Applying Portfolio Theory to Timber Product

by

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ABSTRAK

Sejak seminar analisis min-varians diperkenalkan oleh Markowitz (1952), teori portfolio telah diperkembangkan dalam konteks model pilihan normatif, termasuk bagaimana untuk membentuk portfolio yang optimum.

Analisis ini menggunakan teori portfolio untuk mendapatkan penyelesaian optimum, memaksimumkan keuntungan dan meminimumkan risiko bagi produk balak di Semenanjung Malaysia dan eksport satu produk balak terpilih ke destinasi utama dunia. Masalah ini adalah aplikasi secara langsung pendekatan min-varians Markowitz dan masalah pengoptimuman portfolio dapat diformulakan sebagai pengaturcaraan matematik.

Data berkenaan produk balak dianalisis untuk menghasilkan min-varians efisyensi. Kemudian, kecekapan *frontier* dihasilkan untuk memastikan pulangan risiko-teritlak optimum bagi portfolio tersebut. Akhir sekali, model semi-varians dijanakan untuk menghasilkan nilai optimum dan perbandingan dengan model min-varians dilaksanakan.

ABSTRACT

Since the seminal mean-variance analysis was introduced by Markowitz (1952), the portfolio theory has been expanded in the context of normative choice modeling, including how to form an optimal portfolio.

This study uses portfolio theory to find the optimal, profit maximizing and risk-minimizing combinations of timber product in Peninsular Malaysia and the export of one selected timber product to major destination throughout the world. This problem is a straight forward application of Markowitz mean-variance approach and the optimal portfolio problem can be formulated as mathematical programming.

The data on timber product was analyzed to create mean-variance efficiency. Then, an efficiency frontier was created to ensure optimal risk-adjusted returns of the portfolio. Finally, a semi-variance model was run to generate the optimal values and to make comparison with the mean-variance model.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Financial economics, mathematics, management theory and operations research have derived several techniques to value portfolios. Formal portfolio theory research saw major advances in the context of normative choice modeling, including how to form an optimal portfolio, beginning with Harry Markowitz (1959).

Researchers and portfolio analysts have spent considerable effort developing models showing the appropriate mix of equity investments to optimize risk-adjusted returns. These optimal portfolios often have a mix of stocks, bonds, and cash, often including an international component used to reduce risk or boost returns. Increasingly, institutional investors are looking for other alternative investments to increase the return or lower the risk of their investment portfolios.

A basic premise of economics is that, due to the scarcity of resources, all economic decisions are made in the face of trade-offs. Markowitz identified the trade-off facing the investor: risk versus expected return. The investment decision is not merely which securities to own, but how to divide the investor's wealth amongst securities. This is the problem of "Portfolio Selection;" hence the title of Markowitz's seminal article published in the March 1952 issue of the *Journal of Finance*. In that article and subsequent works, Markowitz extends the techniques of linear programming to develop the critical line algorithm. The critical line algorithm identifies all feasible portfolios that minimize risk (as measured by variance or standard deviation) for a given level of expected return and maximize expected return for a given level of risk. When graphed in standard deviation versus expected return space, these portfolios form the efficient frontier. The efficient frontier represents the trade-off between risk and expected return faced by an investor when forming his portfolio. Most of the efficient frontier represents well diversified portfolios. This is because diversification is a powerful means of achieving risk reduction.

Markowitz developed mean-variance analysis in the context of selecting a portfolio of common stocks. Over the last decade, mean-variance analysis has been increasingly applied to asset allocation. Product allocation is the selection of a portfolio of investments where each component is an asset class rather than an individual security. Mean-variance analysis requires not only knowledge of the expected return and standard deviation on each asset, but also the correlation of returns for each and every pair of assets. Whereas a stock portfolio selection problem might involve hundred of stocks (and hence thousands of correlations), a product selection problem typically involves a handful of asset classes (for example stocks, bonds, cash, real estate, and marketing product). Furthermore, the opportunity to reduce total portfolio risk comes from the lack of correlation across assets. Since stocks generally move together, the benefits of diversification within a stock portfolio are limited. In contrast, the correlation across asset classes is usually low and in some cases negative. Hence, mean-variance is a powerful tool in asset allocation for uncovering large risk reduction opportunities through diversification.

The relatively small data requirements of applying mean-variance analysis to product selection along with the speed and low cost of powerful personal computers (PCs) have led to the commercial development of many PC-based mean-variance optimization software packages for use in product selection. Some of these optimizers do not solve for the entire efficient frontier using the critical line algorithm; instead, they maximize a parametric objective function in mean and variance for a handful of parameter values. Other optimizers implement some form of the critical line algorithm to solve for the entire efficient frontier. The latter approach has the advantage that once the efficient frontier has been found, any number of objective functions with any number of parameter values can be optimized without having to rerun the algorithm.

1.2 Assumptions of Mean-Variance Analysis

As with any model, it is important to understand the assumptions of mean-variance analysis in order to use it effectively. First of all, mean-variance analysis is based on a single period model of investment. At the beginning of the period, the investor allocates his wealth among various asset classes, assigning a nonnegative weight to each asset. During the period, each asset generates a random rate of return so that at the end of the period, his wealth has been changed by the weighted average of the returns. In selecting asset weights, the investor faces a set of linear constraints, one of which is that the weights must sum to one.

Based on the game theory work of Von Neumann and Morgenstern, economic theory postulates that individuals make decisions under uncertainty by maximizing the expected value of an increasing concave utility function of consumption. In a one period model, consumption is end of period wealth. In general, maximizing expected utility of ending period wealth by choosing portfolio weights is a complicated stochastic nonlinear programming problem. Markowitz asserted that if the utility function can be approximated closely enough by a second-order Taylor expansion over a wide range of returns then expected utility will be approximately equal to a function of expected value (mean) and variance of returns. This allows the investor's problem to be restated as a mean-variance optimization problem so that the objective function is a quadratic function of portfolio weights.

The utility function is assumed to be increasing and concave because we assume that (1) investors prefer more consumption to less, and (2) investors are risk averse. In terms of the approximating utility function, this translates into expected utility being increasing in expected return (more is better than less) and decreasing in variance (the less risk the better). Hence, of all feasible portfolios, the investor should only consider those that maximize expected return for a given level of variance, or minimize variance for a given level of expected return. These portfolios form the mean-variance efficient set.

1.3 Assumption to Application of Portfolio Selection

We consider portfolio selection when the following three conditions are satisfied:

i. The investor owns only liquid assets.

ii. He maximize the expected value of $U(C_1, C_2, \ldots, C_T)$, where C_T is the money value of consumption during the ith period (C_T could, alternatively, represent money expenditure deflated by a cost of living index).

iii. The set of available probability distributions of returns from portfolios remains the same through time (if C_T is deflated consumption, then it is 'real return', taking into account changes in price level whose probability distribution is assumed constant).

Later, we consider modifications of these assumptions.

An asset is perfectly liquid if

iv. The price at which it can be sold, at a particular time, always equals the price at which it can be bought at that time; and

v. Any amount can be bought or sold at this price.

Even though securities are not perfectly liquid, they are sufficiently liquid for an analysis based on liquidity to be instructive. The effects of illiquidities, among other things, are consider later.

Conditions (iii) does not imply that the same security offers the same opportunities at all times. The new and promising firm of today may be a well-established or a defunct firm tomorrow – at which time the role of being 'new and promising' is taken over by other firms. The assumption, made at first and modified later, is that the opportunities from the market as a whole remain constant.

Perfectly, liquid assets may be converted into cash, and cash may be converted into liquid assets without loss. If available probability distributions remain the same through time, the investor's opportunities depend only on the value of his portfolio. If we let y_{t+1} be the value of the portfolio at the beginning of period $t + 1$ (i.e. at the end of period t), then, under our present assumptions, the single period utility function

$$U = U((C_t, w_{t+1}, C_1, C_2, \ldots, C_{t-1})$$

can be written as

$$U = U((C_t, y_{t+1}, C_1, C_2, \ldots, C_{t-1})$$

1.4 Objectives of a Portfolio Analysis

It is impossible to derive all possible conclusion concerning portfolios. A portfolio analysis must be based on criteria which serve as a guide to the important and unimportant, the relevant and irrelevant.

The proper choice of criteria depends on the nature of the investor. For some investors, taxes are a prime consideration; for others, such as non-profit corporations, they are irrelevant. Institutional considerations, legal restrictions, relationship between portfolio returns and the cost of living may be important to one investor and not to another. For each type of investor, the details of the portfolio analysis must be suitably selected.

Two objectives, however, are common to all investors for which the techniques are design:

i. They want return to be high, the appropriate definition of return may vary from investor to investor. But, in what sense is appropriate, they prefer more of it to less of it.

ii. They want return to be dependable, stable, not subject to uncertainty. No doubt there are security purchasers who prefer uncertainty, like bettors at a horse race who pay to take chances. The techniques are not for speculators. The techniques are for the investors who, other things being equal, prefer certainty to uncertainty.

The portfolio with highest likely return is not necessarily the one with least uncertainty of return. The most reliable portfolio with an extremely high likely return may be subject to an unacceptably high degree of uncertainty. The portfolio with the least uncertainty may have an undesirably small likely return. Between these extremes would lay portfolios with varying degrees of likely return and uncertainty.

The proper choice among efficient portfolios depends on the willingness and ability of the investor to assume risk. If safety is of extreme importance, likely return must be sacrificed to decrease uncertainty. If a greater degree of uncertainty can be borne, a greater level of likely return can be obtained. An analysis presented here are:

First, separate efficient portfolios from inefficient one's;

Secondly, portrays the combinations of likely return and uncertainty of return available from efficient portfolios;

Thirdly, the investors or investment manager carefully select the combination of likely return and uncertainty that best suits his needs; and

Lastly, determine the portfolio which provides the most suitable combinations of risk and return.

1.5 Illustration of Mean-Variance Analysis

To illustrate mean-variance analysis as it applies to asset allocation, consider an investor whose portfolio is entirely in the U.S. capital markets but is considering going into non-U.S. markets. The first step of the analysis is to divide the world capital markets into broad asset classes. In this example, we have four portfolios: U.S. stocks, non-U.S. stocks, U.S. bonds, and non-U.S. bonds. The second step is to develop capital market assumptions; namely, expected returns and standard deviations for each asset class and correlations between each pair of asset classes. These assumptions are usually derived from historical data on asset class returns and current market conditions. The third step is to generate the efficient frontier by running the critical line algorithm. Markowitz showed that while there are infinitely many efficient portfolios, you only need a limited number of corner portfolios to identify all efficient portfolios. Not all efficient portfolios contain all assets. Moving along the efficient frontier, a corner portfolio is located where an asset weight or slack variable is either added or dropped. Every efficient portfolio is a linear combination of the two corner portfolios immediately adjacent to it. Thus, by locating all corner portfolios, the critical line algorithm generates the entire efficient frontier.

1.6 Background of the Study

This study will focus on application of portfolio theory to improve return to timber product selection and distribution to major destination. Data were obtained from Malaysian Timber Council and are accessible through the website http://www.mtc.com.my/info/index.php?option=com_content&view=category&id=44&Itemid=63 . There are 13 major products available for exports which are logs, sawn timber, sleepers, veneer, mouldings, chipboard/particleboard, fibreboard, plywood, wooden frame, builders joinery & carpentry, wooden furniture, rattan furniture and also others timber products.

The data are available from 1994 to third quarter of 2008. However the complete data set is available for 2002 to selected timber product with volume and FOB (freight on board) value.

For this study, we will focus onsix (6) major products i.e. sawlogs, sawn timber, sleepers, veneer, mouldings and fibreboard, with export Freight on Board value obtained from Peninsular Malaysia. Later, we will focus on one selected timber product i.e. sawn timber to analyze the returns to fifteen (15) major destinations by using the same method as constructed for the first data set, which were taken from 2001 to third quarter 2008.

1.7 Objectives of the Study

The objectives of the study are as follows:

- To minimize the end-of-period variance in portfolio selection
- To generate points along the efficient frontier
- To minimize the semi-variance in the portfolio selection

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Timber product selection is timely, important, and essential in Malaysia, since public and private companies continue to produce higher-volume of timber product over the time. In this study, timber can be considered as product for investors and also as forest/timber plantation which can be utilised by logging operators.

A 'portfolio' is defined simply as a combination of items: securities, assets, or other objects of interest. Portfolio theory is used to derive efficient outcomes, through identification of a set of actions, or choices, that minimize variance for a given level of expected returns, or maximize expected returns, given a level of variance. Decision makers can then use the efficient outcomes to find expected utility-maximizing solutions to a broad class of problems in investment, finance, and resource allocation (Robison and Brake, 1979). Simply put, portfolio theory can be used to maximize profits and minimize risk in a wide variety of settings and choices, including timber product selection in Peninsular Malaysia.

A literature on product selection and variety adoption decisions exists, as reported by Cardozo and Smith (1983) that said, "Results indicate that financial portfolio theory has promise an analytical and planning tool for product portfolio decisions, and suggest how action recommendations based on financial portfolio theory may be modified for product portfolio decisions." Leong and Lim (1991) informed that multiperiod portfolio framework should help marketers in allocating scarce corporate resources to various competing products as well as contribute to develop a body of theory to solve an important problem in marketing management.

Seminal works on plantations begin with Griliches (1957), who evaluated the determinants of hybrid corn adoption in the United States. Heisey and Brennan (1991) studied the demand for wheat replacement seed in Pakistan, and Traxler et al. (1995) documented and analyzed the steady growth of yields of new wheat varieties in Mexico. Smale, Just, and Leathers (1994) summarized several explanations for a relatively slow adjustment to a newly introduced variety, including input fixity and portfolio selection. The use of mixtures of varieties portfolios has also been studied from ecological and pathological perspectives. Garrett and Cox (2008) reported that, "The construction of crop variety mixtures is an example of a technology that draws heavily on ecological ideas and has also contributed to our understanding of disease ecology through experiments that examine the effects of patterns of host variability on disease through time and space" (pp. 1-2).

The study of decision making under risk has a long history, beginning with early decision models of resource allocation that maximized expected returns. Portfolio theory significantly improved our ability to analyze and identify optimal choices under risk by extension of the analysis to include variability, as well as expected returns. Portfolio theory was initially developed by Markowitz (1959) and Tobin (1958), with extensions by Lintner (1965) and Sharpe (1970).

Financial portfolio analysis provides a useful framework for conceptualizing product selection decisions, and implementing variety strategies and product decisions. Variety choices are similar to investment decisions in financial markets, where financial managers allocate money across investment opportunities with relative risks and returns across a set of correlated assets. Since different varieties of product respond differently to environmental conditions, the risks associated with product selection are correlated. Some product will be positively related to other product, and some may be negatively correlated with other products. Because of this correlation, or relationship, there are potential benefits by considering investment in multiple product selection.

The application of portfolio theory to product decisions is new, but applications of portfolio theory to risky decisions in agriculture and timberland allocation has been around for a long time. Collins and Barry (1986) applied Sharpe's (1970) extension of the Markowitz model to a 'single index' portfolio model to study diversification of agricultural activities. The single index model does not require a complete, balanced data set, and is computationally less demanding. Turvey et al. (1985) compared a full variance-covariance (Markowitz) model to a single index model in a case farm in southern Ontario, and found that the single index model a practical alternative in applications to the complete model for deriving mean-variance efficient farm plans.

Robison and Brake (1979) provided a thorough and informative literature review of portfolio theory, with applications to agriculture and agricultural finance. Barry (1980) extended portfolio theory to the Capital Asset Pricing Model (CAPM), and applied the model to farm real estate. More recently, Nyikal and Kosura (2005) used quadratic programming (QP) to solve for the efficient mean-variance frontier to better understand farming decisions in Kenyan agriculture. Another recent application of portfolio theory was conducted by Redmond and Cubbage (1988), who applied the capital asset pricing model (CAPM) to timber asset investments in the United States. Figge (2002) summarized the literature on how portfolio theory has been applied to biodiversity, and Sanchirico et al. (2005) use portfolio theory to develop optimal management of fisheries.

The portfolio approach used in these previous studies will be applied to timber product selection and the distribution of a single product to major destination through out the world.

2.2 Mean - Variance (MV) Efficiency

The model used to estimate the efficiency frontier for product selection is the model developed by Markowitz to study investments, and later applied to timber product selection in Peninsular Malaysia.

Markowitz (1959) developed portfolio theory as a systematic method of minimizing risk for a given level of expenditure. To derive an efficient portfolio of timber product selection, measures of expected returns (average product) and variance of product are required for each product, together with all of the pairwise covariances across all products. The efficient mean-variance frontier for a portfolio of timber product is derived by solving a sequence of quadratic programming problems. Based on an investor's preferences for higher return and less risk, a particular point on the efficiency frontier can be identified as the 'optimal' portfolio of product selection.

We assume that a timber producer or timber investor is given total volume (X), and desires to choose the optimal allocation of timber product selection. Thus the decision variable is, x_i the percentage of total volume allocated to selection i, where i = 1, ..., n, and $\sum_{i=1} x_i = X$. Quadratic programming is used to solve for the efficiency frontier of mean-variance (MV) combinations. This frontier is defined as the maximum mean for a given level of variance, or the minimum variation for a given x_i mean product. If we define y_i as the mean product of selection i, then the total product is simply the weighted average product, that is $\sum_{i=1} x_i y_i$.

The variance of total timber selection for the entire product (V) is defined in equation (1),

$$V = \sum_{i=1} {}_j \sum {}_k x_j x_k \sigma_{jk} \tag{2.1}$$

where:

x_j is the level of activity j, which is the percentage of volume allocated to product j,

σ_{jk} is the covariance of selection between the jth and kth product selection, and equal to variance when $j = k$.

Hazell and Norton (1986) emphasize the intuition embedded in equation (1): the total variance product or varieties allocated (V) is an aggregate of the variability of individual varieties and covariance relationships between the products. Two conclusions are useful to better understand the portfolio approach to timber product selection:

(i) Combinations of product that have negative covariate selection will result in a more stable aggregate selection for the entire farm than specialized strategies of allocating single varieties, and

(ii) A product that is risky in terms of its own variance may still be attractive if its returns are negatively covariate with other product chosen.

The mean-variance efficiency frontier is calculated by minimizing total variance (V) for each possible level of mean product or yields (y_i), as given in equation (2).

Minimize $V = \sum_{i=1} {}_j \sum {}_k x_j x_k \sigma_{jk}$, (2.2)

subject to:

$\sum_{i=1} x_j y_j = \lambda$ and (2.3)

$$x_j \geq 0 \text{ for all } j \tag{2.4}$$

The sum of the mean product in equation (3) is set equal to the parameter λ, defined as the target product level, which is varied over the feasible range to obtain a sequence of solutions of increasing mean product and variance, until the maximum possible mean product is obtained.

Equation (2) is quadratic in x_j, resulting in the use of the Lingo Release 11.0.0.23 program to solve the nonlinear equation. The tool uses the nonlinear optimization code available through example in the program and through its website at www.lindo.com . Chapter 4 will describe the data utilized in the portfolio model.

CHAPTER 3

METHODOLOGY

3.1 Introduction

In this chapter, the chosen methodology for the purposes of this study will be discussed. Section 3.2 will define the descriptive statistics. Section 3.3 will define the Mean - Variance (MV) Efficiency Analysis together with the assumptions pertaining to this case. Subsequently in Section 3.4, the efficient – frontier generated will be defined while taking into account the selected assumptions. Next, Section 3.5 will discuss the semi-variance model as a measure of risk and finally in Section 3.6 a method will be selected to solve the problem.

3.2 Descriptive Statistics

3.2.1 Mean and variance

The mean or arithmetic mean is the standard average, often simply called the mean. It can be written as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where x_i is the observed value

$i = 1, 2, 3 \ldots, n$

The mean is the arithmetic average of a set of values, or distribution; however, for skewed distributions, the mean is not necessarily the same as the middle value (median), or the most likely (mode). For example, if the mean income of a group of people is skewed upwards by a small number of people with very large incomes, the majority of people will have incomes lower than the mean. By contrast, the median income is the level at which half the population is below it and half is above. The mode income is the most likely income, and favors a larger number of people with lower incomes. The median or mode are often more intuitive measures of such data.

If a random variable X has expected value (mean) $\mu = E(X)$, then the variance $Var(X)$ of X is given by:

$$V(X) = E\ [(X - \mu_i)]^2$$

This definition encompasses random variables that are discrete, continuous, or neither. Of all the points about which squared deviations could have been calculated, the mean produces the minimum value for the averaged sum of squared deviations.

The variance of random variable X is typically designated as $Var(X)$, or simply σ^2 (pronounced 'sigma squared'). If a distribution does not have an expected value, as is the case for the Cauchy distribution, it does not have a variance either. Many other distributions for which the expected value does exist do not have a finite variance because the relevant integral diverges. An example is a Pareto distribution whose Pareto index k satisfies $1 < k \leq 2$.

3.2.2 Correlation

In probability theory and statistics, correlation (often measured as a correlation coefficient) indicates the strength and direction of a linear relationship between two random variables. That is in contrast with the usage of the term in colloquial speech, denoting any relationship, which is not necessarily linear. In general statistical usage, correlation or co-relation refers to the departure of two random variables from independence. In this broad sense there are several coefficients, measuring the degree of correlation, adapted to the nature of the data.

A number of different coefficients are used for different situations. The best known is the Pearson product-moment correlation coefficient, which is obtained by dividing the covariance of the two variables by the product of their standard deviations. Despite its name, it was first introduced by Francis Galton.

The correlation coefficient $\rho_{X,Y}$ between two random variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as:

$$\rho_{X,Y} = \frac{\text{cov }(X,Y)}{\sigma_X \sigma_Y} = \frac{E\,[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} \tag{3.1}$$

where E is the expected value operator and cov means covariance, and

cov (X,Y) is the covariance between two real-valued random variables X and Y, with expected values μ_X and μ_Y.

A widely used alternative notation is

$$corr(X,Y) = \rho_{X,Y}$$

Since μ_x= E(X), $\sigma_X^2 = \text{E}[(X - \text{E}(X))^2] = \text{E}(X^2) - \grave{\text{E}}^2(X)$

and likewise for Y, and since

$$E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y),$$

we may also write

$$\rho_{X,Y} = \frac{E\ (XY) - E(X)E(Y)}{\sqrt{E\ (X^2) - E^2(X)}\sqrt{E\ (Y^2) - E^2(Y)}} \tag{3.2}$$

The correlation is defined only if both of the standard deviations are finite and both of them are nonzero. It is a corollary of the Cauchy-Schwarz inequality that the correlation cannot exceed 1 in absolute value.

The correlation is 1 in the case of an increasing linear relationship, -1 in the case of a decreasing linear relationship, and some value in between in all other cases, indicating the degree of linear dependence between the variables. The closer the coefficient is to either -1 or 1, the stronger the correlation between the variables.

If the variables are independent then the correlation is 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables. Here is an example: Suppose the random variable X is uniformly distributed on the interval from -1 to 1, and $Y = X^2$. Then Y is completely determined by X, so that X and Y are dependent, but their correlation is zero; they are uncorrelated. However, in the special case when X and Y are jointly normal, uncorrelatedness is equivalent to independence.

A correlation between two variables is diluted in the presence of measurement error around estimates of one or both variables, in which case disattenuation provides a more accurate coefficient.

3.3 The Mean-Variance (MV) Efficiency

A quadratic utility function can be assumed to be able to describe an investor's risk / reward preference. This theory thus assumes that only the expected return and the volatility (i.e., mean return and standard deviation) matter to the investor. To the investor, other characteristics of the distribution of returns, such as its skewness (measures the level of asymmetry in the distribution) or kurtosis (measure of the thickness or so-called 'fat tail'), is of little or no concern.

Volatility, as a proxy for risk, is used as a parameter in this theory, whereas return is deemed an expectation on the future. This agrees with the efficient market hypothesis and most of the traditional conclusions in finance such as Black and Scholes European Option Pricing (martingale measure: which means that the best forecast for tomorrow is the price of today).

Under the model:

- Portfolio return is the proportion-weighted combination of the constituent assets' returns.

- Portfolio volatility is a function of the correlation ρ of the component assets. The change in volatility is non-linear as the weighting of the component assets changes.

In general,

- The returns on individual securities or asset $r_1, r_2, \ldots, r_n$ are jointly distributed random variables, and the return on the portfolio is

$$R = \sum_{i=1}^{n} w_i r_i \tag{3.3}$$

and the expected return on the portfolio as a whole is given by:

$$E(r) = \sum_{i=1}^{n} w_i E(r_i) \tag{3.4}$$

where w_i is the weighting of component asset i

and $\mu_i = E(r_i)$ (3.5)

for $i = 1, \cdots, n$

- Portfolio variance:

$$\sigma_p^2 = \sum_{i=1} w_i^2 \sigma_i^2 \sum_{i=1} \sum_{j=1} w_i w_j \sigma_i \sigma_j \rho_{ij} \tag{3.6}$$

where $i \neq j$. Alternatively the expression can be written as:

$$\sigma_p^2 = \sum_{i=1} \sum_{j=1} w_i w_j \sigma_i \sigma_j \rho_{ij} \tag{3.7}$$

where $\rho_{ij} = 1$ for $i = j$.

- Portfolio volatility:

$$\sigma_p = \sqrt{\sigma_p^2} \tag{3.8}$$

3.3.1 Portfolio of Two Assets

For a two asset portfolio:-

- Portfolio return:

$$E(R_p) = w_A E(R_A) + (1 - w_A) E(R_B) = w_A E(R_A) + w_B E(R_B) \tag{3.9}$$

- Portfolio variance:

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{AB} \tag{3.10}$$

matrices are preferred for calculations of the efficient frontier. In matrix form, for a given 'risk tolerance' $q \in [0, \infty)$, the efficient front is found by minimizing the following expression:

$$\frac{1}{2} \cdot w^T \sum w - q * R^T w \tag{3.11}$$

where

w is a vector of portfolio weights. Each $w_i \geq 0$ and

$$\sum_i w_i = 1 \tag{3.12}$$

for $\quad i = 1, \cdots, n$.

Σ is the covariance matrix for the assets in the portfolio,