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A Test Of Symmetry

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When data are nonnormal in form classical procedures for assessing treatment group equality are prone to distortions in rates of Type I error and power to detect effects. Replacing the usual means with trimmed means reduces rates of Type I error and increases sensitivity to detect effects. If data are skewed, say to the right, then it has been postulated that asymmetric trimming, to the right, should be better at controlling rates of Type I error and power to detect effects than symmetric trimming from both tails of the data distribution. Keselman, Wilcox, Othman and Fradette (2002) found that Babu, Padmanabhan and Puri's (1999) test for symmetry when combined with a heteroscedastic statistic which compared either symmetrically or asymmetrically determined means provided excellent Type I error control even when data were extremely heterogeneous and very nonnormal in form. In this paper, we present a detailed discussion of the Babu et al. procedure as well as a numerical example demonstrating its use.

Key words: Symmetry, Preliminary test

Introduction

Keselman, Wilcox, Othman and Fradette (2002) found that by utilizing a test for symmetry prior to testing for equality of trimmed means they were able to achieve excellent Type I error control even though data were extremely heterogeneous and very nonnormal in form. In particular, they used a test for symmetry first proposed by Hogg, Fisher, and Randles (1975) and subsequently modified by Babu, Padmanaban and Puri (1999) in order to determine whether data should be trimmed symmetrically or asymmetrically. Asymmetric trimming has been theorized to be potentially advantageous when the distributions are known to be skewed, a situation likely to be realized with behavioral science data (See De Wet & van Wyk, 1979; Micceri, 1989; Tiku, 1980, 1982; Wilcox, 1995). That is, theoretical considerations suggest that when data are say skewed to the right then in order to achieve robustness to nonnormality and greater sensitivity to detect effects one should trim data just from the upper tail of the data distribution. Indeed, Keselman et al. found that by combining a test for mean equality with a preliminary test for symmetry Type I error rates could be substantially improved for the nonnormal and heterogeneous distributions they examined. Because space considerations prevented them from providing a full description of the symmetry test we present the method herein and illustrate its application with a numerical example.

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Theoretical Background

The Babu et al. (1999) procedure is based, in part, on the work of Hogg et al. (1975). Specifically, for these authors, the hypothesis of interest was H_0: $\theta = 0$ against H_A: $\theta > 0$, where θ is the location parameter of interest. They proposed a test to detect the nature of the underlying distribution before proceeding with (nonparametric) tests of H_0.

In particular, they defined $Y_1, Y_2, \ldots, Y_m$ as a random sample from F(y), and $Y_{m+1}, Y_{m+2}, \ldots, Y_n$ as a random sample from F(y - θ). Then $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the ordered statistics of the combined random samples and Y_{med} is the median of the combined samples.

Hogg et al.'s (1975) procedure to detect the nature of the underlying distribution is composed of two tests, a test of the heaviness of the tail of the distribution using the Q_2 statistic and a test of symmetry using the Q_1 statistic. Their work was based on papers by Uthoff (1970, 1973). Hogg et al. (1975) chose a test statistic enumerated by Uthoff (1973, Equation 2) as a basis to define their Q_2 index. This index determined whether the tail of the underlying distribution is light or heavy. They first approximated it as

$$\frac{Y_{(n)} - Y_{(1)}}{2\Sigma\left|Y_{(i)} - Y_{med}\right| / n}.$$

They transformed this ratio into

$$Q_2 = \frac{(U_{0.05} - L_{0.05})}{(U_{0.5} - L_{0.5})},$$

where $U_{0.05}$ and $L_{0.05}$ are, respectively, the means of the upper and lower 5% of the order statistics of the sample and $U_{0.5}$ and $L_{0.5}$ are, respectively, the means of the upper and lower 50% of the order statistics of the combined sample.

Again, based on the work of Uthoff (1970, Equation 1), Hogg et al. (1975) derived their Q_1 index:

$$Q_1 = \frac{(U_{0.05} - MID)}{(MID - L_{0.05})},$$

where MID is the mean of the middle 50% of the combined sample. Thus, this index determines the symmetry of the underlying distribution.

Babu et al. (1999) extended the use of these two indices to more than two groups. They proposed that both indices be calculated within the groups and weighted means of these indices be the overall estimates of Q_2 and Q_1. They also proposed adjustments to the Q_1 index whereby the amount of data needed to calculate the index depended on the outcome of the calculation of the Q_2 index.

Determination of Symmetry

Consider the problem of comparing distributions $F_1 = F_2 = \ldots = F_J$. One way of approaching this problem is to consider the one-way ANOVA problem of comparing means $\mu_1 = \mu_2 = \ldots = \mu_J$ from J distributions $F_1(y) = F(y - \mu_1)$, $F_2(y) = F(y - \mu_2)$, … , $F_J(y) = F(y - \mu_J)$. When the distributions are unknown and one cannot assume that they are normal with equal variances, Babu et al. (1999) suggested the following procedure to determine heavy-tailedness and symmetry prior to applying the appropriate test on the location parameters:

Let $Y_{ij} = (Y_{1j}, Y_{2j}, \ldots, Y_{n_j j})$ be a sample from an unknown distribution F_j. Let $Y_{(1)j} \le Y_{(2)j} \le \cdots \le Y_{(n_j)j}$ represent the ordered observations associated with the j^{th} group. Let γ be the proportion of the data in the sample that are of interest as either the proportion of data to be trimmed or the proportion of data to be used in the calculation of several intermediate variables leading to two statistics, namely Q_2 and Q_1. Let $g = [\gamma n_j] + 1$, where [x] represents the greatest integer less than γn_j and $r = g - \gamma n_j$. It is important to note that trimming here, and the amount trimmed, is just for purposes of assessing symmetry.

Q_2 Index

Prior to determining the symmetry of the distributions, the nature of their tails is examined. The Q_2 index determines whether $F_1(y), F_2(y), \ldots, F_J(y)$ are normal-tailed, heavy-tailed or very heavy-tailed. Tail classification is determined in the following manner:

1. Define $U_{\gamma j}$ and $L_{\gamma j}$ as the means of the upper and lower γn_j order statistics, respectively, of the sample Y_j.

Case 1. If $\gamma n_j \le 1$,
then $U_{\gamma j} = Y_{(n_j)j}$ and $L_{\gamma j} = Y_{(1)j}$.

Case 2. If $\gamma n_j > 1$
then

$$U_{\gamma,j} = \frac{1}{\gamma n_j}\left(\sum_{j=n_j-g+2}^{n_j} Y_{(i)j} + (1-r)Y_{(n_j-g+1),j}\right) \text{ and}$$

$$L_{\gamma,j} = \frac{1}{\gamma n_j}\left(\sum_{i=1}^{g-1} Y_{(i)j} + (1-r)Y_{(g)j}\right)$$

2. Calculate $U_{0.05,\,j}$ and $L_{0.05,\,j}$ as the mean of the upper and lower $0.05n_j$ order statistics of Y_j, respectively.
3. Calculate $U_{0.5,\,j}$ and $L_{0.5,\,j}$ as the mean of the upper and lower $0.5n_j$ order statistics of Y_j, respectively.
4. For each j, set $Q_{2,j} = (U_{0.05,j} - L_{0.05,j}) / (U_{0.5,j} - L_{0.5,j})$.
5. Using $Q_{2,j}$, j = 1, 2,…, J, from # 4 compute

$$Q_2 = \left(\sum_{j=1}^{J} n_j Q_{2,j}\right) \Big/ \left(\sum_{j=1}^{J} n_j\right)$$

6. If $Q_2 < 3$ then F is classified as normal-tailed. If $3 \le Q_2 < 5$ then F is classified as heavy-tailed. If $Q_2 \ge 5$ then F is classified as very heavy-tailed.

Q_1 Index

Once the nature of the tails of the distributions is known, the Q_1 index, which determines the symmetry of the distributions, is calculated. To calculate the Q_1 index one should:
1. Based on Q_2, determine the number of sample points in each sample Y_j to be used. Define this as n_j*. (This is the Babu et al., 1999, modification of the Hogg et al., 1975, proposal for computing Q_1.) Specifically, if $Q_2 < 3$ then use all sample points in Y_j. If $3 \le Q_2 < 5$ then trim the top and bottom 10% of the sample points and use the middle 80% in Y_j. If $Q_2 \ge 5$ then trim the top and bottom 20% of the sample points and use the middle 60% in Y_j.
2. Let MID_j to be the mean of the middle 50% of the order statistics of the sample points in sample Y_j defined in #1. According to A. R. Padmanaban (personal communication, June 26, 2001), MID_j is calculated in the following manner:

Discard the top and bottom 25% of the order statistics of Y_j.

The remainder is the middle 50% of the order statistics of Y_j.

Hence, $g^* = [\,0.25n_j^*\,] + 1$ and $r^* = g^* - 0.25n_j^*$. Therefore, MIDj is given by

$$MID_j = \frac{1}{0.5n_j^*}\left[\sum_{i=g^*+1}^{n_j^*-g^*} Y_{(i)j} + r^*(Y_{(g^*)j} + Y_{(n_j^*-g^*+1)j})\right].$$

3. For each j, set

$$Q_{1,j} = (U_{0.05,j} - MID_j)/(MID_j - L_{0.05,j}).$$

4. Using $Q_{1,j}$, j = 1, 2,…, J, from # 3 compute

$$Q_1 = \left(\sum_{j=1}^{J} n_j^* Q_{1,j}\right) \Big/ \left(\sum_{j=1}^{J} n_j^*\right).$$

5. If $Q_1 < ½$, F is deemed to be left skewed. If $½ \le Q_1 \le 2$, then F is considered to be symmetric. If $Q_1 > 2$, then F is designated as right skewed.

Computational Example

Suppose we want to test the null hypothesis, Ho: $F_1(x) = F_2(x) = F_3(x)$ based on the following data set.

Table 1. Data set.

Groups	Order Statistics	n_j
1	30 32 32 34 35 35 39 40 40 41 42 48 50 52 99	15
2	35 36 40 40 41 42 43 49 56 64	10
3	48 48 51 51 51 55 55 60 63 83	10

Note: The tabled values were chosen so that the data would be classified as heavy-tailed.

Calculating Q_2 (Tail thickness)

Notice that $0.05n_j < 1$ for j = 1, 2, 3. Therefore, $U_{0.05,\,1} = Y_{(15,\,1)} = 99$, $U_{0.05,\,2} = Y_{(10,\,2)} = 64$, $U_{0.05,\,3} = Y_{(10,\,3)} = 83$, and $L_{0.05,\,1} = Y_{(1,\,1)} = 30$,

$L_{0.05,2} = Y_{(1,2)} = 35$, and $L_{0.05,3} = Y_{(1,3)} = 48$. When $\gamma = 0.5$, the calculations for $U_{0.5,j}$, $L_{0.5,j}$ and $Q_{2,j}$ for each group are as follows:

Group 1
$n_1 = 15$, $0.5\, n_1 = 7.5$, $g = 8$ and $r = 0.5$.

$$U_{0.5,1} = \frac{1}{7.5}\left(\sum_{i=9}^{15} Y_{(i1)} + 0.5Y_{(8,1)}\right)$$
$$= \frac{1}{7.5}((40+41+\cdots+99)+(0.5)40)$$
$$= 52.2667$$

$$L_{0.5,1} = \frac{1}{7.5}\left(\sum_{i=1}^{7} Y_{(i1)} + 0.5Y_{(8,1)}\right)$$
$$= \frac{1}{7.5}((30+32+\cdots+39)+(0.5)40)$$
$$= 34.2667$$

$$Q_{2,1} = \frac{(99-30)}{(52.2667-34.2667)} = 3.8333$$

Group 2

$n_2 = 10$, $0.5\, n_2 = 5$, $g = 6$ and $r = 0$.

$$U_{0.5,2} = \frac{1}{5}\left(\sum_{i=6}^{10} Y_{(i2)} + (0)Y_{(5,2)}\right)$$
$$= \frac{1}{5}((42+43+\cdots+64)+0)$$
$$= 50.8$$

$$L_{0.5,2} = \frac{1}{5}\left(\sum_{i=1}^{5} Y_{(i2)} + (0)Y_{(6,2)}\right)$$
$$= \frac{1}{5}((35+36+\cdots+41)+0)$$
$$= 38.4$$

$$Q_{2,2} = \frac{(64-35)}{(50.8-38.4)} = 2.3387$$

Group 3
$n_3 = 10$, $0.5\, n_3 = 5$, $g = 6$ and $r = 0$.

$$U_{0.5,3} = \frac{1}{5}\left(\sum_{i=6}^{10} Y_{(i3)} + (0)Y_{(5,3)}\right)$$
$$= \frac{1}{5}((55+55+\cdots+83)+0)$$
$$= 63.2$$

$$L_{0.5,3} = \frac{1}{5}\left(\sum_{i=1}^{5} Y_{(i3)} + (0)Y_{(6,3)}\right)$$
$$= \frac{1}{5}((48+48+\cdots+51)+0)$$
$$= 49.8$$

$$Q_{2,3} = \frac{(83-48)}{(63.2-49.8)} = 2.6119$$

Therefore,

$$Q_2 = \frac{(15(3.8333)+10(2.3387)+10(2.6119))}{(15+10+10)}$$
$$= 3.0573$$

and F is classified as heavy-tailed.

Calculating Q1

Because F if classified as heavy-tailed, we have to symmetrically trim 10% of the data before calculating Q_1.

Notice that $0.05\, n_j^* < 1$ for j = 1, 2, 3. Therefore:

$$U^*_{0.05,1} = Y^*_{(13,1)} = 52,\ U^*_{0.05,2} = Y^*_{(8,2)} = 56,$$

$$U^*_{0.05,3} = Y^*_{(8,3)} = 63, \text{ and } L^*_{0.05,1} = Y^*_{(1,1)} = 32,$$

$$L^*_{0.05,2} = Y^*_{(1,2)} = 36,\ L^*_{0.05,3} = Y^*_{(1,3)} = 48.$$

Let us calculate MID_j and $Q_{1,j}$, for j = 1, 2, 3.

Table 2. 10% Trimming.

Groups	Order Statistics Following 10% Symmetric Trimming	n_j^*
1	32 32 34 35 35 39 40 40 41 42 48 50 52	13
2	36 40 40 41 42 43 49 56	8
3	48 51 51 51 55 55 60 63	8

Group 1

$n_1^* = 13,\ 0.25n_1^* = 3.25,\ g^* = 4, \text{and } r^* = 0.75$.

$$\begin{aligned} MID_1 &= \frac{1}{6.5}\left(\sum_{i=5}^{9} Y_{(i1)}^* + 0.75(Y_{(4,1)}^* + Y_{(10,1)}^*)\right) \\ &= \frac{1}{6.5}((35+39+40+40+41)+(0.75)(35+42)) \\ &= 38.8846 \end{aligned}$$

$$Q_{1,1} = \frac{(52-38.8846)}{(38.8846-32)} = 1.905$$

Group 2

$n_2^* = 8,\ 0.25n_2^* = 2,\ g^* = 3, \text{and } r^* = 0$

$$\begin{aligned} MID_2 &= \frac{1}{4}\left(\sum_{i=3}^{6} Y_{(i2)}^*\right) \\ &= \frac{1}{4}(40+41+42+43) \\ &= 41.5 \end{aligned}$$

$$Q_{1,2} = \frac{(56-41.5)}{(41.5-36)} = 2.6364$$

Group 3

$n_3^* = 8,\ 0.25n_3^* = 2,\ g^* = 3, \text{and } r^* = 0$.

$$\begin{aligned} MID_3 &= \frac{1}{4}\left(\sum_{i=3}^{6} Y_{(i3)}^*\right) \\ &= \frac{1}{4}(51+51+55+55) \\ &= 53 \end{aligned}$$

$$Q_{1,3} = \frac{(63-53)}{(53-48)} = 2$$

Therefore,

$$\begin{aligned} Q_1 &= \frac{(13(1.905)+8(2.6364)+8(2))}{(13+8+8)} \\ &= 2.133 \end{aligned}$$

and F is classified as right skewed.

Discussion

As indicated in our introduction, Keselman et al. (2002) found that by first applying the Babu et al. (1999) procedure prior to testing for treatment group equality with sample symmetrically or asymmetrically determined trimmed means one could achieve excellent control over Type I errors even though data were obtained from very heterogenous distributions that were extremely nonnormal in form. Accordingly, they recommended that users adopt the Babu et al. (1999) test for symmetry.

It is also interesting to note that Babu et al. (1999) used the preliminary test for symmetry in order to determine whether groups should be compared on their symmmetrically determined trimmed means, when distributions were deemed symmetric, or on their medians, when distributions were deemed asymmetric. Thus, a test for symmetry can be beneficial in many different applications.

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