SPLINES FOR TWO-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

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SPLINES FOR TWO-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

by

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LIST OF ABBREVIATIONS

BCBIM	Bicubic B-spline Interpolation Method
BCTBIM	Bicubic Trigonometric B-spline Interpolation Method
BEM	Boundary Element Method
BVPs	boundary value problems
DLWE	Discontinuous Lagendre Multiwavelet Element
FDM	Finite Difference Method
FEM	Finite Element Method
FMM	Fast Multipole Method
HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method
ISOM	Isogeometrical Method
MLPG	Meshless Local Petrov-Galerkin
MLS	Moving Least Squares
NHPM	New Homotopy Perturbation Method
ODEs	ordinary differential equations
PDE	partial differential equation
PDEs	partial differential equations
RBF	Radial Basis Function
VIM	Variational Iteration Method

LIST OF SYMBOLS

A, B, C, D, E, F	integer constants
a	endpoint in x-direction
b	endpoint in y-direction
C_i^j	real constant
c	given real constant in the heat and wave equations
E	error variable
$E(x, y)$	function of error by subtracting exaction solution from the approximate solution
$e(x, y)$	function of error in operator notation
$E(x_i, y_j)$	function of error at point (x_i, y_j)
f	given function of x and y
g	arbitrary function of x and y
l	$\{b, t\}$
L	$\{B, T\}$
L_2	two-norm
L_∞	infinity-norm
lim	limit
F	functional
i, j	integer index

m	number of partition in x-direction
n	number of partition in y-direction
p, q	given functions of x and y that specify the initial conditions
p_1, p_2, p_3, p_4	given functions of x and y that specify the boundary conditions
$S(x, y)$	function of approximate solution
$S(x_i, y_j)$	function of approximate solution at point (x_i, y_j)
t	independent time variable
t_0	particular time
u	dependent variable
$u(x, y)$	function of exact solution
$u(x, y, t)$	function of exact solution
$u(x_i, y_j)$	function of exact solution at point x_i, y_j
w	integer
x, y	independent spatial variables
x_i	discretized x
y_j	discretized y
β, γ	mode numbers in the Fourier harmonics
$\hbar$	auxiliary parameter for Homotopy Analysis Method
Δt	step-size in t-direction
Δx	step-size in x-direction

Δy	step-size in y-direction
ι	$\sqrt{-1}$
∇	Laplace operator
ϕ	scalar potential
θ	angle in radians

SPLIN UNTUK PERSAMAAN PEMBEZAAN SEPARA DUA DIMENSI

ABSTRAK

Di dalam tesis ini, dua kaedah berasaskan splin dibangunkan untuk menyelesaikan persamaan pembezaan separa dua dimensi. Kaedah-kaedah tersebut adalah Kaedah Interpolasi Splin-B Bikubik (KISB) dan Kaedah Interpolasi Splin-B Trigonometri Bikubik (KISTB). Kajian ini adalah kesinambungan daripada perkembangan terkini di dalam penggunaan kedua-dua splin terhadap masalah-masalah satu dimensi. Pendekatan KISB dan KISTB adalah serupa kecuali pada penggunaan fungsi asas splin yang berbeza, iaitu splin-B kubik dan splin-B trigonometri kubik. Bagi masalah dengan pembolehubah masa, masa tersebut dipecahkan menggunakan Kaedah Beza Terhingga yang biasa. Pembolehubah ruang pula dipecahkan menggunakan fungsi permukaan splin bikubik. Dengan menambah syarat-syarat permulaan dan sempadan, satu sistem persamaan linear yang *underdetermined* akan terhasil. Sistem ini kemudiannya diselesaikan menggunakan Kaedah Kuasa Dua Terkecil. Persamaan-persamaan ini diselesaikan menurut jenis-jenisnya, iaitu persamaan Poisson, persamaan haba, dan persamaan gelombang. Persamaan-persamaan ini ialah persamaan yang paling mudah masing-masing daripada persamaan pembezaan separa eliptik, parabolik, dan hiperbolik. Untuk persamaan Poisson, KISB didapati menghasilkan keputusan yang setanding dengan keputusan daripada Kaedah Unsur Terhingga. KISB menghasilkan keputusan yang lebih tepat berbading KISTB kecuali pada masalah yang mempunyai penyelesaian tepat berbentuk trigonometri. Skim KISB dibuktikan konsisten dan stabil tidak bersyarat manakala skim KISTB dibuktikan stabil bersyarat. Keputusan berangka bagi KISB dan KISTB didapati tertumpu secara sublinear pada arah x dan y. Untuk persamaan haba, KISB didapati menghasilkan keputusan yang lebih tepat berbanding dengan

KISTB bagi kedua-dua contoh yang mempunyai penyelesaian tepat yang berbentuk geometri dan bukan geometri. Sebaliknya, untuk persamaan gelombang, KISTB didapati menghasilkan keputusan yang lebih baik berbanding KISB. Jadi, untuk persamaan haba dan gelombang, KISB and KISTB tidak semestinya menghasilkan keputusan yang lebih tepat antara satu sama lain. Skim KISB untuk persamaan haba dan gelombang dibuktikan konsisten dan stabil tidak bersyarat manakala skim KISTB dibuktikan stabil bersyarat. Keputusan berangka bagi kedua-dua kaedah didapati tertumpur secara sublinear pada arah x, y, dan t. Kaedah-kaedah ini kemudiannya diuji pada persamaan pembezaan separa yang lebih umum dan terkenal dengan hasil yang memberangsangkan. Persaman-persaman tersebut adalah persamaan resapan air lintang, persamaan Burgers, dan persamaan hiperbola linear yang mempunyai banyak kegunaan di dalam bidang mekanik bendalir dan fenomena gelombang.

SPLINES FOR TWO-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT

In this thesis, two spline-based methods are developed to solve two-dimensional partial differential equations. The methods are Bicubic B-spline Interpolation Method (BCBIM) and Bicubic Trigonometric B-spline Interpolation Method (BCTBIM). This study is a continuation of recent developments in the application of both splines on the one-dimensional problems. The approach of BCBIM and BCTBIM are similar except for the use of different spline basis functions, namely cubic B-spline and cubic trigonometric B-spline, respectively. For problems with time variable, the time is discretized using the usual Finite Difference Method. The spatial variables are discretized using the corresponding bicubic spline surface function. By adding the initial and boundary conditions, an underdetermined system of linear equations results. This system is then solved using the method of Least Squares. The equations are dealt according to its types, namely Poisson's, heat, and wave equations. These equations are the simplest form of elliptic, parabolic, and hyperbolic partial differential equations, respectively. For Poisson's equations, BCBIM is found to produce comparable results with that of Finite Element Method. BCBIM generates slightly more accurate results than BCTBIM except for problems with trigonometric exact solutions. BCBIM scheme is proved to be consistent and unconditionally stable whereas BCTBIM conditionally stable.The numerical results of BCBIM and BCTBIM are found to be sublinearly convergent in directions x and y. For the heat equation, BCBIM is found to produce more accurate results than BCTBIM for both examples with trigonometric and non-trigonometric exact solutions. Otherwise, for the wave equation, BCTBIM is found to produce better results than BCBIM. Therefore, for the heat and wave equations, BCBIM and BCTBIM do not necessarily produce more accurate results than each other. Similarly, BCBIM schemes for the heat and wave equations are proved to be consistent and unconditionaly stable whereas BCTBIM schemes are shown to be conditionally stable. The numerical results from both BCBIM and BCTBIM are found to be sublinearly convergent in directions x, y, and t. These methods are then tested to more general and well-known partial differential equations with promising results. The equations are the advection-diffusion, Burgers' equations, and linear hyperbolic equation that have many applications in the fields of fluid mechanics and wave phenomena.

CHAPTER 1

INTRODUCTION

1.1 Second Order Two-Dimensional Partial Differential Equations (PDEs)

Many physical phenomena are modeled by partial differential equations (PDEs). Some examples are vibrations of solids, flow of fluids, diffusion of chemicals, spread of heat, structure of molecules, interactions of photons and electrons, and radiation of electromagnetic waves (Strauss, 1992a). This thesis deals with second order two-dimensional PDEs.

Suppose that $u(x,y,t)$ is a function and its derivatives are denoted by the subsripts; $\frac{\partial u}{\partial x} = u_x$, and so on. The general form of second order two-dimensional PDEs is as follows.

$$F(t,x,y,u,u_t,u_{tx},u_{ty},u_{tt},u_x,u_{xy},u_{xx},u_y,u_{yy}) = 0. \tag{1.1}$$

In general, this equation has infinitely many solutions. Initial and boundary conditions are imposed on the equation to ensure the uniqueness of the solution. An initial condition gives the function of u at a particular time t_0. From the physical model, this could be extracted from the initial state of the problem. It is either of the form $u(x,y,t_0)$ or its derivative with respect to time. On the other hand, the boundary conditions specifies the function of u at the boundaries of the equation. Again, it could be expressed in a function form, u, or its correspoding spatial derivatives evaluated at the boundaries (Strauss, 1992b).

Well-posed problems involving PDEs are defined on a domain with a set of initial or boundary conditions that has the following three properties:

(i) existence,

(ii) uniqueness, and

(iii) stability.

A solution of a two-dimensional partial differential equation (PDE) is a function $u(x, y, t)$ that satisfies (1.1). The existence and uniqueness properties guarantee that there exists at least and at most one solution to the problem, respectively. The stability property makes certain that the solution behaves in a stable manner according to the data of the problem. That is, a small perturbation on the data would result in a small change on the solution (Strauss, 1992c).

Most equations arised from the physical phenomena are complicated that it becomes difficult to find the exact mathematical solutions. Therefore, many numerical methods have been established to find the approximations to the solutions. Studies on improving and extending the established methods as well as developing new methods to solve these equations are active areas of research in Numerical Analysis.

One of the established numerical methods is known as splines. Splines have been used extensively to solve ordinary differential equations (ODEs) and one-dimensional PDEs (Khan, 2004; Nikolis, 2004; Dag et al., 2005; Caglar et al., 2006; Abd Hamid et al., 2010, 2011, 2012; Goh et al., 2011, 2012; Abbas et al., 2014; Siddiqi and Arshed, 2014). Splines also had been incorporated into other numerical methods to solve two-dimensional PDEs (Hassani et al., 2009; Mohanty et al., 2013; Žitňan, 2013; Mittal and Bhatia, 2014). The main advantage of using splines is that the resulting approximate solutions will be in analytical forms. Therefore, numerical solution at any discrete point can be generated from the approximate analytical solution. In my study, the results of which are reported in this thesis, splines are implemented in a direct manner to solve two-dimensional PDEs.

1.2 Motivations of Study

There are two major motivations of this study. The first one is the masters work by Abd Hamid (2010) that compares four types of splines in solving second order linear ODEs. They are cubic B-spline, cubic trigonometric B-spline, cubic Beta-spline, and extended cubic B-spline. This study concluded that trigonometric B-spline produces slightly more accurate results than B-spline for problems of trigonometric nature. This claim is also supported by the work of Abbas et al. (2014) that applied trigonometric B-spline on the one-dimensional hyperbolic PDEs. The second one is the doctoral work by Goh et al. (2012) that investigated the applicability of B-spline in solving one-dimensional PDEs. This study concluded that B-spline produces more accurate results than the forward time centered space Finite Difference Method (FDM) for these equations. The order of convergence of the method was also calculated and the method was proved to be stable.

From both studies, the idea of extending the application of B-spline to two-dimensional PDEs emerged. So far, B-spline curve, the one-dimensional version of B-spline function, has been applied to solve any differential equation. B-spline is found to produce results that are in good agreement with the exact solutions. Furthermore, B-spline possesses nice properties that can be used to simplify the calculations. The resulting B-spline scheme is also stable and consistent (Caglar et al., 2006; Goh et al., 2011, 2012). However, in Computer Aided Geometric Design, B-spline can be expanded to generate two-dimensional surfaces. Therefore, if the dimension of the PDEs is increased, the same can be done with B-spline. With that, questions on accuracy, stability, and consistency of the B-spline surface need to be addressed. Will the numerical scheme that uses B-spline surface inherit the nice properties that B-spline curve has? If not, what are the limitations? Besides, the approach of B-spline and trigonometric B-spline will be exactly the same except for the use of different surface function. B-spline is composed of polynomials whereas trigonometric B-spline is of trigonometric polynomials.

Since trigonometric B-spline method produces better results for problems with trigonometric exact solutions, it is interesting to figure out whether it will be the same in the two-dimensional case.

So far as is known, these ideas have not been studied yet in the literature.

1.3 Problem Statement

Based on the motivation, the problem statement can be stated as follows: There should be an array of solution tools utilizing splines surfaces for solving two-dimensional PDEs. This is at present somewhat lacking and this gap should be filled. Our aim is to develop methods based on bicubic B-spline and trigonometric B-spline to add to the array of tools.

1.4 Research Questions

The relevant research questions are:

1. What is the method of bicubic B-spline for solving two-dimensional partial differential equations?
2. What is the method of bicubic trigonometric B-spline for solving two-dimensional partial differential equations?
3. What are the differences in methodology between these two methods?
4. Are the methods consistent?
5. Are the methods stable?
6. Do the methods produce convergent results numerically?
7. How do the numerical results differ between the two methods?

8. How do the numerical results differ between the three types of PDEs?
9. Do the methods produce accurate results if compared to other numerical methods?
10. Which method is better in producing accurate and convergent results?
11. What are the limitations of the methods?

1.5 Aims and Objectives of Study

The aim of this thesis is to solve two-dimensional PDEs using B-spline and trigonometric B-spline. With this aim in mind, the objectives are

1. to solve the two-dimensional Poisson's equation using B-spline and trigonometric B-spline,
2. to solve the two-dimensional heat and wave equations using B-spline and trigonometric B-spline,
3. to find the truncation error for the numerical schemes developed in 1 and 2,
4. to check the stability of the numerical schemes developed in 2,
5. to determine the rate of convergence of both schemes, and
6. to extend the application of B-spline and trigonometric B-spline to more complicated two-dimensional problems.

Upon achieving the aim and objectives, a better understanding of the behavior and properties of B-spline and trigonometric B-spline methods in solving two-dimensional PDEs will be obtained.

1.6 Methodology

In order to achieve objectives 1 and 2, the approaches proposed by Abd Hamid (2010) and Goh (2013) are followed closely. The spline surface is always presumed to be the solution to the equation in the spatial direction. The time derivatives will be treated similar to the FDM. By discretizing and collocating the equation, a system of linear equations results. This system is solved and the obtained values can be used to generate approximate analytical solution for the equation. *Mathematica 8* is used to carry out the calculations and to produce the numerical as well as graphical results.

For objectives 3 and 4, Taylor series is incorporated in calculating the order of convergence and Von Neumann stability analysis is used to prove the stability of the scheme. For the last objective, some linearization techniques are used for the nonlinear equations.

1.7 Structure of Thesis

This thesis contains eight chapters altogether. Chapter 1 provides an overview and key factors of the study. Chapter 2 covers a survey of recent numerical methods, especially involving splines, to solve the two-dimensional PDEs. A brief history on the application of splines interpolation for solving the ODEs and PDEs is also included. Chapter 3 discusses on the definition and some relevant properties of cubic B-spline and cubic trigonometric B-spline. This chapter provides simplifications of the spline surfaces that are useful in Chapters 4 to 6. This chapter also covers the error formulas that will be used for the numerical experiments in Chapters 4 to 7.

Chapter 4, 5, and 6 develop the B-spline and trigonometric B-spline methods for the two-dimensional Poisson's, heat, and wave equations, respectively. In each chapter, the truncation error and rate of convergence are figured out. In Chapters 5 and 6, the stability analysis is also carried out. Chapter 7 extends the application of B-spline and trigonometric B-spline to more complicated problems. Lastly, Chapter 8 concludes this study and mentions briefly on the possible future work.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The use of splines for solving differential equations has been an active area of research in the past ten years or so. The applications of splines in solving differential equations started in 1968 and has since expanded (Bickley, 1968). Splines have been used widely to solve ODEs and one-dimensional PDEs. Splines have also been incorporated into other numerical methods to improve the efficiency and accuracy of the results. Recently, the application of splines has been broadened into two-dimensional PDEs.

This chapter will review the literature on the methods for solving two-dimensional PDEs, especially ones involving splines. It is divided into three sections. The first section will cover a brief history of the direct application of splines in solving differential equations. The second section will survey recent methods that utilize splines to solve two-dimensional PDEs. The third section will cover other recent methods available from the literature for solving the same equations. By the end of the chapter, we hope to provide enough information on the trend of the methods for solving the equations and establish the novelty of our study.

2.2 Spline Interpolation Method: A Brief History

The earliest work of using spline interpolation method for solving differential equations dated back in the 1960s. In this study, Bickley solved the simplest form of differential equation, that is, the second order linear two-point boundary value problems (BVPs). The essence of this method is that a spline function can be arbitrarily defined on a certain domain. Therefore, the spline function is generated on the domain of the problem and the boundary conditions are imposed on it. By discretizing the differential equation accordingly and solving the resulting system of linear equations, an approximate analytical solution to the problem is obtained. The experiments were carried out and the results were claimed to be encouraging (Bickley, 1968). Shortly after, the algorithm was further modified, improved, generalized and analyzed (Albasiny and Hoskins, 1969; Fyfe, 1969). Studies in this area then revolved around implementing splines, specifically monomial splines, in other types of problems, as in Al-Said (1998), Khan (2004), and the references therein.

However, Caglar et al. (2006) proposed the use of cubic B-spline, a more stable representation of cubic spline, to solve second order linear two-point boundary value problems. The approach from Albasiny and Hoskins (1969) was adopted and thus the resulting coefficient matrix was in the tridiagonal form, which can be solved very quickly using any software such as MATLAB or Mathematica (Caglar et al., 2006). This study was further extended by applying other types of splines, namely cubic trigonometric B-spline, cubic Beta-spline, extended cubic B-spline and quartic B-spline, to solve the same problem. It was found that cubic trigonometric B-spline produced more accurate results than that of cubic B-spline if the exact solution was trigonometric. Moreover, since the other three splines have free parameters, the results were improved significantly for optimum values of parameters (Abd Hamid, 2010; Abd Hamid et al., 2011, 2012). Therefore, being a simple and straight-forward problem, linear two-point BVPs can be solved using various types of splines and the accuracy of the results depends on the types of problems and the splines used. There are many other types of splines that are yet to be experimented with such as wavelet spline and alpha-spline.

Goh et al. (2011) continued this study in a different direction where cubic B-spline was proposed to solve one-dimensional heat and wave equations. The finite difference approach was applied to discretize the time derivative while cubic B-spline was applied as an interpolation function in the space dimension. The truncation error and the stability of the method were also investigated. This study concluded that cubic B-spline produces more accurate results compared to that of FDM for small space-steps. Similarly, cubic B-spline produces approximate analytical solution at each time level whereas FDM only provides discrete solutions (Goh et al., 2011; Goh, 2013). This method was further applied to the more complicated problems such as advection diffusion problems and telegraph equations with promising results (Goh et al., 2012; Goh, 2013). From these studies, it can be deduced that spline interpolation method adjusted nicely in one-dimensional equation with time variable. Similar to the two-point BVPs, this method also produces better results than the FDM. Moreover, this method is consistent and stable which ensure the convergence of the results.

Abbas et al. (2014) then applied this method using cubic trigonometric B-spline on the one-dimensional hyperbolic problems. The results were compared to that of cubic B-spline and again, it was found that cubic trigonometric B-spline produces more accurate results if the exact solution was a trigonometric function. These findings give support to our research questions. If cubic trigonometric B-spline was a superior method than cubic B-spline for solving two-point boundary value problems and one-dimensional hyperbolic equation with trigonometric exact solution, will it be the same with two-dimensional PDEs?

2.3 Splines-Based Methods for Two-Dimensional PDEs

Splines have also been incorporated into other numerical methods to solve two-dimensional PDEs. This is done in order to improve the accuracy of the results. Boundary Element Method (BEM) is one of them. This method originally made use of the Green's function to construct the solution of any differential equation in form of an integral equation. Then, it fitted the given boundary conditions into the integral equation. The updated integral equation could then be used to numerically produce results at any discrete point (Ang, 2007). This method had been upgraded with the use of splines.

BEM without numerical integration was proposed by Shen (2001) for the Laplace's equations on a plane domain with a polygonal boundary. This is an indirect BEM involving the use of splines. There are two schemes, namely first and second order. For both schemes, double-layer potentials are used to approximate the solution. A potential is an integral of the density function and the derivatives of the fundamental solution. The double-layer refers to the integration over a specified area. These potentials can be differentiated directly. Solving the problem is equivalent to solving the double-layer density function. On each side of the polygonal boundary, the double-layer density function is taken to be a spline function. The first order scheme is able to solve problems with Dirichlet boundary conditions. This scheme is claimed to be stable, produces accurate results and the resulting collocation equations are well-conditioned on a convex domain. A domain is convex if any line formed by joining two points from the domain lies entirely in the domain. On the other hand, the second order scheme is applied to solve problems with Neumann or mixed boundary conditions. The scheme is shown to adopt a small number of computations. It is then used to solve singular problems without any special treatment next to the singular points. The results are highly accurate except for a few elements near the singular points. However, in fracture mechanics, estimating the strength of singularity is important. Using this method, the strength of the singularity is estimated as good as other methods with special treatment (Shen, 2001).

Besides, the BEM was assimilated with the isogeometric concept and the Fast Multipole Method (FMM) in order to solve two-dimensional Laplace's equation focusing on the external Neumann problems. According to the concept of isogeometric analysis, BEM is first developed where a closed B-spline curve is employed to represent the closed boundary. This is a better alternative than the standard fast BEM that uses piecewise-constant elements. Then, the standard FMM is applied to the present isogeometric BEM. As a result, the computational complexity reduces from $O(n^2)$ to $O(n)$, where n represents the number of control points that defines the closed boundary. A benchmark test was carried out where results were generated using the isogeometric BEM only. It was confirmed that the FMM does accelerates the isogeometric BEM with more accurate results. This method was also successfully tested on a large-scale problem with over a million degree of freedom (Takahashi and Matsumoto, 2012).

An attempt of using isogeometrical analysis in the renowned FEM had been successfully done to solve Laplace's equations. In FEM, the boundaries of the equations are approximated. Therefore, the imposition of the essential boundary conditions cannot be accomplished exactly. The proposed method defines the boundaries with more precision and the boundary conditions are satisfied all along the boundary, not limited to a few discretization boundary points. Splines and Non-uniform Rational B-splines surfaces are generated and applied in the method. This method produces much more accurate results compared to that of FEM. Furthermore, the results are not sensitive to the position of control points and knot vectors, making the method "suitable for an adaptive solution and applicable to finite strain problems with geometrical nonlinearity" (Hassani et al., 2009). Up to this point, it can be observed that spline had been used as a tool in the respective methods to upgrade the method.

There are a few studies that focused on the use of splines in itself to solve the two-dimensional PDEs. One of them is the high accuracy cubic spline approximation. This method was developed to solve the two-dimensional quasi-linear elliptic BVPs. This method incorporates the nine-point compact discretization of order four in the x direction and of order two in the y direction. The approach is based on cubic spline approximation. This method is convergent with $O(\Delta x^2 + \Delta y^2)$. Moreover, it is able to handle Poissons' equations in polar coordinates as well as the two-dimensional Burgers' equation. It was shown that this method produces more accurate results than the previously developed nine-point FDM (Mohanty et al., 2013).

Other than that, the B-spline collocation method was also proposed for Poissons' equations on complex shaped planar domains in a simple and stable way. The main technique is the use of approximate Fekete points on the domains. Fekete points are near optimal points with respect to accuracy of interpolations and conditioning of the corresponding interpolation matrices developed by Sommariva and Vianello (Žitňan, 2011). The method was experimented on amoeba-like domain, star shaped domain and a square with eight holes subject to Dirichlet boundary conditions. The resulting system of linear equation is over-determined but well-conditioned. The study concludes that "there are no theoretical and computational obstacles to apply the method with highly complex shape, i.e., highly non-convex and highly multiple-connected domains. This method may also be generalized for solving arbitrary second and fourth order problems with variable coefficients exhibiting sufficiently smooth solution". However, two challenges are faced in solving these problems. Firstly, the amoeba-like domain requires large sets of trial points that leads to a time-consuming procedure. Secondly, the selection of near-optimal trial points for all test problems also requires a lot of computational time (Žitňan, 2013).

Moreover, Mittal and Bhatia (2014) solved the second order two-dimensional hyperbolic telegraph equation using modified cubic B-spline. This method first converts the equation into a coupled system, that is, into two first order two-dimensional PDEs. Then, the modified cubic B-spline function and its derivatives are used to discretize the equations in the spatial directions. Upon doing that, the system would reduce to a system of ODEs. This system is solved using the SSP-RK43 scheme, which is a variation of the Runge-Kutta method. The stability of the methods was proven by the matrix stability analysis. The results were found to be acceptable and in good agreement with earlier studies. This method was claimed to be using much less computational time than earlier studies, simple, efficient, producing very accurate numerical results in considerably smaller number of nodes and saving computational effort.

However, all of these studies made use of the curve representation of splines in their approach. Up to our knowledge, there is no study yet in applying the surface representation of splines directly in the method. It is interesting to know whether the application of the said surface function will increase the complexity with or without benefit.

2.4 Other Numerical Methods for Two-Dimensional PDEs

There are a huge number of other numerical methods for solving two-dimensional PDEs that do not involve splines. The objective of this section is to provide an overview of available numerical methods and their corresponding approaches.

The classical methods available to solve this equation are the FDM and the FEM. These methods are relatively fast, accurate, and easy to implement. Improvements on this method have always been made in order to either increase the accuracy or shorten the computational time. Finite Difference - Explicit $(1,3)$ is one of the examples. This approach transforms the heat equation into two problems, a one-dimensional non-local boundary value problem and a two-dimensional classical problem with Neumann's boudary conditions. These two problems were solved using the FDM (Dehghan, 2000).

Another similar approach is the Exact Difference Schemes. These schemes were developed to solve the nonlinear two-dimensional convection-diffusion-reaction equation, which is an initial-boundary value problem. A difference scheme is said to be exact if the approximation error is zero at the grid points. Therefore, a difference scheme approximating the equation is constructed and is found to be exact for the travelling-wave solutions with no reaction term. However, this technique can also be applied for a wide class of solutions, not limited to the traveling waves. The scheme was found to be conditionally monotone and stable. The results are also highly accurate (Lapinska-Chrzczonowicz and Matus, 2014).

Some of the more popular methods in the literature are those having a series solution. They come in many series names such as variational iteration, Adomian, and homotopy. In the homotopy case, Homotopy Analysis Method (HAM) was the generalized version of Variational Iteration Method (VIM). HAM contains an auxiliary parameter $\hbar$. When $\hbar = -1$, this method reduces to the VIM. This parameter provides a simple way to adjust and control the convergence region of the solution series for large values of the variables x and y. These methods had been used to solve Laplace's equation with Dirichlet and Neumann boundary conditions. For values of x and y that were larger than the given domain, accurate results were obtained from HAM. For these values, VIM produced extremely inaccurate results. It was shown that HAM gives more accurate results and uses less calculations than VIM and Homotopy Perturbation Method (HPM). HAM also handles linear and nonlinear problems without any assumption and restriction and gives fast convergence (Inc, 2007).

Moreover, the New Homotopy Perturbation Method (NHPM) was developed from the HPM to solve the nonlinear two-dimensional wave equations. Similar to the HPM, a suitable homotopy is constructed for the problems and a suitable initial guess is chosen to start off the calculation. In the NHPM, a different way of choosing and calculating the initial guess than the original HPM is presented. The NHPM is a more effective method that produces more accurate results. In all three examples presented in this study, the NHPM produced exact solutions to the problems while the HPM approximate solutions (Biazar and Eslami, 2013).

Another mainstream method is known as BEM. This method is used widely in the literature to solve two-dimensional PDEs. Some of them were fused with splines, as discussed in the previous section, and some with other methods. Recent work involved in the development of analytical boundary element integration, which originates from the BEM. The authors incorporated the Galerkin Vector Method into the approach. The integrals that appeared in the BEM are expressed by analytical integration. The integrals are computed for constant and linear elements in BEM. This method produces six different integrals on the boundary. They were solved analytically using either Mathematical Tables or the MATLAB symbolic solver. The results are then employed in the BEM approach. Following this approach, numerical schemes and coordinate transformations can be avoided. The results were compared with solutions from the numerical method. It was found that this method is comparable to the original BEM in simplicity and efficiency. However, the accuracy of the results were better for some cases, depending on the types of boundary conditions. This method can also be used for multiple domain cases (Ghadimi et al., 2010).

There are also a number of works involving Lagendre polynomials or Galerkin method. These methods are usually combined with the wavelet theory (Hashish et al., 2009; Zheng et al., 2011; Techapirom and Luadsong, 2013; Khalil and Khan, 2014). Lastly, some isolated works include the Domain Decomposition, the lattice Boltzman, the Taylor Matrix, and the Radial Basis Function methods (Mai-Duy and Tran-Cong, 2008; Duan et al., 2007; Bülbül and Sezer, 2011; Abbasbandy et al., 2014).

2.5 Conclusions

In summary, there are quite a number of numerical methods already developed to solve two-dimensional PDE. The popular methods that always come up in the literature search are BEM, HPM, and variations of them. Some methods had been developed further to solve the more general PDE which is fractional PDE. However, so far as we are aware, there is no study hitherto that applies splines surface directly as an interpolant function to the equations as what we are doing in this thesis.

CHAPTER 3

RESEARCH TOOLS: SPLINES SURFACES AND ERROR FORMULAS

3.1 Introduction

This chapter explains equations and knowledge needed for the methods proposed in the following three chapters. Two types of spline surfaces are discussed; bicubic B-spline and bicubic trigonometric B-spline. The discussion will cover the general equation of the spline surface and its simplifications. The simplifications of the surface and its derivatives are pertinent to solving partial differential equations using splines. Furthermore, the last section lists out all the formulas that are used throughout the thesis in calculating errors.

3.2 Bicubic B-spline Surface

B-spline surface is constructed from a linear combination of some recursive functions, called B-spline basis. The derivation of B-spline basis, its properties, and the construction of B-spline surfaces are discussed in many curves and surfaces books, such as Agoston (2005) and Salomon (2006).

Suppose that $\{x_i\}$ is a uniform partition of an interval in the x-axis with

$$x_{i+1} = x_i + \Delta x, \qquad i \in \mathbb{Z},$$

where Δx is the step size of the partition. B-spline basis of order k with degree $k-1$ is calculated as follows:

$$B_i^k(x) = \frac{x - x_i}{x_{i+k-1} - x_i} B_i^{k-1}(x) + \frac{x_{i+k} - x}{x_{i+k} - x_{i+1}} B_{i+1}^{k-1}(x), \tag{3.1}$$

where

$$B_i^1(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

In this thesis, third degree B-spline is used. Therefore, calculating (3.1) up to $k = 4$ leads to

$$B_i^4(x) = \frac{1}{6\Delta x^3} \begin{cases} (x - x_i)^3, & x \in [x_i, x_{i+1}], \\ \Delta x^3 + 3\Delta x^2(x - x_{i+1}) + 3\Delta x(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & x \in [x_{i+1}, x_{i+2}], \\ \Delta x^3 + 3\Delta x^2(x_{i+3} - x) + 3\Delta x(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4} - x)^3, & x \in [x_{i+3}, x_{i+4}]. \end{cases} \tag{3.2}$$

Since the basis $B_i^4(x)$ is a piecewise polynomial of degree 3, it is called cubic B-spline basis. The basis has second-order parametric continuity property. That is, the first and second derivatives of the basis are continuous. Plots of $B_i^4(x)$ in general and with values are shown in Figures 3.1 and 3.2, respectively.

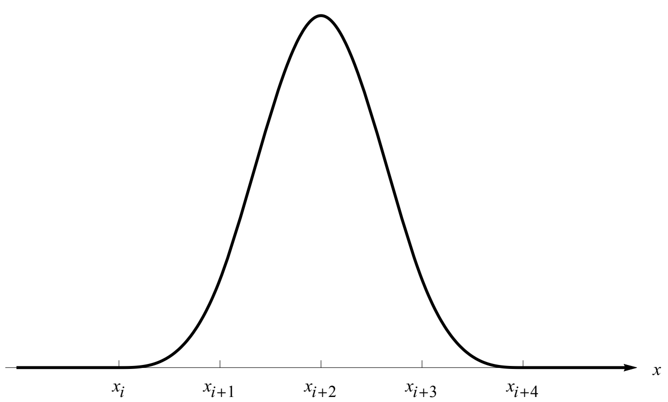

Figure 3.1: Cubic B-spline basis, $B_i^4(x)$

When evaluating cubic B-spline basis in (3.2) at x_i, there are three nonzero bases, namely $B_{i-3}^4(x_i)$, $B_{i-2}^4(x_i)$, and $B_{i-1}^4(x_i)$. This can be seen from Figure 3.3. The nonzero values are given in (3.3).

$$B_{i-3}^4(x_i) = \frac{1}{6}, \qquad B_{i-2}^4(x_i) = \frac{2}{3}, \qquad B_{i-1}^4(x_i) = \frac{1}{6} \tag{3.3}$$

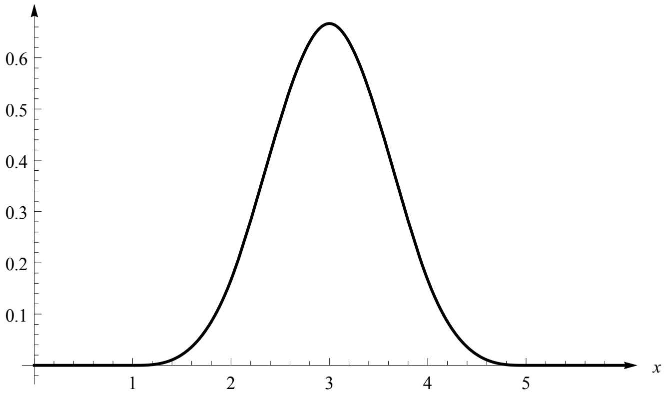

Figure 3.2: Cubic B-spline basis, $B_i^4(x)$, with $i = 0$, $x_0 = 1$, and $\Delta x = 1$

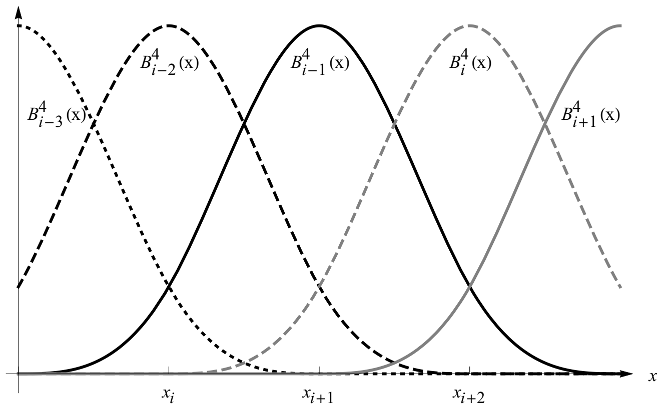

Figure 3.3: A family of cubic B-spline bases

Since this thesis deals with second-order PDEs, the first and second derivatives of the basis are involved. By taking the first derivative of cubic B-spline basis in (3.2) with respect to x, we have

$$\frac{d}{dx}\left[B_i^4(x)\right] = \frac{1}{2\Delta x^3}\begin{cases} (x-x_i)^2, & x \in [x_i, x_{i+1}], \\ \Delta x^2 + 2\Delta x(x - x_{i+1}) - 3(x-x_{i+1})^2, & x \in [x_{i+1}, x_{i+2}], \\ -\Delta x^2 - 2\Delta x(x_{i+3} - x) + 3\Delta x(x_{i+3} - x)^2, & x \in [x_{i+2}, x_{i+3}], \\ -(x_{i+4} - x)^2, & x \in [x_{i+3}, x_{i+4}]. \end{cases} \tag{3.4}$$

The plot of (3.4) is displayed in Figure 3.4. It can be seen that $\frac{d}{dx}\left[B_i^4(x)\right]$ is continuous. The plot of a family of $\frac{d}{dx}\left[B_i^4(x)\right]$ is shown in Figure 3.5. From the figure, it can be observed that

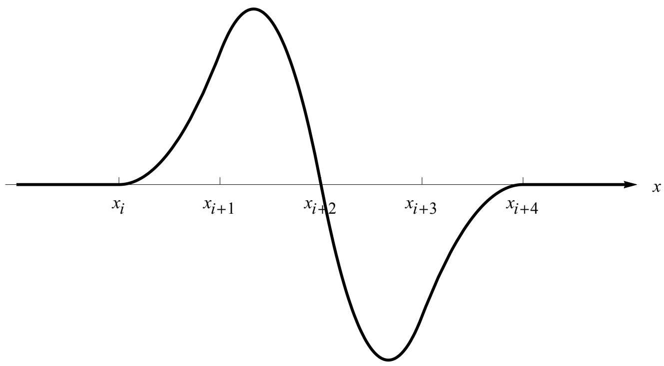

Figure 3.4: The first derivative of cubic B-spline basis, $\frac{d}{dx}\left[B_i^4(x)\right]$

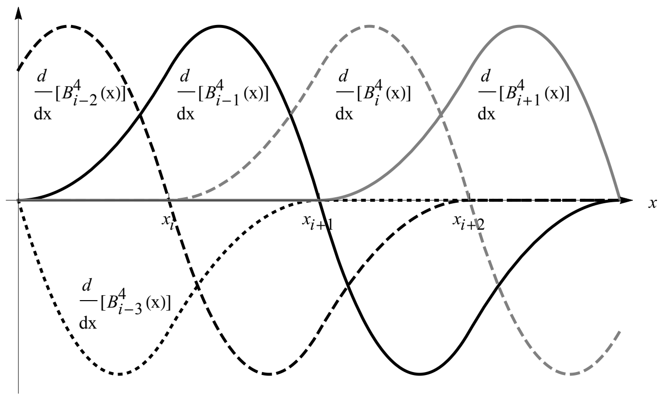

Figure 3.5: A family of the first derivatives of cubic B-spline bases

there are also three nonzero values at x_i. The values are calculated using (3.4) to be

$$\frac{d}{dx}\left[B_{i-3}^4(x_i)\right] = -\frac{1}{2\Delta x}, \quad \frac{d}{dx}\left[B_{i-2}^4(x_i)\right] = 0, \quad \text{and} \quad \frac{d}{dx}\left[B_{i-1}^4(x_i)\right] = \frac{1}{2\Delta x}. \tag{3.5}$$

The second derivative of cubic B-spline basis in (3.2) with respect to x is calculated as follows

$$\frac{d^2}{dx^2}\left[B_i^4(x)\right] = \frac{1}{\Delta x^3}\begin{cases} (x - x_i), & x \in (x_i, x_{i+1}), \\ \Delta x - 3[x - x_{i+1}], & x \in (x_{i+1}, x_{i+2}), \\ \Delta x - 3[x_{i+3} - x], & x \in (x_{i+2}, x_{i+3}), \\ (x_{i+4} - x), & x \in [x_{i+3}, x_{i+4}]. \end{cases} \tag{3.6}$$

The plot of (3.6) is displayed in Figure 3.6. It can be seen that $\frac{d^2}{dx^2}\left[B_i^4(x)\right]$ is also continuous. The plot of a family of $\frac{d^2}{dx^2}\left[B_i^4(x)\right]$ is shown in Figure 3.7. From the figure, it shows three

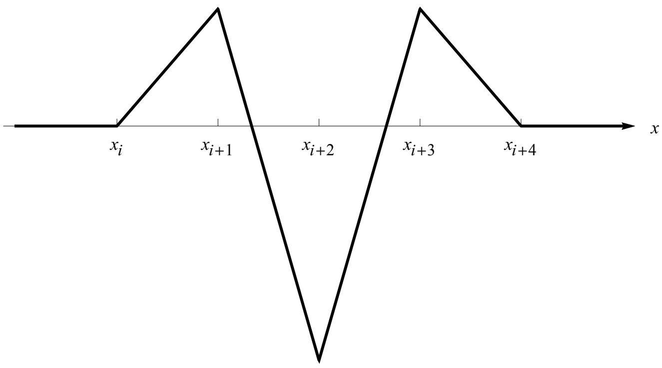

Figure 3.6: The second derivative of cubic B-spline basis, $\frac{d^2}{dx^2}\left[B_i^4(x)\right]$

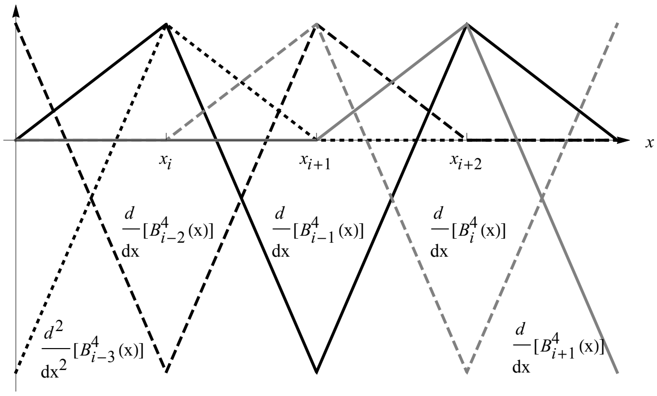

Figure 3.7: A family of the second derivatives of cubic B-spline bases

nonzero values at x_i. The values are calculated using (3.6) to be

$$\frac{d^2}{dx^2}\left[B_{i-3}^4(x_i)\right] = \frac{1}{\Delta x^2}, \quad \frac{d^2}{dx^2}\left[B_{i-2}^4(x_i)\right] = -\frac{2}{\Delta x^2}, \quad \text{and} \quad \frac{d^2}{dx^2}\left[B_{i-1}^4(x_i)\right] = \frac{1}{\Delta x^2}. \qquad (3.7)$$

Similarly, suppose that $\{y_j\}$is a uniform partition of an interval in the y-axis with

$$y_{j+1} = y_j + \Delta y, \qquad j \in \mathbb{Z},$$

where Δy is the step size of the partition. B-spline basis of order k with degree $k-1$ is calculated as follows

$$B_j^k(y) = \frac{x - y_j}{y_{j+k-1} - y_j} B_j^{k-1}(y) + \frac{y_{j+k} - y}{y_{j+k} - y_{j+1}} B_{j+1}^{k-1}(y), \qquad (3.8)$$

where

$$B_j^1(y) = \begin{cases} 1, & y \in [y_j, y_{j+1}], \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding cubic B-spline basis is

$$B_j^4(y) = \frac{1}{6\Delta y^3} \begin{cases} (y-y_j)^3, & y \in [y_j, y_{j+1}], \\ \Delta y^3 + 3\Delta y^2(y-y_{j+1}) + 3\Delta y(y-y_{j+1})^2 - 3(y-y_{j+1})^3, & y \in [y_{j+1}, y_{j+2}], \\ \Delta y^3 + 3\Delta y^2(y_{j+3}-y) + 3\Delta y(y_{j+3}-y)^2 - 3(y_{j+3}-y)^3, & y \in [y_{j+2}, y_{j+3}], \\ (y_{j+4}-y)^3, & y \in [y_{j+3}, y_{j+4}]. \end{cases} \tag{3.9}$$

The corresponding nonzero values when evaluating (3.9) and its derivatives at y_j are given in (3.10), (3.11), and (3.12), respectively.

$$B_{j-3}^4(y_j)\frac{1}{6}, \qquad B_{j-2}^4(y_j) = \frac{2}{3}, \qquad B_{j-1}^4(y_j) = \frac{1}{6} \tag{3.10}$$

$$\frac{d}{dy}\left[B_{j-3}^4(y_j)\right] = -\frac{1}{2\Delta y}, \quad \frac{d}{dy}\left[B_{j-2}^4(y_j)\right] = 0, \quad \frac{d}{dy}\left[B_{j-1}^4(y_j)\right] = \frac{1}{2\Delta y} \tag{3.11}$$

$$\frac{d^2}{dy^2}\left[B_{j-3}^4(y_j)\right] = \frac{1}{\Delta y^2}, \quad \frac{d^2}{dy^2}\left[B_{j-2}^4(y_j)\right] = -\frac{2}{\Delta y^2}, \quad \frac{d^2}{dy^2}\left[B_{j-1}^4(y_j)\right] = \frac{1}{\Delta y^2} \tag{3.12}$$

An arbitrary B-spline surface equation, $S_B(x,y)$, can be generated from bases (3.2) and (3.9) by the following equation:

$$S_B(x,y) = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} C_{i,j}B_i^4(x)B_j^4(y), \quad x \in [x_0, x_m], \quad y \in [y_0, y_n], \quad m,n \geq 1, \tag{3.13}$$

where $C_{i,j}$ are unknown coefficients. Since the surface is built from two cubic B-spline bases, the surface is known as bicubic B-spline surface. A plot of the surface basis, $B_4^i(x)B_4^j(y)$ is shown in Figure 3.8. Evaluating $S_B(x,y)$, as in (3.13), at (x_i, y_j) and applying the simplifications

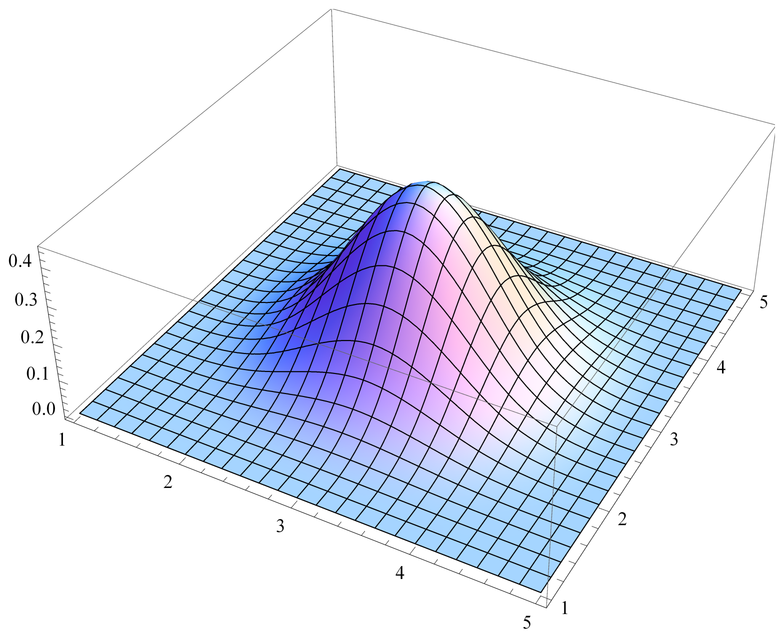

Figure 3.8: Bicubic B-spline basis, $B_i^4(x)B_j^4(y)$ with $i=0$, $x_0=1$, $\Delta x=1$, $y_0=1$, and $\Delta y=1$

in (3.3) and (3.10), we have

$$
\begin{aligned}
S_B(x_i,y_i) = b_1 &\left(b_3C_{i-3}^{j-3}+b_4C_{i-3}^{j-2}+b_3C_{i-3}^{j-1}\right) \\
&+b_2\left(b_3C_{i-2}^{j-3}+b_4C_{i-2}^{j-2}+b_3C_{i-2}^{j-1}\right) \\
&\quad +b_1\left(b_3C_{i-1}^{j-3}+b_4C_{i-1}^{j-2}+b_3C_{i-1}^{j-1}\right),
\end{aligned} \tag{3.14}
$$

where

$$
b_1=\frac{1}{6}, \qquad b_2=\frac{2}{3}, \qquad b_3=\frac{1}{6}, \qquad \text{and} \qquad b_4=\frac{2}{3}.
$$

By taking the first derivative of $S_B(x,y)$ with respect to x and y and evaluating them at (x_i,y_j) using (3.5) and (3.11), the following equations (3.15) and (3.16) are obtained, respectively.

$$
\begin{aligned}
\frac{\partial}{\partial x}S_B(x_i,y_j) = -b_x &\left(b_1C_{i-3}^{j-3}+b_2C_{i-3}^{j-2}+b_1C_{i-3}^{j-1}\right) \\
&+b_x\left(b_1C_{i-1}^{j-3}+b_2C_{i-1}^{j-2}+b_1C_{i-1}^{j-1}\right),
\end{aligned} \tag{3.15}
$$

$$
b_x=\frac{1}{2\Delta x}.
$$

$$
\begin{aligned}
\frac{\partial}{\partial y} S_B(x_i, y_j) = b_1 &\left(-b_y C_{i-3}^{j-3} + b_y C_{i-3}^{j-1}\right) \\
&+ b_2 \left(-b_y C_{i-2}^{j-3} + b_y C_{i-2}^{j-1}\right) \\
&\quad + b_1 \left(-b_y C_{i-1}^{j-3} + b_y C_{i-1}^{j-1}\right), \\
b_y &= \frac{1}{2\Delta y}.
\end{aligned} \tag{3.16}
$$

By taking the second derivative of $S_B(x, y)$ with respect to x and y and evaluating them at (x_i, y_j) using (3.7) and (3.12), the following equations are obtained:

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} S_B(x_i, y_j) = b_{xx_1} &\left(b_1 C_{i-3}^{j-3} + b_2 C_{i-3}^{j-2} + b_1 C_{i-3}^{j-1}\right) \\
&+ b_{xx_2} \left(b_1 C_{i-2}^{j-3} + b_2 C_{i-2}^{j-2} + b_1 C_{i-2}^{j-1}\right) \\
&\quad + b_{xx_1} \left(b_1 C_{i-1}^{j-3} + b_2 C_{i-1}^{j-2} + b_1 C_{i-1}^{j-1}\right),
\end{aligned} \tag{3.17}
$$

$$
b_{xx_1} = \frac{1}{\Delta x^2}, \qquad b_{xx_2} = -\frac{2}{\Delta x^2}.
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} S_B(x_i, y_j) = b_1 &\left(b_{yy_1} C_{i-3}^{j-3} + b_{yy_2} C_{i-3}^{j-2} + b_{yy_1} C_{i-3}^{j-1}\right) \\
&+ b_2 \left(b_{yy_1} C_{i-2}^{j-3} + b_{yy_2} C_{i-2}^{j-2} + b_{yy_1} C_{i-2}^{j-1}\right) \\
&\quad + b_1 \left(b_{yy_1} C_{i-1}^{j-3} + b_{yy_2} C_{i-1}^{j-2} + b_{yy_1} C_{i-1}^{j-1}\right),
\end{aligned} \tag{3.18}
$$

$$
b_{yy_1} = \frac{1}{\Delta y^2}, \qquad b_{yy_2} = -\frac{2}{\Delta y^2}.
$$

The simplifications of bicubic B-spline basis and its derivatives at (x_i, y_j) are extensively used in solving two-dimensional PDEs using bicubic B-spline.