Proc

ETM 2

Experimental & Theoretical Mechanics

Sanur - Bali

March, 18-19, 2002
Grand Bali Beach Hotel

Sanur, Bali - INDONESIA

AUN/SEED-Net

SHAPE ANALYSIS OF TRUSS STRUCTURES UNDER THERMAL LOADING WITH CONSTRAINTS IN MEMBER STRAINS

K.K.Choong and Y.K.Chong
School of Civil Engineering, Universiti Sains Malaysia, Engineering Campus
14300 Nibong Tebal, Seberang Perai Selatan, Pulau Pinang, Malaysia
Phone: (604)-5937788 ext.6225, e-mail: cekkc@eng.usm.my

Abstract

Truss structures can experience thermal loading due to heating from external sources. Changes in temperature will induce stresses in the members and cause deformation of the structure. For long span truss structures, effect of such thermal loading might be especially critical in the sense that excessive deformation or failure due to over-stressing of members might occur. This paper is about shape analysis of truss structures subjected to thermal loading with constraints in member strains. Strain is chosen as the constraint because it can be easily and directly measured. By constraining the member strains to certain prescribed values, stresses caused by thermal loading can also be controlled. The governing equations for shape analysis is formulated by combining stiffness equation for truss structure and constraint equations in member strains. The existence condition of solution formulated with the use of generalized inverse matrix is adopted as the basis of analysis. Newton-Raphson method is used in the shape analysis process to obtain the shape satisfying the constraints. Results of analysis carried out on three truss examples show that the adopted solution strategy formulated with the use of generalized inverse is applicable and worthy of further development.

Key words: Generalized inverse, existence condition for solutions, shape analysis, constraints, strains, thermal loading.

1. FOREWORD

Thermal loading due to change in temperature can be considered as a kind of external loading in structural analysis. Stresses will be induced in the structure if free expansion of structural members is restrained. Effect of stresses as well as deformation due to thermal loading might be especially critical in large-span lightweight truss structures. Fig. 1 shows an example where members in large-span lightweight truss structures might experience different changes in temperatures.

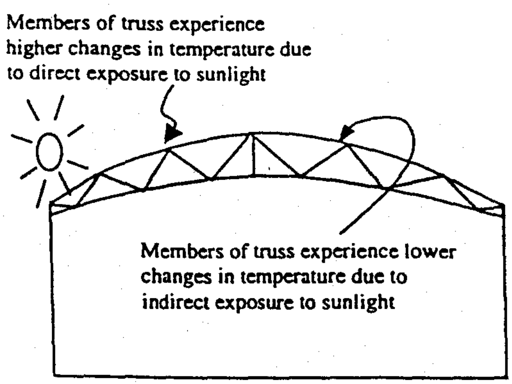

Fig.1 Change in temperature in a large-span roof structure due to uneven exposure to sunlight

Shape analysis constitutes an important step in the design process of truss structures. If x = a vector representing the shape of structures and g_i =i^{th} constraint function, then the process of shape analysis with constraints could be expressed as follows :

obtain x under the condition $g_i(x) \leq 0$, $i = 1, 2,..$ (1)

During shape analysis, topology of structure is kept unchanged with the shape of structure being the only design variable. In this research study, member strain has been chosen as the constraint due to the reason that strain could be easily measured. Overstressing or excessive deformation of truss structure under the effect of thermal loading could be avoided by limiting the strain to a suitable level.

Results of literature review has shown that shape analysis with constraints has been carried out by various researchers[1-5]. However, shape analysis of truss structrues under thermal loading with member strains as constraints has not yet been studied. Hence the present research work has been carried out with the objective of studying the applicability of a solution algorithm for shape analysis of truss structures under thermal loading with members strains as constraints by the use of generalized inverse.

This paper consists of four sections. Background to the research study is explained in section 1. Section 2 describes the basic governing equations as well as the solution algorithm. Results of three numerical examples are presented in section 3. Section 4 summarizes the work and outlines topics for future research.

2. BASIC FORMULATION

Shape analysis considered in this study involves deformation analysis of truss structures under thermal loading with constraints in member strains. Hence, two sets of equations which must be solved simultaneously are necessary : i. force-displacement and ii. strain-displacement relations. A brief description of the basic equations are given below. More detail explanation could be found in Ref.[7].

2.1 Governing equations

Finite element method is used in the formulation of force-displacement relation. Effect of temperature changes is treated as equivalent external forces during analysis. Let us assume that the numbers of members and degrees of freedom of a truss structure are m and n respectively. Using a two-node truss element, force(F)-displacement(U) relation could be expressed as follows :

$$F = KU \qquad \text{...(2)}$$

where $F=\Sigma\theta$, $K=\Sigma k$, θ: equivalent member nodal forces due to temperature change, K : structure stiffness matrix, k : member stiffness matrix, and Σ represents assembly process symbolically. k and θ are given by the following two equations respectively :

$$k = \frac{E_e A_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \qquad \text{...(3)}$$

$$\theta = E_e A_e \alpha \Delta T \begin{Bmatrix} -l \\ -m \\ l \\ m \end{Bmatrix} \qquad \text{...(4)}$$

where E_e, A_e, l_e : Young's modulus, cross-sectional area and length of member e respectively, l, m : directional cosine of member axis with respect to global x and y axes respectively, α: coefficient of thermal expansion and ΔT:change in temperature.

Relation between member axial strain and nodal displacements could be expressed as follows :

$$\varepsilon = \frac{1}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \text{...(5)}$$

where u_j, v_j : global x and y displacement of node j (j=1,2) respectively. Eq.(5) could be rewritten using matrix notation as follows :

$$\varepsilon = B_e u_e \qquad \text{...(6)}$$

Assuming that strains in $p(\leq m)$ members are constrained to the prescribed values, then the p constraint equations could be expressed as follows :

$$\varepsilon = BU \qquad \text{...(7)}$$

where ε : vector of member strains with size $p\times 1$, $B=\Sigma B_e$ and U : vector of structure nodal displacements.

Basic equations for shape analysis with constraint in member strains are obtained by combining Eq.(2) and (7) into the following augmented form :

$$\begin{bmatrix} F \\ \hline \varepsilon \end{bmatrix} = \begin{bmatrix} K \\ \hline B \end{bmatrix} U \qquad \text{...(8)}$$

Denoting

$$A = \begin{bmatrix} K \\ \hline B \end{bmatrix} \text{ and } b = \begin{bmatrix} F \\ \hline \varepsilon \end{bmatrix} \qquad \text{...(9)}$$

Eq.(8) could then be written in the following more compact form :

$$AU = b \qquad \text{...(10)}$$

Matrix A on the lefthand side of Eq.(10) is a rectangular matrix with size $(n+p)\times n$. Hence Eq.(10) could not be solved by using ordinary inverse matrix. As mentioned earlier, shape analysis is carried out using shape of structure x as design variables with all other parameters kept constant. Thus matrix A is a function of x, i.e. $A=A(x)$. Since the shape of structure satisfying the prescribed member strains is to be determined and remains unknown in the beginning of analysis, Eq.(10) will not be satisfied in general. An initial shape has to be assumed and corrected by means of iterative calculation until the required shape is obtained.

2.2 Solution algorithm

Condition for the existing of solution for Eq.(10) is used as the basis of shape analysis. This condition could be expressed using generalized inverse as follows :

$$(I_f - AA^{+})\, b = 0 \qquad \text{...(11)}$$

where I_j: identity matrix of size $j\times j$ and A^+ : Moore-Penrose generalized inverse[1] for A. For size A of $(n+p)\times n$, A^+ will be of size $n\times(n+p)$. Here, generalized inverse[6] is adopted due to the reason that A is not a square matrix. Since A is a function of shape of structure x, then the task here now is to find x such that Eq.(11) is satisfied. Since x is not known beforehand, iterative calculation needs to be carried out. Here, Newton-Raphson iterative scheme is adopted.

The lefthand side of Eq.(11) is first denoted as follows :

$$g(x)= (I_j-AA^+)b \qquad ...(12)$$

Using Taylor series expansion and by retaining only the linear term, $g(x)$ could be written in the following form for the purpose of iterative calculation :

$$g(x_{i+1}) = g(x_i) + \nabla g(x_i)(x_{i+1} - x_i) \qquad ...(13)$$

where

$$[\nabla g(x_i)]_{jk} = \frac{\partial g_k(x_i)}{\partial x_j} \qquad ...(14)$$

$\nabla g(x_n)$ is the Jacobian matrix with size $(n+p)\times q$ where q : number of design variables and subscript in vector x represents iterative step. Assuming that the required shape is obtained at $(i+1)^{th}$ iterative step, then the correction to x vector could be obtained from Eq.(13) as follows :

$$\Delta x_i = -[\nabla g(x_i)]^+ g(x_i) \qquad ...(15)$$

and

$$x_{i+1} = x_i + \Delta x_i \qquad ...(16)$$

The evaluation of Jacobian matrix $\nabla g(x_n)$ involves the calculation of differentiation of both A and its generalized inverse A^+. Remembering that $g(x)$ is given by Eq.(12) and carrying out the differentiation with respect to x, it can be shown that the Jacobian matrix could be written as follows :

$$\frac{\partial g(x)}{\partial x} = \frac{\partial\{A(x)A^+(x)\}}{\partial x} b(x) + \left[A(x)A^+(x) - I_j\right]\frac{\partial b(x)}{\partial x} \qquad ...(17)$$

Partial differentiation appearing in the first term on the righthand side of Eq.(17) could be evaluated using the following expression :

$$\frac{\partial\{A(x)A^+(x)\}}{\partial x} = \left[I_j - A(x)A^+(x)\right]\frac{\partial A(x)}{\partial x}A^+(x) + \frac{\partial A(x)}{\partial x}A^+(x)\left[I_j - A(x)A^+(x)\right] \qquad ...(18)$$

2.3 Summary of the solution process

(a) An initial shape x_0 is assumed.
(b) Compute K and B in Eq.(8).
(c) Constraints on member strains are prescribed(ε in Eq.(8)).
(d) From the given change in temperature patterns, compute F in Eq.(8).
(e) Form matrix A and vector b in Eq.(9).
(f) Evaluate Jacobian matrix $\nabla g(x_n)$ by using Eq.(17) and (18).
(g) Evaluate $g(x_i)$ and solve Eq.(15) for Δx_i.
(h) Update the shape of structure using Eq.(16).
(i) Repeat the steps (a) to (h) until the required shape is obtained. Replace x_0 in step (a) with x_i for $i\geq 1$.

Criteria for convergence of solution is required in step no. (i) above. In this research study, the following convergent criteria has been adopted :

$$\left[\frac{\varepsilon_c}{\|\varepsilon_c\|}\cdot\frac{\varepsilon_t}{\|\varepsilon_t\|} - 1.0\right] \leq \xi \qquad ...(19)$$

where ε_c : vector of member strains at the end of each iteration, ε_t : vector of prescribed member strains and ξ : specified convergent tolerance. A value of 1×10^{-5} has been chosen for ξ.

3. NUMERICAL EXAMPLES

Three plane truss structures, with Young's modulus E=200 x 10^9 N/m^2, member cross-sectional area A=0.05 m^2 and coefficient of thermal expansion α=12 x 10^{-6} / ^{0}C , have been analysed. In each of the analysis, a target shape x_{tar} is first identified. The truss structure with x_{tar} is then analysed subjected to the given patterns of temperature change. The resulting member strains ε_c are then adopted as the constraints. Shape analysis is then started with an initial shape x_0 in which the joint coordinates are deviated uniformly from x_{tar} by three different levels of percentage : 1%, 3% and 5%. Iterative calculation is then carried out until either the convergent criteria is satisfied or maximum number of iteration specified is exceeded.

3.1 Numerical example 1 : a three-member plane truss

Fig.2 shows a three-member plane truss which has been adopted as the first example in which ΔT_1=10°C and ΔT_2=20°C. The target shape and prescribed member strains are tabulated in Table 1 and 2 respectively.

[1] See Appendix *A*

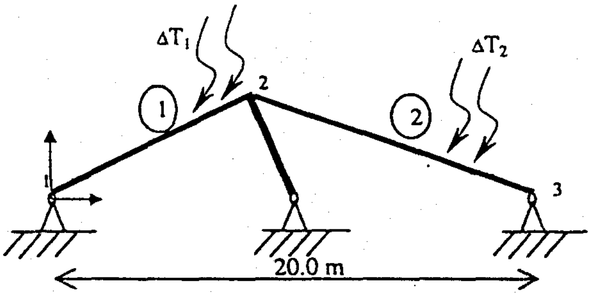

Fig.2 Numerical example 1 : a three-member truss

Table 1 Joint coordintes of the target shape for numerical example 1

Node	Coordinates (x,y)
1	(0,0)
2	(8,3)
3	(20,0)
4	(10,0)

Table 2 Prescribed member strains for numerical example 1

Member	Prescribed member strain
1	-23.72170×10^{-6}
2	31.93136×10^{-6}
3	121.30050×10^{-6}

Results of analysis are shown in Table 3 and 4. Ite. In Table 3 and hereafter denotes number of iteration required to achieve convergence. Percentage deviation in both Table 3, 4 and hereafter denotes amount with which initial shape is deviated from target shape.

Table 3 Results of shape analysis for numerical example 1 : coordinates of joint 2

Percentage deviation	Initial shape	Converged shape	Target shape
1.0	X=8.08	X=8.000	X=8.00
(ite.=1)	Y=3.03	Y=3.001	Y=3.00
3.0	X=8.24	X=8.001	X=8.00
(ite.=1)	Y=3.09	Y=3.012	Y=3.00
5.0	X=8.40	X=8.002	X=8.00
(ite.=1)	Y=3.15	Y=3.034	Y=3.00

Table 4 Results of shape analysis for numerical example 1 : member strains

Percentage deviation	Member	Member strains in converged shape
1.0	1	-23.72×10^{-6} (-23.722×10^{-6})
	2	31.95×10^{-6} (31.9312×10^{-6})
	3	121.3×10^{-6} (121.301×10^{-6})
3.0	1	-23.68×10^{-6} (-23.722×10^{-6})
	2	32.06×10^{-6} (31.9312×10^{-6})
	3	121.5×10^{-6} (121.301×10^{-6})
5.0	1	-23.63×10^{-6} (-23.722×10^{-6})
	2	32.31×10^{-6} (31.9312×10^{-6})
	3	121.9×10^{-6} (121.301×10^{-6})

(Figures in parenthesis under the column member strains in converged shape represent target values)

3.2 Numerical example 2 : a five-member truss

Fig.3 shows a five-member plane truss which has been analysed as the second example in which ΔT_1=10°C and ΔT_2=20°C. Coordinates of both joints 2 and 3 are used as design variables.

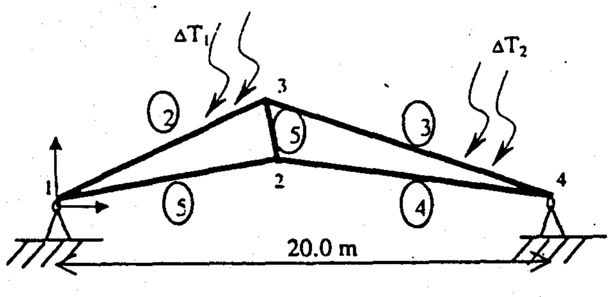

Fig.3 Numerical example 2 : a five-member truss

The target shape and prescribed member strains are tabulated in Table 5 and 6 respectively.

Table 5 Joint coordinates of target shape for numerical example 2

Node	Coordinates : (x,y)
1	(0,0)
2	(8,1)
3	(7,2)
4	(20,0)

Table 6 The constraints : prescribed member strains, for numerical example 2

Member	Prescribed member strain
1	74.59469×10^{-6}
2	86.32100×10^{-6}
3	190.21551×10^{-6}
4	91.15596×10^{-6}
5	23.79050×10^{-6}

Results of analysis showing joint coordinates and member strains in converged shape are shown in Table 7 and 8 respectively.

Table 7 Results of shape analysis for numerical example 2 : coordinates of joints 2 and 3

Percentage deviation	Node	Initial shape	Converged shape
1.0 (ite.=1)	2	X=8.04	X=8.004(8.0)
		Y=1.01	Y=1.001(1.0)
	3	X=7.07	X=7.004(7.0)
		Y=2.02	Y=2.003(2.0)
3.0 (ite.=1)	2	X=8.24	X=8.000(8.0)
		Y=1.03	Y=1.000(1.0)
	3	X=7.21	X=6.999(7.0)
		Y=2.06	Y=2.001(2.0)
5.0 (ite.=1)	2	X=8.40	X=7.999(8.0)
		Y=1.05	Y=1.001(1.0)
	3	X=7.35	X=6.998(7.0)
		Y=2.10	Y=2.002(2.0)

(Figures in parenthesis under the column converged shape represent target values)

Table 8 Results of shape analysis for numerical example 2 : member strains

Percentage deviation	Member	Member strains in converged shape
1.0	1	74.59×10^{-6} (74.595×10^{-6})
	2	86.32×10^{-6} (86.321×10^{-6})
	3	190.2×10^{-6} (190.216×10^{-6})
	4	91.15×10^{-6} (91.156×10^{-6})
	5	23.81×10^{-6} (23.791×10^{-6})
3.0	1	7.459×10^{-6} (74.595×10^{-6})
	2	86.32×10^{-6} (86.321×10^{-6})
	3	190.2×10^{-6} (190.216×10^{-6})
	4	91.17×10^{-6} (91.156×10^{-6})
	5	23.80×10^{-6} (23.791×10^{-6})
5.0	1	74.59×10^{-6} (74.595×10^{-6})
	2	86.32×10^{-6} (86.321×10^{-6})
	3	190.2×10^{-6} (190.216×10^{-6})
	4	91.19×10^{-6} (91.156×10^{-6})
	5	23.83×10^{-6} (23.791×10^{-6})

(Figures in parenthesis under the column member strains in converged shape represent target values)

3.3 Numerical example 3 : a 11-member plane truss

Fig.4 shows a 11-member plane truss which has been analysed as the third example. In this example, only coordinates of nodes 3 and 5 are allowed to change. Coordinates of nodes 2, 4 and 6 remain fixed during the analysis.

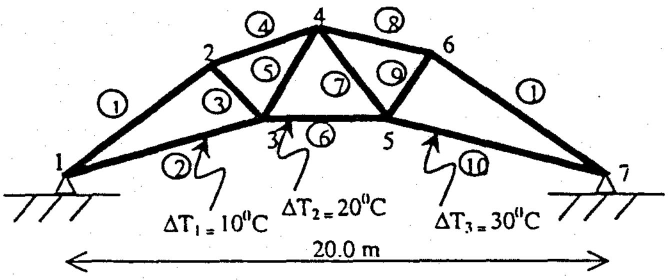

Fig.4 Numerical example 3 : a 11-member truss

Joint coordinates and prescribed member strains are given in Table 9 and 10 respectively.

Table 9 Joint coordinates of target shape for numerical example 3

Node	Coordinates : (x,y)
1	(0,0)
2	(4,4)
3	(6,3)
4	(10,5)
5	(14,3)
6	(16,4)
7	(20,0)

Table 10 Prescribed member strains for numerical example 3

Member	Prescribed member strains
1	130.73110×10^{-6}
2	262.42893×10^{-6}
3	-129.1900×10^{-6}
4	210.86089×10^{-6}
5	-94.20543×10^{-6}
6	58.66208×10^{-6}
7	-44.13117×10^{-6}
8	165.45533×10^{-6}
9	-101.37098×10^{-6}
10	214.49785×10^{-6}
11	71.88832×10^{-6}

The results of analysis showing the joint coordinates and member strains of the converged shape are shown in Table 11 and 12 respectively. In Table 12, only the results correspond to percentage deviation 5% are listed.

Table 11 Results of analysis for numerical example 3 : joint coordinates of converged shape

Percentage deviation	Node	Initial shape	Converged shape
1.0 (ite.=2)	3	6.06	6.000(6.0)
		3.03	3.000(3.0)
	5	14.14	14.00(14.0)
		3.03	3.000(3.0)
3.0 (ite.=3)	3	6.18	6.000(6.0)
		3.09	3.000(3.0)
	5	14.42	14.000(14.0)
		3.09	3.000(3.0)
5.0 (ite.=4)	3	6.3	6.000(6.0)
		3.15	3.000(3.0)
	5	14.7	14.000(14.0)
		3.15	3.000(3.0)

(Figures in parenthesis under the column converged shape represent target values)

Table 12 Results of analysis for numerical example 3 : member strains in converged shape

Percentage deviation	Member	Member strains in converged shape
5.0	1	130.8×10^{-6} (130.731×10^{-6})
	2	262.5×10^{-6} (262.429×10^{-6})
	3	-129.2×10^{-6} (-129.190×10^{-6})
	4	210.9×10^{-6} (210.861×10^{-6})
	5	-94.25×10^{-6} (-94.205×10^{-6})

	6	58.65×10^{-6} (58.662×10^{-6})
	7	-44.11×10^{-6} (-44.131×10^{-6})
	8	165.5×10^{-6} (165.455×10^{-6})
	9	-101.4×10^{-6} (-101.371×10^{-6})
	10	214.5×10^{-6} (214.498×10^{-6})
	11	71.89×10^{-6} (71.888×10^{-6})

(Figures in parenthesis under the column member strains in converged shape represent target values)

3.4 Discussions

From the results presented, it can be seen that convergence has been achieved after 1 iteration for numerical example 1, 2(Table 3 and 7) and after 2-4 iterations(Table 11) for numerical example 3. Comparison between joint coordinates and member strains of converged shape with the corresponding values of target shape for all three numerical examples have shown that the solution algorithm yield results with high accuracy (Table 3,4,7,8,11,12).

4. CONCLUSIONS

Shape analysis of truss structures under thermal loading with constraints in member strains has been studied. A solution algorithm involving the use of generalized inverse and Newton-Raphson iteration scheme has been adopted and its applicability investigated. Results obtained from analysis carried out on three simple plane truss structures have shown that the solution algorithm adopted yield solutions with sufficient accuracy. Based on the above, it can be concluded that the adopted solution algorithm is applicable to shape analysis problems and worthy of further development.

Among areas that need further research works are as follows :

(a) Verification of applicability of solution algorithm to problems with higher degrees of freedom including 3D problems.

(b) Investigation on sensitivity of final converged shape with respect to initial assumed shape.

(c) Potential extension of the solution algorithm to problems involving shape control of truss structures with member strains as constraints.

REFERENCES

[1] F. L. Guan, "Shape analysis with constraint in displacement mode", *Research Report on Spatial Structures*, Institute of Industrial Science, Universitiy of Tokyo, Japan and Technical Research Centre of Takenaka Corporation, Japan(in Japanese), 1990.

[2] Hangai, Y. dan Kawaguchi, K., "Analysis for shape-finding of unstable link structures in the unstable state", *Proc. of the International Colloquium on Space Structures for Sports Buildings*, Beijing, edited by T. T. Lan dan Y. Zhilian, Science Press and Elsevier Applied Science, page 104-111, 1987.

[3] Tanaka, H. dan Hangai, Y., "Rigid body displacement and stabilization conditions of unstable truss structures, shells, membranes and space frames", *Proc. of IASS Symposium*, Osaka, Volume 2, edited by K. Heki, Elsevier Science Publishers, 1988.

[4] Yasuhiko Hangai, "Shape analysis of structures", Theoretical and Applied Mechanics, Volume 39, *Proc. of 39th Japan National Congress for Applied Mechanics*, edited by JNCTAM, Science Council of Japan,University of Tokyo Press, pg 11-28, 1990.

[5] Yasuhiko Hangai, "Shape analysis of structures", *Collection of Technical Papers for UM Seminar on Shell and Spatial Structures 1997 – Research and Practice in the field of Shell and Spatial Strucures* – ed. KK Choong and Hashim Abdul Razak, Universiti Malaya, Kuala Lumpur, 1997.

[6] Yasuhiko Hangai, Kenichi Kawaguchi, *Keitai Kaiseki (Shape Analysis)*, Baifukan (in Japanese), 1991.

[7] Y.K.Chong, *Analisis Bentuk Struktur Kekuda Terhadap Beban Suhu Dengan Kekangan Dalam Terikan Anggota Dengan Menggunakan Songsangan Teritlak*, MSc disertation, School of Civil Engineering, Universiti Sains Malaysia, 2001.

APPENDIX A

Let A be a matrix of size $m\times n$ ($m\neq n$ in general). A matrix, denoted as A^{+} with size $n\times m$, which satisfies the following four conditions is called the Moore-Penrose generalized inverse for A :

$$AA^{+}A = A \qquad \text{...(A1)}$$
$$A^{+}AA^{+} = A+ \qquad \text{... (A2)}$$
$$(AA^{+})^{T} = AA+ \qquad \text{...(A3)}$$
$$(AA^{+})^{T} = (AA^{+}) \qquad \text{...(A4)}$$

Generalized inverse exists for all matrix including singular matrix.