$g(z) = \sqrt{f(z^2)} = z + \frac{1}{2}a_2 z^3 + \left(\frac{1}{2}a_3 - \frac{1}{8}a_2^2\right) z^5 + \cdots$$

in $\mathcal{S}$. An inversion to g produce a function

$$h(z) = \frac{1}{g(1/z)} = z - \frac{1}{2}a_2 \frac{1}{z} + a_3 \frac{1}{z^3} + \cdots$$

in Σ_0. By the area theorem, it follows that

$$\sum_{n=1}^{\infty} n|b_n|^2 = \left|-\frac{a_2}{2}\right|^2 + 3|a_3|^2 + \cdots \leq 1,$$

and so $|-a_2/2|^2 \leq 1$ or $|a_2| \leq 2$, as required. If $a_2 = 2e^{i\alpha}$, for some real α, it is clear that the coefficient $b_n = 0$ for all $n \geq 2$. This implies that h has the form $h(z) = z - e^{i\alpha}/z$. Hence,

$$g(z) = \frac{1}{h(1/z)} = \frac{1}{1/z - e^{i\alpha}z} = \frac{z}{1 - e^{i\alpha}z^2}.$$

Since $f(z^2) = g^2(z) = z^2/(1-e^{i\alpha}z^2)^2$, and thus $f(z) = z/(1-e^{i\alpha}z)^2$ is a rotation of the Koebe function. $\square$

The Bieberbach inequality $|a_2| \leq 2$ can be used to prove other properties of functions f in the class $\mathcal{S}$. The famous covering theorem due to Koebe, that is, the Koebe one-quarter theorem is an important application of Bieberbach theorem. It ensures that the image of $\mathbb{U}$ under every f in $\mathcal{S}$ contains an open disk centered at the origin with radius $1/4$. The following proof can be found in [20, p. 31].

Proof of Theorem 1.5 (Koebe One-Quarter Theorem). Every function $f \in \mathcal{S}$ satisfies $|a_2| \leq 2$ by Bieberbach theorem. Suppose that $\omega \notin f(\mathbb{U})$, and the omitted value transformation yields the function

$$g(z) = \frac{\omega f(z)}{\omega - f(z)} = z + \left(a_2 + \frac{1}{\omega}\right) z^2 + \cdots$$

in $\mathcal{S}$. From Bieberbach theorem, it follows that $|a_2 + 1/\omega| \leq 2$ and the triangle inequality yields

$$\left|\frac{1}{\omega}\right| - |a_2| \leq \left|a_2 + \frac{1}{\omega}\right| \leq 2.$$

Since $|a_2| \leq 2$, it is clear that $|1/\omega| \leq 4$, or $|\omega| \geq 1/4$. If $|\omega| = 1/4$, then $|a_2| = 2$, and hence f is some rotation of the Koebe function. $\square$

The proof shows that the Koebe function and its rotations are the only functions in the class $\mathcal{S}$ which omit a value of modulus $1/4$. Thus the range of every other function in $\mathcal{S}$ covers a disk of larger radius.

Bieberbach inequality $|a_2| \leq 2$ also has application to establish the estimate leading to the fundamental theorem about univalent functions, that is, the Koebe distortion theorem. It yields bounds on $|f'(z)|$ as f ranges over the class $\mathcal{S}$. The following proof can be found in [24, p. 15].

Proof of Theorem 1.6 (Distortion Theorem). Suppose that $f \in \mathcal{S}$ and let

$$w(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta} = z + (1 - |z|^2)\zeta - \bar{z}(1 - |z|^2)\zeta^2 + \cdots, \qquad (\zeta \in \mathbb{U}),$$

be a Möbius transformation of $\mathbb{U}$ onto $\mathbb{U}$ with $w(0) = z$ and $w'(0) = 1 - |z|^2$. Then the disk automorphism transformation yields the function

$$g(\zeta) = \frac{f(w(\zeta)) - f(z)}{(1 - |z|^2) f'(z)} = \zeta + \left[\frac{(1 - |z|^2) f''(z)}{2f'(z)} - \bar{z} \right] \zeta^2 + \cdots, \qquad (\zeta \in \mathbb{U}),$$

in $\mathcal{S}$. By Bieberbach theorem, it follows that

$$\left| \frac{(1 - |z|^2) f''(z)}{2f'(z)} - \bar{z} \right| \leq 2.$$

Multiplying by $2|z|/(1 - |z|^2)$ to the latter inequality yields

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{4|z|}{1 - |z|^2}.$$

Note that the inequality $|\tau| \leq \xi$ implies that $-\xi \leq \text{Re}\,\{\tau\} \leq \xi$. Thus

$$\frac{2|z|^2}{1 - |z|^2} - \frac{4|z|}{1 - |z|^2} \leq \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} \leq \frac{2|z|^2}{1 - |z|^2} + \frac{4|z|}{1 - |z|^2}. \qquad (1.10)$$

Since $f'(z) \neq 0$ and $f'(0) = 1$, there exists an analytic branch of $\log f'$ such that $\log f'(z)|_{z=0} = 0$. For $z = re^{i\theta}$, it follows that

$$\frac{\partial}{\partial r} \log |f'(z)| = \frac{\partial}{\partial r} \text{Re}\,\{\log f'(z)\} = \frac{1}{r} \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\}.$$

It is evident from (1.10) that

$$\frac{2r^2-4r}{1-r^2} \leq r\frac{\partial}{\partial r}\log\left|f'\left(re^{i\theta}\right)\right| \leq \frac{2r^2+4r}{1-r^2},$$

or

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r}\log\left|f'\left(re^{i\theta}\right)\right| \leq \frac{2r+4}{1-r^2}.$$

Integrating the last inequality with respect to r gives

$$\int_0^r \frac{2u-4}{1-u^2}\,du \leq \log\left|f'\left(re^{i\theta}\right)\right| \leq \int_0^r \frac{2u+4}{1-u^2}\,du.$$

Since

$$\int_0^r \frac{2u-4}{1-u^2}\,du = \int_0^r \frac{1}{u-1} - \frac{3}{1+u}\,du = \log(1-r) - 3\log(1+r),$$

and

$$\int_0^r \frac{2u+4}{1-u^2}\,du = \int_0^r \frac{1}{1+u} + \frac{3}{1-u}\,du = \log(1+r) - 3\log(1-r),$$

it is clear that

$$\log\left(\frac{1-r}{(1+r)^3}\right) \leq \log\left|f'\left(re^{i\theta}\right)\right| \leq \log\left(\frac{1+r}{(1-r)^3}\right).$$

Since $\log|f'(0)| = \log 1 = 0$, exponentiating both sides yields

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

□

In view of the influence of the second coefficient on the investigation of geometric properties of the class $\mathcal{S}$, the class of analytic functions with a fixed initial coefficient will be investigated in this thesis. Let $\mathcal{H}_\beta[a,n]$ be the class consisting of all analytic functions f in $\mathbb{U}$ of the form

$$f(z) = a + \beta z^n + a_{n+1}z^{n+1} + \cdots$$

with a fixed initial coefficient β in $\mathbb{C}$. Since its rotation $e^{-i\alpha}f(e^{i\alpha}z)$ is in $\mathcal{H}_\beta[a,n]$, choose α such that $\beta > 0$. In the other words, since $f \in \mathcal{H}_\beta[a,n]$ is rotationally invariance, β is assumed to be a non-negative real number.

Further, let $\mathcal{A}_{n,b}$ be the class consisting of all normalized analytic functions $f \in \mathcal{A}_n$ in $\mathbb{U}$ of the form

$$f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$$

where the coefficient $a_{n+1} = b$ is a fixed non-negative real number. Write $\mathcal{A}_{1,b}$ as $\mathcal{A}_b$. Thus, the subclass of $\mathcal{A}_b$ consisting of univalent functions is denoted by $\mathcal{S}_b$ and satisfy $\mathcal{S}_b \subset \mathcal{S}$. For $0 \leq \alpha < 1$, let $\mathcal{CV}_b(\alpha)$ and $\mathcal{ST}_b(\alpha)$ be the classes of convex and starlike functions of order α in $\mathcal{S}_b$, respectively. When $\alpha = 0$, these classes are denoted by $\mathcal{CV}_b := \mathcal{CV}_b(0)$ and $\mathcal{ST}_b := \mathcal{ST}_b(0)$.

1.4 Differential Subordination

A differential subordination in the complex plane is a generalization of a differential inequality on the real line. Obtaining information about the properties of a function from its derivatives plays an important role in functions of a real variable. In the study of complex-valued functions, there are differential implications that are characterizing the functions. A simple example is the Noshiro-Warschawski theorem (Theorem 1.8) in Section 1.1.

In the view of the principle of subordination between analytic functions, let f and g be a member of $\mathcal{H}(\mathbb{U})$. Then, the function f is said to be *subordinate* to g in $\mathbb{U}$, written as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z), \qquad (z \in \mathbb{U}),$$

if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The basic notations, definitions and theorems stated in this section can be found in the monograph by Miller and Mocanu, which is the main reference that provides a comprehensive discussion on differential subordination. To develop the main idea of Miller and Mocanu's theory on differential subordination, let p be analytic in $\mathbb{U}$ with $p(0) = a$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Let Ω and $\triangle$ be any subsets in $\mathbb{C}$ and consider the differential implication:

$$\left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \qquad \Rightarrow \qquad p(\mathbb{U}) \subset \triangle. \tag{1.11}$$

The following definition is required to formulate the fundamental result in the theory of differential subordination.

Definition 1.4. [35, Definition 2.2b, p. 21] *Denote by Q the set of functions q that are analytic and univalent in $\overline{\mathbb{U}}\backslash E(q)$, where*

$$E(q) := \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.

By the definition of Q, a suitably defined class of functions Ψ as below is a basis to develop the fundamental result in the theory of differential subordination.

Definition 1.5. (Admissibility Condition) [35, Definition 2.3a, p. 27] *Let Ω be a* *domain in $\mathbb{C}$, $q \in Q$, and n be a positive integer. The class of admissible functions $\Psi_n(\Omega, q)$ consists of functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfying the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega \tag{1.12}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, and

$$\mathrm{Re}\,\left(\frac{t}{s} + 1\right) \geq m \mathrm{Re}\,\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

for $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \backslash E(q)$ and $m \geq n$. In particular, $\Psi_1(\Omega, q) := \Psi(\Omega, q)$.

If $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$, then the admissibility condition (1.12) reduces to

$$\psi\left(q(\zeta), m\zeta q'(\zeta); z\right) \notin \Omega$$

for $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \backslash E(q)$ and $m \geq n$.

The next theorem is the fundamental result in the theory of first and second-order differential subordination.

Theorem 1.18. [35, Theorem 2.3b, p. 28] *Let $\psi \in \Psi_n(\Omega, q)$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

In view of this theorem, the differential implication of (1.11) is equivalent to

$$\left\{\psi\left(p(z), zp'(z), z^2 p''(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \qquad \Rightarrow \qquad p(z) \prec q(z),$$

by assuming that $\triangle \neq \mathbb{C}$ is a simply connected domain containing the point a and there is a conformal mapping q of $\mathbb{U}$ onto $\triangle$ satisfying $q(0) = a$.

In the special case when $\Omega \neq \mathbb{C}$ is also a simply connected domain, then $\Omega = h(\mathbb{U})$

where h is a conformal mapping of $\mathbb{U}$ onto Ω such that $h(0) = \psi(a,0,0;0)$. In addition, suppose that the function $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ is analytic in $\mathbb{U}$. In this case, the differential implication of (1.11) is rewritten as

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \qquad \Rightarrow \qquad p(z) \prec q(z).$$

Denote this class by $\Psi_n(h(\mathbb{U}), q)$ or $\Psi_n(h,q)$ and the following result is an immediate consequence of Theorem 1.18.

Theorem 1.19. [35, Theorem 2.3c, p. 30] *Let $\psi \in \Psi_n(h,q)$ with $q(0) = a$. If $p \in \mathcal{H}[a,n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in $\mathbb{U}$, and*

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z),$$

then $p(z) \prec q(z)$.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.13}$$

then p is called a *solution* of the differential subordination. A univalent function q is called a *dominant of the solution* of the differential subordination if $p(z) \prec q(z)$ for all p satisfying (1.13). A dominant $\tilde{q}$ satisfying $\tilde{q} \prec q$ for all q of (1.13) is said to be the *best dominant* of (1.13). The best dominant is unique up to a rotation of $\mathbb{U}$. If $p(z) \in \mathcal{H}[a,n]$, then $p(z)$ will be called an (a,n)-*solution*, $q(z)$ an (a,n)-*dominant*, and $\tilde{q}(z)$ the *best* (a,n)-*dominant*.

The more general version of (1.13) is given by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \tag{1.14}$$

where $\Omega \subset \mathbb{C}$ is a simply connected domain containing $h(\mathbb{U})$. Even though $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in $\mathbb{U}$, the condition in (1.14) shall also be referred as a second-order differential subordination. The same definition of solution, dominant and best dominant as given above can be extended to this generalization.

1.5 Integral Operators

The study of operators plays an important role in geometric function theory. Over the past few decades, many authors have employed various methods to study different types of integral operator $\mathbf{I}$ mapping subsets of $\mathcal{S}$ into $\mathcal{S}$. In this section, some integral operators which map certain subsets $\mathcal{A}$ into $\mathcal{S}$ are given. Noting that an integral operator is sometimes called an integral transformation.

The study of operators can be traced back to 1915 due to Alexander [1]. He introduced an operator $\mathbf{A} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathbf{A}[f](z) := \int_0^z \frac{f(t)}{t}\, dt,$$

and the operator is now known as Alexander operator. By the Alexander theorem, it is evident that $\mathbf{A}$ is in $\mathcal{CV}$ if and only if $z\mathbf{A}'[f](z) = f(z)$ is in $\mathcal{ST}$.

In 1960, Biernacki [13] conjectured that $f \in \mathcal{S}$ implies $\mathbf{A} \in \mathcal{S}$, but this turned out to be wrong as subsequently, in 1963, Krzyz and Lewandowski [27] disproved it by giving the following counterexample:

$$f(z) = ze^{(i-1)\mathrm{Log}(1-iz)} \equiv \frac{z}{(1-iz)^{1-i}}, \tag{1.15}$$

where Log denotes the principal branch of the logarithm. A function $f \in \mathcal{A}$ is called