SOLUTIONS OF POLYNOMIAL EQUATIONS

By

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ABSTRAK

Penentuan punca polinomial adalah antara masalah matematik yang paling kuno. Perbincangan utama dalam projek ini adalah tentang kaedah-kaedah mencari satu punca atau punca-punca persamaan polinomial. Sejarah mengenai penyelesaian persamaan polinomial telah mempengaruhi perkembangan kaedah-kaedah dalam penyelesaian persamaan polinomial. Kaedah melengkapkan kuasa dua, kaedah graf dan kaedah pemfaktoran telah diperkenalkan untuk menyelesaikan persamaan quadratik. Rumus Quadratik diperolehi daripada proses kaedah melengkapkan kuasa dua. Rumus Quadratik boleh digunakan untuk menyelesaikan sebarang persamaan quadratik. Dalam kaedah pemfaktoran, pembahagian sintetik merupakan kaedah singkat untuk membahagikan polinomial yang mempunyai kuasa yang tinggi dengan polinomial linear berbentuk $x - c$. Kaedah Cardano dan kaedah Ferrari digunakan untuk mencari punca-punca persamaan cubik dan persamaan quartik masing-masing. Akan tetapi, kedua-dua kaedah tersebut adalah agak rumit. Penyelesaian penghampiran bagi sebarang persamaan polinomial boleh dicari sama ada dengan kaedah Pembahagian Dua Sama atau kaedah Newton. Kita mendapati bahawa kadar penumpuan kaedah Newton adalah lebih cepat berbanding dengan kaedah Pembahagian Dua Sama. Persamaan polinomial telah muncul dalam sains, kejuruteraan, perniagaan, ekonomi dan lain-lain bidang. Dengan itu, kita menamatkan projek ini dengan memperkenalkan beberapa aplikasi punca-punca persamaan polinomial dalam kehidupan dunia sebenar.

ABSTRACT

Determining the roots of polynomial equations is among the oldest problems in mathematics. The major problem discussed in this project is that the methods of finding one or more roots of polynomial equations. The history of solutions of polynomial equations has influenced the development of the methods in solving the polynomial equations. Completing the square, graphical method and factorization are introduced to solve the quadratic equations. Quadratic formula can be obtained from the process of completing the square and it gives the solution of any quadratic equation. Synthetic division in section factorization is a shortcut method for dividing a polynomial of higher degree by a linear factor of the form $x - c$. Cardano's method and Ferrari's method can be used to find the roots of cubic equations and quartic equations respectively. However, we found that these two methods are complicated. We can find approximate solutions of any polynomial equations either by Bisection method or Newton's method. We noticed that Newton's method converges rapidly compared with Bisection method. In particular, polynomial equations arising in the sciences, engineering, business, economic and the other fields. Hence, we end by introduced several applications of the roots of polynomial equations in these fields in our real life situations.

CHAPTER 1

INTRODUCTION

One of the oldest and maybe for centuries the only area of study in algebra had been polynomial equations (http://members.tripod.com/~PetrilisD/). Polynomials are built from terms called monomials, which consist of a constant called the coefficient multiplied by one or more variables that usually represented by letters. Each variable may have a constant positive whole number exponent. For example, $4x^3$ is a monomial. The coefficient is 4, the variable is x and the degree of x is three. A polynomial is a sum of one or more monomial terms. A polynomial in one variable with constant coefficients has a general form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 \qquad (a_n \neq 0)$$

where n is a non-negative integer, x is a variable and $a_n, a_{n-1,\ldots,} a_1, a_0$ are real numbers which called coefficients of a polynomial. The highest power of x is called the degree of the polynomial.

The number a_0 is called the constant term of the polynomial. If $n = 0$ in the polynomial above, then the polynomial consists of just one term, namely a_0. Such a polynomial is called a constant polynomial. If $a_0 \neq 0$, the constant polynomial a_0 has degree zero. If $n = 1$, the resulting polynomial

$$a_1 x + a_0$$

is called a linear polynomial. Its degree is one if $a_1 \neq 0$. Each polynomial of degree two has the form

$$a_2x^2 + a_1x + a_0, \qquad a_2 \neq 0$$

We have called such algebraic expressions quadratic polynomial. A polynomial of degree three is called a cubic polynomial and has the form

$$a_3x^3 + a_2x^2 + a_1x + a_0, \qquad a_3 \neq 0$$

Polynomials of degree 4, 5, 6 are called quartic, quintic and sextic.

A general form of polynomial equation as shown below. Let $P(x)$ be a polynomial of degree $n \geq 1$ and with real or complex coefficient, then

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_2x^2 + a_1x + a_0 = 0 \tag{1.1}$$

The problem of solving a polynomial equation (1.1) consists in finding all its roots, i.e. the roots of the polynomial P(x).

The objectives of this project are to study some of the theorems that related in the solutions of polynomial equations, to study the methods for finding the roots of quadratic, cubic and quartic polynomial equations, to study the common numerical methods in finding approximate roots of polynomial equations and to illustrate the important of root-finding with several applications.

CHAPTER 2

A BRIEF HISTORY OF SOLUTIONS OF POLYNOMIAL EQUATIONS

Polynomial equations have a lengthy history (Stewart, 1973, p. xiii). Determining the roots of polynomial equation (1.1) is among the oldest problems in mathematics (http://en.wikipedia.org/wiki/polynomial). How do we determine the roots of polynomial equation? The earliest solutions to quadratic equations involving an unknown are found in Babylonian mathematical texts that date back to about 2000 B.C.. In about 400 B.C., they developed an algorithmic approach to solving problems that gives to a quadratic equation. This method is about on the method of completing the square. But at this time the Babylonians did not have algebraic notation to express their solution.

The next big step was done by the ancient Greeks. About 500 B.C. in ancient Greeks, the Pythagoreans proved that the equation $x^2 = 2$ has no rational solution, that is, that its solution must use a radical and not only arithmetic operations. The ancient Greeks mathematician Diophantus solved quadratic equation but giving only one root, even when both roots were positive.

Between 598 and 665 A.D., Brahmagupta, an Indian mathematician advanced the Babylonian methods to almost modern methods. Indian and Chinese mathematicians recognized negative roots to quadratic equations.

An Arab mathematician, Abu Abdullah Muhammad ibn Musa al-Khwarizmi, in his book *Hisab Aljabr w'al-muqabala* gave numerical examples of several categories of quadratic equations, using geometry and algebraic methods. The title refers to two basic operations on equations. The first is al-jabr which means the restoration, referring to the process of moving a subtracted quantity to the other side of an equation. For instance, the equation

$$x^2 = 40x - 4x^2$$

is converted to

$$5x^2 = 40x$$

by al-jabr (Karpinski, p. 105, in Tignol, p. 16). The second basic operation al muqabala means opposition , it is a simplification procedure by which like terms are removed from both sides of an equation (Karpinski, p. 109, in Tignol, p. 17). For instance, al muqabala changes

$$50 + x^2 = 29 + 10x$$

into

$$21 + x^2 = 10x$$

In this work, al-Khwarizmi initiates reducing the old methods for solving equations to a few standardized procedures. In problems involving several of the unknowns, he systematically sets up an equation for one of the unknowns and he solves the three types of quadratic equations:

$$x^2 + ax = b, \qquad x^2 + b = ax, \qquad x^2 = ax + b$$

by completing of the square, giving the two positive solutions for the type $x^2 + b = ax$.

Abraham Bar Hiyya Ha-Nasi, often known by the Latin name Savasorda, is famed for his book *Liber embadorum* published in 1145 which is the first book published in Europe to give the complete solution of the quadratic equation (http://en.wikipedia.org/wiki/Quadratic_equation).

Some polynomials, such that $f(x) = x^2 + 1$, do not have any roots among the real numbers. However, the set of allowed candidates is expanded to the complex numbers, every polynomial has at least one distinct root; this follows from the fundamental theorem of algebra. Peter Rothe, in his book *Arithmetica Philosophica,* published in 1608, wrote that a polynomial equation of degree n with real coefficients may have n solutions (http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra).

At the end of the 16^{th} century, the mathematical notation and symbolism was introduced by amateur-mathematician Francois Viète in France. In 1637, when René Descartes published La Géométrie, modern mathematics was born and the quadratic formula has adopted the form we know today.

Cubic equations were known to the ancient Indians and ancient Greeks since the 5^{th} century. In the 11^{th} century, the Persian poet-mathematician Omar Khayyam (1048-1131) made significant progress in the theory of cubic equations. In an early paper he wrote regarding cubic equations. He discovered that a cubic equation can have more than one solution (http://en.wikipedia.org/wiki/Cubic_equation).

The Renaissance mathematicians at Bologna discovered that the solution of the cubic could be reduced to that of three basic types: $x^3 + px = q$, $x^3 = px + q$, $x^3 + q = px$.

They were forced to distinguish these cases because negative numbers were not known at that time. Professor Scipio del Ferro found a method for solving these three types of cubic equations but he doesn't publish his solution. After his death in 1526 his method passed to his student, Antanio Maria Fior. In 1530, Niccolo Fontana (nicknamed Tartaglia) received two problems in cubic equations (Burton, 1999, p.295), namely:

i. Find a number whose cube added to three times its square makes 5: that is, find a value of x satisfying the equation $x^3 + 3x^2 = 5$.

ii. Find three numbers, the second of which exceeds the first by 2 and the third of which exceeds the second by 2 also and whose product is 1 000; that is, solve the equation $x(x+2)(x+4) = 1000$, or equivalently, $x^3 + 6x^2 + 8x = 1000$.

In 1535, Tartaglia announced that he could solve these two problems. He was soon challenged by Fiore to a public problem solving contest. Tartaglia received questions in the form $x^3 + mx = n$, for which he knew the answer. Fiore received questions in the form $x^3 + mx = n$, which proved to be too difficult for him to solve and Tartaglia won the contest.

Later, Tartaglia was persuaded by Girolamo Cardano to reveal his secret for solving cubic equations. In 1545, Girolamo Cardano gives the complete solution of cubics in his book, *Ars Magna*. This book also includes the solution of the quartic equation, due to Ludovico Ferrari, by deducing them to a cubic.

All the formulae discovered had one striking property which can be illustrated y Fontana's solution of $x^3 + px = q$:

$$x=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}}$$

The expression is built up from the coefficients by repeated addition, subtraction, multiplication, division and extraction of roots. Such expressions are called radical expressions.

Since all equations of degree less than or equal to four (degree ≤ 4) were now solved, it was natural to ask how to solve equations with degree 5 or greater than 5 by radicals. In the 18^{th} century, Josheph Louis Lagrange shows that polynomial of degree 5 or more cannot be solved by the method used for quadratics, cubics and quartics. In 19^{th} century, Niels Henrik Abel publishes *Proof of The Impossibility of Generally Solving Algebraic Equations of a Degree Higher Than the Fourth* (http://library.wolfram.com/examples/quintic/timeline.html). In 1824, Abel have proved that the general quintic equation was insoluble by radicals (Stewart, 1973, p.xv).

With no hope left for the exact solution formulae, then come out with the method of numerical analysis to find the approximate roots of nonlinear equations. Such methods are Bisection method, Newton's method or by one of the many more modern methods of approximating solutions. Newton's method was described by Isaac Newton. Newton's method was first published in 1685 in *A Treatise of Algebra both Historical and Practical* by John Wallis. In 1690, Joseph Raphson published a simplified description in *Analysis aequationum universalis*. Raphson again viewed Newton's method purely as an algebraic method and restricted its use to polynomials. Numerically solving a polynomial equations is easily done on computer.

CHAPTER 3

QUADRATIC EQUATIONS

An equation of the form

$$ax^2 + bx + c = 0 \qquad (a \neq 0) \tag{3.1}$$

where *a, b* and *c* are real numbers, is called a second degree, or quadratic equation in the variable *x*. Quadratic equations are called quadratic because quadratus is Latin for "square", in the leading term the variable is squared. In this chapter, we will discuss the three methods of solving such an equation: completing the square, graphical method and factorization. These three methods are shown as follow.

3.1 Trial and improvement method

For simple quadratic equations, we can determine the roots through the trial and improvement method. To save time when trying, we usually consider the factors of the constant *c* in the equation (3.1).

Example 3.1.1

Determine the roots of the quadratic $x^2 - 5x + 6 = 0$ by the trial and improvement method.

Solution:

Let $x = 1,\ x^2 - 5x + 6 = (1)^2 - 5(1) + 6$

$= 1 - 5 + 6$

$= 2$

Let $x = 2,\ x^2 - 5x + 6 = (2)^2 - 5(2) + 6$

$= 4 - 10 + 6$

$= 0$

Let $x = 3,\ x^2 - 5x + 6 = (3)^2 - 5(3) + 6$

$= 9 - 15 + 6$

$= 0$

Therefore, $x = 2$ and $x = 3$ are the roots of the equation $x^2 - 5x + 6 = 0$.

3.2 Factorization

Before go through to factorization method, let us look at a few related theorems.

Theorem 3.1 Division Algorithm

For each polynomial $P(x)$ of degree greater than 0 and each number c, there exists a unique polynomial $Q(x)$ of degree 1 less than $P(x)$ and a unique number R such that

$$P(x) = (x\text{-}c)Q(x) + R \qquad (3.2)$$

The polynomial $Q(x)$ is called the quotient, $x - c$ is the divisor and R is the remainder.

Theorem 3.2 The Remainder Theorem

Let $P(x)$ be a polynomial. If $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

Proof:

We use the division algorithm in Theorem 3.2 to prove the remainder theorem. The equation (3.2) is an identity in x and is true for all real number x. In particular, it is true when $x = c$. Thus, if $x = c$, the equation (3.2) becomes

$$\begin{aligned} P(c) &= (c - c)Q(x) + R \\ &= 0 \, . \, Q(x) + R \\ &= 0 + R \\ &= R \end{aligned}$$

In words, the value of a polynomial $P(x)$ at $x = c$ is the same as the remainder R obtained when we divide $P(x)$ by $x - c$. Thus, equation (3.4) takes the form

$$P(x) = (x - c)Q(x) + P(c) \tag{3.3}$$

An important and useful consequence of the Remainder Theorem is the Factor Theorem.

Theorem 3.3 Factor Theorem

$(x - c)$ is a factor of the polynomial $P(x)$ if and only if c is a root of $P(x)$.

Proof:

The Factor Theorem consists of two separate statements:

i. If $P(c) = 0$, then $x - c$ is a factor of $P(x)$.

ii. If $x - c$ is a factor of $P(x)$, then $P(c) = 0$.

Thus, the proof requires two parts.

Part (i):

Suppose that $P(c) = 0$. Then by equation (3.4), we have

$$P(x) = (x\text{-}c)Q(x)$$

Hence, $x - c$ is a factor of $P(x)$ if $P(c) = 0$.

Part (ii):

Suppose that $x - c$ is a factor of $P(x)$. Then there is a polynomial $P(c)$ such that

$$P(x) = (x\text{-}c)Q(x)$$

Replacing x by c, we find that

$$\begin{aligned} P(c) &= (c - c)Q(c) \\ &= 0.Q(c) \\ &= 0 \end{aligned}$$

This complete the proof.

To solve the quadratic equation by Factorization,

i. write the polynomial equation in standard form with zero on the right side, such that $ax^2 + bx + c = 0$.

ii. factor the left side.

iii. set each factor equal to zero and solve the resulting linear equations. Applying the following property: $ab = 0$ if and only if $a = 0$ or $b = 0$.

Example 3.2.1

Solve $x^2 - 3x - 4 = 0$.

Solution:

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x - 4 = 0 \qquad \text{or} \qquad x + 1 = 0$$

$$x = 4 \qquad\qquad\qquad x = -1$$

Therefore, the solutions to the equation $x^2 - 3x - 4 = 0$ are $x = 4$ and $x = -1$.

3.3 Completing the square

To solve a quadratic equation by completing the square, we follow the steps below.

First divide both sides of the equation (3.1) by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Rewrite so that the x^2 and x terms are on the left side and the constant is on the other side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

The equation is now in a form in which we can conveniently complete the square. We would like to make the left side of the last equation a perfect square of the form $(x+k)^2 = x^2 + 2kx + k^2$. If we let $2k = \frac{b}{a}$, then $k = \frac{b}{2a}$ and $k^2 = \frac{b^2}{4a^2}$. Hence, we continue as follow:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

The left side is now a perfect square of $\left(x + \frac{b}{2a}\right)^2$. The right side can be written as a single fraction, with common denominator $4a^2$. This gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square roots of both sides of equations,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify the radical, we get

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Isolating x, gives

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence, the solution set of the quadratic equation (3.1) is $\left\{\frac{-b+\sqrt{b^2-4ac}}{2a},\frac{-b-\sqrt{b^2-4ac}}{2a}\right\}$.

Example 3.3.1

Solve $x^2-6x-16=0$ by completing the square.

Solution:

$$x^2-6x=16$$

$$x^2-6x+\frac{(-6)^2}{4}=16+\frac{(-6)^2}{4}$$

$$\left(x-\frac{6}{2}\right)^2=25$$

$$(x-3)^2=25$$

$$x-3=\pm\sqrt{25}$$

$$x-3=\pm 5$$

$$x-3=5 \quad \text{or} \quad x-3=-5$$

$$x=5+3 \quad \text{or} \quad x=-5+3$$

$$x = 8 \qquad \text{or} \qquad x = -2$$

Therefore, the two solutions of equation $x^2 - 6x - 16 = 0$ are $x = 8$ and $x = -2$.

3.4 Quadratic formula

We use the process of completing the square to solve the general quadratic equation (3.1), we obtain a quadratic formula. This formula gives the solution of any quadratic equation. We can use this formula in the following way:

i. write the equation in the standard form $ax^2 + bx + c = 0$.

ii. if necessary, clear the equation of fractions to simplify calculations.

iii. identify a, b and c.

iv. substitute the values of a, b and c into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

v. simplify the right side.

Example 3.4.1

Solve $6x^2 - 5x - 4 = 0$ by quadratic formula.

Solution:

a = 6, b = -5 and c = -4.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)}$$

$$= \frac{5 \pm \sqrt{25 + 96}}{12}$$

$$= \frac{5 \pm \sqrt{121}}{12}$$

$$= \frac{5 \pm 11}{12}$$

$$x = \frac{5+11}{12} \qquad \text{or} \qquad x = \frac{5-11}{12}$$

$$x = \frac{4}{3} \qquad \text{or} \qquad x = -\frac{1}{2}$$

Therefore, the solutions of the equation $6x^2 - 5x - 4 = 0$ are $x = \frac{4}{3}$ and $x = -\frac{1}{2}$.

Example 3.4.2

Solve $x^2 - 2x + 4 = 0$.

Solution:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$$x = 1 + i\sqrt{3} \qquad \text{or} \qquad x = 1 - i\sqrt{3}$$

Therefore, the solutions of equation $x^2 - 2x + 4 = 0$ are $x = 1 + i\sqrt{3}$ and $x = 1 - i\sqrt{3}$.

CHAPTER 4

CUBIC EQUATIONS

This chapter discusses about solution of cubic equations. Cardano's method and factorization (synthetic division) are used to solve the cubic equations.

4.1 Cardano's method

Let starts with the equation

$$x^3 + ax^2 + bx + c = 0 \qquad (a \neq 0) \tag{4.1}$$

where a, b and c are real numbers.

To solve the general cubic equation (4.1), change the it is reasonable by attempting to eliminate the ax^2 term. We let

$$x = y - \frac{a}{3} \tag{4.2}$$

and substituting into equation (4.1). We have,

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$$

$$y^3 - \frac{a^2 y}{3} + \frac{2a^3}{27} + by - \frac{ab}{3} + c = 0$$

$$y^3 + \left(b - \frac{a^2}{3}\right)y + \frac{2a^3}{27} - \frac{ab}{3} + c = 0$$

To make it simple, we let $p=b-\frac{a^2}{3}$ and $q=\frac{2a^3}{27}-\frac{ab}{3}+c$. So, it gives

$$y^3+py+q=0 \tag{4.3}$$

We have transformed equation (4.1) into equation (4.3) which looks much simple. If we can solve equation (4.3), then we get the answer of equation (4.1) via equation (4.2). Let

$$y=u+v \tag{4.4}$$

Substituting (4.4) into equation (4.3)

$$(u+v)^3+p(u+v)+q=0$$

$$u^3+v^3+3uv(u+v)+p(u+v)+q=0$$

$$u^3+v^3+(u+v)(3uv+p)+q=0$$

If we put $v=-\frac{p}{3u}$, we get

$$u^3+\left(-\frac{p}{3u}\right)^3+q=0$$

$$u^3-\frac{p^3}{27u^3}+q=0$$

Multiplying both sides of this equation by u^3 gives the quadratic equation in u^3.

$$\left(u^3\right)^2+qu^3-\frac{p^3}{27}=0$$

By quadratic formula :

$$u^3=-\frac{q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2+\frac{p^3}{27}}$$

$$u=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2+\frac{p^3}{27}}}$$

To find v, substitute u into $v = -\dfrac{p}{3u}$

$$v = -\frac{p}{3}\left[\frac{1}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}}\right]$$

$$= -\frac{p}{3}\left[\frac{1}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}}\right]\left[\frac{\sqrt[3]{-\frac{q}{2} \mp \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}}{\sqrt[3]{-\frac{q}{2} \mp \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}}\right]$$

$$= \sqrt[3]{-\frac{q}{2} \mp \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}$$

From equation (4.4), we have

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}$$

There is only one solution for y.

Equation (4.3) should have three solutions. To find the other two solutions, let us fix $y_0 = u + v$. So, $y - y_0$ is a factor of equation (4.3). By synthetic division, put $c = y_0$,

		1	0	p	q
$c = y_0$,	+		y_0	y_0^2	$y_0^3 + y_0 p$
		1	y_0	$y_0^2 + p$	$y_0^3 + y_0 p + q$

Note that as y_0 is a factor of equation (4.3), so

$$y_0^3 + y_0 p + q = 0$$

Thus,

$$y^3 + py + q = (y - y_0)(y^2 + y_0 y + (p + y_0^2))$$

The other two solutions of equation (4.3) are solutions of the quadratic equation $y^2 + y_0 y + p + y_0^2 = 0$. By quadratic formula,

$$y = \frac{-y_0 \pm \sqrt{y_0^2 - 4(p + y_0^2)}}{2}$$

$$= \frac{1}{2}\left[-y_0 \pm \sqrt{-3y_0^2 - 4p}\right]$$

$$= \frac{1}{2}\left[-(u+v) \pm \sqrt{-3(u+v)^2 - 4(-3uv)}\right]$$

$$= \frac{1}{2}\left[-u - v \pm \sqrt{-3(u^2 + v^2 - 2uv)}\right]$$

$$= \frac{1}{2}\left[-u - v \pm \sqrt{3(u-v)^2 i}\right]$$

$$= \frac{1}{2}\left[-u - v \pm (u-v)\sqrt{3i}\right]$$

So,

$$y = \frac{-u + \sqrt{3}ui}{2} + \left(\frac{-v - \sqrt{3}vi}{2}\right) \qquad \text{or} \qquad y = \frac{-u - \sqrt{3}ui}{2} + \left(\frac{-v + \sqrt{3}vi}{2}\right)$$

$$= u\left(\frac{-1 + \sqrt{3}i}{2}\right) + v\left(\frac{-1 - \sqrt{3}i}{2}\right) \qquad\qquad = u\left(\frac{-1 - \sqrt{3}i}{2}\right) + v\left(\frac{-1 + \sqrt{3}i}{2}\right)$$

Let $w = \frac{-1 + \sqrt{3}i}{2}$ then $\frac{-1 - \sqrt{3}i}{2} = w^2$, we have

$$y = uw + vw^2 \qquad \text{or} \qquad y = uw^2 + vw$$

So, the three solutions of equation (4.3) are

$$y_1 = u + v, \qquad y_2 = uw + vw^2, \qquad y_3 = uw^2 + vw \tag{4.5}$$

where $$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}, \qquad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}}$$

and $$w = \frac{-1 + \sqrt{3}i}{2}$$

Hence, equation (4.1) can be solved using equation (4.2).

Example 4.1.1

Solve $x^3 + 3x + 4 = 0$.

Solution:

$$p = 3, q = 4$$

Applying (4.5) to get the values of x_1, x_2 and x_3,

$$x_1 = \sqrt[3]{-\frac{4}{2} + \sqrt{\left(\frac{4}{2}\right)^2 + \frac{3^3}{27}}} + \sqrt[3]{-\frac{4}{2} - \sqrt{\left(\frac{4}{2}\right)^2 + \frac{3^3}{27}}}$$

$$= \sqrt[3]{-2 + \sqrt{4+1}} + \sqrt[3]{-2 - \sqrt{4+1}}$$

$$= \sqrt[3]{-2 + \sqrt{5}} + \sqrt[3]{-2 - \sqrt{5}}$$

$$= -1$$

$$x_2 = \left(\frac{-1 + \sqrt{3}i}{2}\right)\left(\sqrt[3]{-2 + \sqrt{5}}\right) + \left(\frac{-1 - \sqrt{3}i}{2}\right)\left(\sqrt[3]{-2 - \sqrt{5}}\right)$$

$$x_3 = \left(\frac{-1 - \sqrt{3}i}{2}\right)\left(\sqrt[3]{-2 + \sqrt{5}}\right) + \left(\frac{-1 + \sqrt{3}i}{2}\right)\left(\sqrt[3]{-2 - \sqrt{5}}\right)$$

Example 4.1.2

Solve $x^3 - 3x^2 - 6x - 4 = 0$.

Solution:

To find the solution of this equation by Cardano's formula, first transform the equation into the form of equation (4.3).

$$p = -6 - \frac{(-3)^2}{3}$$

$$= -9$$

$$q = \frac{2(-3)^3}{27} - \frac{(-3)(-6)}{3} + (-4)$$

$$= -12$$

This gives,

$$y^3 - 9y - 12 = 0$$

Applying (4.5) to get the value of y_1, y_2 and y_3.

$$y_1 = \sqrt[3]{-\frac{(-12)}{2} + \sqrt{\left(\frac{-12}{2}\right)^2 + \frac{(-9)^3}{27}}} + \sqrt[3]{-\frac{(-12)}{2} - \sqrt{\left(\frac{-12}{2}\right)^2 + \frac{(-9)^3}{27}}}$$

$$= \sqrt[3]{6 + \sqrt{36 + (-27)}} + \sqrt[3]{6 - \sqrt{36 + (-27)}}$$

$$= \sqrt[3]{6 + \sqrt{9}} + \sqrt[3]{6 - \sqrt{9}}$$

$$= \sqrt[3]{6 + 3} + \sqrt[3]{6 - 3}$$

$$= \sqrt[3]{9} + \sqrt[3]{3}$$

Applying (4.2) by substituting $y_1 = \sqrt[3]{9} + \sqrt[3]{3}$ to get the value of x_1,

$$x_1 = \sqrt[3]{9} + \sqrt[3]{3} - \frac{(-3)}{3}$$

$$= \sqrt[3]{9} + \sqrt[3]{3} + 1$$

Now, find the value of y_2,

$$y_2 = \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\sqrt[3]{9}\right) + \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\sqrt[3]{3}\right)$$

Then, substituting the value of y_2 in (4.2) to get the value of x_2,

$$x_2 = \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\sqrt[3]{9}\right) + \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\sqrt[3]{3}\right) + 1$$

$$= -\frac{1}{2}\left(\sqrt[3]{9} + \sqrt[3]{3} - 2\right) + \frac{\sqrt{3}i}{2}\left(\sqrt[3]{9} - \sqrt[3]{3}\right)$$

Continue our calculation to get the value of y_3,

$$y_3 = \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\sqrt[3]{9}\right) + \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\sqrt[3]{3}\right)$$

Applying (4.2) by substituting the value of y_3 to get the value of x_3,

$$x_3 = \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\sqrt[3]{9}\right) + \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\sqrt[3]{3}\right) + 1$$

$$= -\frac{1}{2}\left(\sqrt[3]{9} + \sqrt[3]{3} - 2\right) - \frac{\sqrt{3}i}{2}\left(\sqrt[3]{9} - \sqrt[3]{3}\right)$$

Therefore, the solutions of $x^3 - 3x^2 - 6x - 4 = 0$ are:

$x_1 = \sqrt[3]{9} + \sqrt[3]{3} + 1$,

$x_2 = -\frac{1}{2}\left(\sqrt[3]{9} + \sqrt[3]{3} - 2\right) + \frac{\sqrt{3}i}{2}\left(\sqrt[3]{9} - \sqrt[3]{3}\right)$ and $x_3 = -\frac{1}{2}\left(\sqrt[3]{9} + \sqrt[3]{3} - 2\right) - \frac{\sqrt{3}i}{2}\left(\sqrt[3]{9} - \sqrt[3]{3}\right)$

4.2 Factorization (synthetic division)

Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form $x - c$, where c is a root of the polynomial. It can be used instead of the standard long division algorithm. This method reduces the polynomial and the linear factor into a set of numeric values. After these values are processed, the resulting set of numeric outputs is used to construct the polynomial quotient and the polynomial remainder. To see how it works, let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x + a_1 \tag{4.6}$$

By the normal process of division, the quotient of equation (4.6) with first coefficient b, say

$$b_n x^{n-1} + b_{n-1} x^{n-2} + b_{n-2} x^{n-3} + \ldots + b_2 x + b_1$$

Suppose that the remainder is R, then by definition of quotient and remainder, we have

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x + a_1$$

$$\equiv (x - c)(b_n x^{n-1} + b_{n-1} x^{n-2} + b_{n-2} x^{n-3} + \ldots + b_2 x + b_1) + R$$

$$\equiv b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_2 x^2 + b_1 x - c b_n x^{n-1} - c b_{x-1} x^{x-2} - c b_{n-2} x^{n-3} - \ldots$$
$$- c b_2 x - c b_1 + R$$

$$\equiv b_n x^n + (b_{n-1} - c b_n) x^{n-1} + (b_{n-2} - c b_{n-1}) x^{n-2} + \ldots + (b_1 - c b_2) x + R - c b_1$$

an identity, true for all values of x.

The coefficients of the various powers of x on both sides must be equal, and so,

$$a_n = b_n,\ a_{n-1} = b_{n-1} - c b_n,\ a_{n-2} = b_{n-2} - c b_{n-1},\ \ldots, a_2 = b_1 - c b_2,\ a_1 = R - c b_1$$

The coefficients b_n, b_{n-1}, b_{n-2},... and the remainder R may be calculated successively by the relations

$$b_n = a_n,\ b_{n-1} = a_{n-1} + c b_n,\ b_{n-2} = a_{n-2} + c b_{n-1}, \ldots,\ b_1 = a_2 + c b_2,\ R = a_1 + c b_1$$

The calculation can be effected rapidly by the following pattern,

	a_n	a_{n-1}	a_{n-2}	...	a_2	a_1
+		cb_n	cb_{n-1}	...	cb_2	cb_1
	b_n	b_{n-1}	b_{n-2}	...	b_1	R

According to the Factor Theorem, if $x - c$ is a factor, then the remainder of equation (4.6) is 0. So, the remainder $R = 0$.

Example 4.2.1

Solve $4x^3 + 4x^2 - 11x - 6 = 0$.

Solution:

Let $(x + 2)$ is one of the factor of equation. So, put $c = -2$

		4	4	-11	-6
c = -2,	+		-8	-8	6
		4	-4	-3	0

The bottom line of the synthetic division indicates that the given polynomial can now be factored as follows:

$$4x^3 + 4x^2 - 11x - 6 = 0$$

$$(x+2)(4x^2 - 4x - 3) = 0$$

Since the quadratic equation factors as

$$4x^2 - 4x - 3 = (2x - 3)(2x + 1)$$

the complete factorization of the given polynomial is