

**SATU KAJIAN KEATAS CIRI-CIRI NORMA VEKTOR DAN MatriKS  
DALAM ANALISIS PENGOPTIMUMAN BENTUK**

*A STUDY ABOUT THE CHARACTERISTIC OF VECTOR AND MATRIX NORMS  
IN SHAPE OPTIMIZATION ANALYSIS*

Oleh

THEN SIN EAU

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**SARJANA SAINS KEJURUTERAAN (STRUKTUR)**

## ABSTRAK

Rekabentuk struktur 'optimum' atau 'terbaik' adalah berdasarkan kepada objektif seperti berat struktur, kos pembinaan, keindahan, kepercayaan atau gabungan antara mereka. Objektif ini telah membawa kepada enam kategori kajian yang luas iaitu kriteria kaedah pengoptimuman, pengoptimuman bentuk, teknik pemrograman matematik, pengoptimuman jenis kebolehppercayaan, pengoptimuman jenis pelbagai objektif, pembangunan pemrograman matematik dan pengoptimuman struktur secara pratik yang melibatkan perisian yang banyak.

Kajian disertasi ini menumpukan kepada pengoptimuman bentuk dengan objektif mengkaji ciri-ciri norma vektor dan norma matriks dalam analisis pengoptimuman bentuk. Oleh kerana dalam pengoptimuman bentuk struktur pelbagai darjah kebebasan dengan pemaksimuman kekukuhan atau peminimuman anjakan sebagai fungsi objektif, di mana kekukuhan adalah dalam bentuk matriks dan anjakan adalah dalam bentuk vektor, maka timbulnya keperluan norma vektor dan matriks untuk tujuan mewakili saiz vektor dan matriks masing-masing. Dan ini telah membawa kepada kepentingan objektif kajian disertasi ini. Dalam kajian ini, tiga jenis contoh berangka telah dianalisis. Contoh berangka pertama dipilih dengan struktur kekuda ringkas yang mempunyai satu darjah kebebasan, manakala dua contoh berangka yang lain mempunyai dua darjah kebebasan.

Daripada keputusan yang diperolehi, didapati norma vektor untuk anjakan dapat memberikan bentuk optimum kekukuhan dengan lebih baik daripada norma matriks untuk kedua-dua matriks kekukuhan dan songsangan matriks kekukuhan. Antara tiga jenis norma vektor yang dikaji, didapati vektor norma *Euclidean* (atau norma- $l_2$ ) dan

norma- $l_\infty$  adalah lebih baik daripada norma- $l_1$  untuk mewakili struktur optimum. Antara empat jenis norma matriks yang dikaji pula, matriks norma *Euclidean* (atau norma  $-l_2$ ) dan norma *spektral* adalah lebih baik daripada kedua-dua norma jumlah baris maksimum (atau norma jumlah lajur maksimum akibat kesimetrian penjuru matriks kekukuhan) dan norma  $-l_1$ . Norma matriks *Euclidean* dan *spektral* tersebut juga menjelaskan bahawa norma matriks untuk songsangan matriks kekukuhan adalah lebih baik untuk mewakili kekukuhan struktur daripada norma matriks untuk matriks kekukuhan.

# **PENENTUAN RAGAM TEGASAN KESEIMBANGAN DIRI UNTUK STRUKTUR BERDASARKAN KONSEP RODA BASIKAL**

Tan Teck Chai  
2001/2002

## **Abstrak**

Tujuan projek tahun akhir ini ialah untuk menganalisis penentuan ragam tegasan struktur yang berkonsepkan roda basikal yang berada dalam keseimbangan diri. Program komputer *TCSELF* telah digunakan dalam analisis struktur tersebut. Laporan ini mengandungi lapan Bab, ianya merangkumi pengenalan kepada struktur berkonsepkan roda basikal, teori kepada penyelesaian matriks songsang 'Moore Penrose' teritlak yang digunakan pada program komputer *TCSELF*, pengenalan kepada program komputer *TCSELF* dan langkah-langkah menggunakan program tersebut dan seterusnya analisis dijalankan pada 11 model dari keadaan bukaan simetri di pertengahan struktur kepada tidak simetri secara langsung. Laporan ini juga merangkumi rekabentuk dan pembinaan sebuah struktur berbentuk oktagon yang berkonsepkan roda basikal. Seterusnya perbincangan ke atas keputusan analisis akan disertakan. Laporan ini diakhiri dengan cadangan-cadangan pembaikan program pada masa akan datang. Senarai rujukan dan lampiran keputusan analisis program komputer *TCSELF* akan terletak pada akhir laporan ini.

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**FIFTH INTERNATIONAL CONFERENCE ON  
SPACE STRUCTURES**

19-21 August 2002  
University of Surrey, Guildford, Surrey, UK



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**A Study about Characteristics of Vector and  
Matrix Norms in Shape Optimization Analysis**

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# A Study about Characteristics of Vector and Matrix Norms in Shape Optimization Analysis

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## ABSTRACT

The load resisting capabilities of spatial structures are influenced to a great extent by their shapes. When dealing with load resisting capability, two important points of against what and what kind of resisting capacity are required need to be taken into consideration. The question of what kind of resisting capability generally involves the measurement of maxima or minima of a certain quantity. When an  $n$  degrees of freedom problem is considered, such quantity involved is in general a vector or matrix quantity. In order to measure the 'maxima' or 'minima' of a vector or matrix quantity, norm is normally used. For 'maximum stiffness problem' with shape as the design variable, use of different type of norm might yield different result. This research has been carried out with the objective of investigating the characteristics of three different vector norms and five different matrix norms by studying the 'shapes' obtained from analysis carried out on three simple truss structure. From this preliminary research study, it has been found that vector norm is more suitable than matrix norm for use in 'maximum stiffness problem'. Of the three vector studied, Euclidean norm is the most suitable to be used.

## INTRODUCTION

Spatial structures such as shells and space frames belong to the category of 'shape-resistant type structures'. Load resisting capacities of such kind of structures against external disturbance are influenced to a great extent by their shapes. Hence results obtained from research works on 'optimized shapes' provide a kind of guideline for the selection of shape during a design process.

The notion of 'shape-resistant' could be expressed mathematically as follows :

$$\text{Resisting capability} = F(\text{shape}) \quad \dots(1)$$

If resisting capability and shape of structure are denoted as  $R$  and  $x$ , respectively, then Eq.(1) could be rewritten as follows :

$$R = F(x) \quad \dots(2)$$

Two important points in the treatment of resisting capability are the questions of resistance against what and what kind of resistance are required. If the first point of against what is represented using  $f$ , then Eq.(2) above will become as follows :

$$R = F(x, f) \quad \dots(3)$$

Let's consider an illustrative example of 'maximum stiffness problem'. In the stress analysis of a structure, the following equation is used :

$$f = Kd \quad \dots(4)$$

where  $K$ ,  $d$  and  $f$  are stiffness matrix, displacement and external load vectors, respectively. Since  $K$  is dependent on shape, hence  $K=K(x)$ . The basic equation to be used in the analysis of 'maximum stiffness problem' is as follows :

$$d = K^{-1} f \quad \dots(5)$$

If comparison between Eqs.(3) and 5 are made, it can be seen that  $R$  is substituted by  $d$  which shows that the content of resisting capability  $R$  is equivalent to displacement  $d$ . This resisting capability  $d$  is measured against prescribed external load vector  $f$ .

The second important point of what kind of resistance will now be formulated. With reference to Eq.(5), the question of what kind of resistance corresponds to maximum stiffness kind of resistance in maximum stiffness problem. In the case of a single degree of freedom(d.o.f.) problem, Eq.(5) will become  $d=K^{-1}(x) f$  and the maximum stiffness problem could be stated as follows : for a prescribed  $f$ , [ find  $x$  which maximizes  $K$  ] or equivalently [ find  $x$  which minimizes  $d$  ]. If the preceding statements are extended to the case of a multiple d.o.f. problem, then they will become [ find  $x$  which maximizes  $K$  ] or equivalently [ find  $x$  which minimizes  $d$  ], respectively. The preceding statements show that definition of maximum  $K$  or minimum  $d$  is necessary. Mathematically, 'size' of vectors and matrices could be measured using the quantity called 'norm'. Since types of norm that exist in the literature are numerous[Ref.1], the question of which norm should be used arises.

Considering the above facts, it can be said that 'maximum stiffness problems' are problems with wide ranges meaning that characteristics of norms could be expressed in terms of shapes obtained in shape optimization analysis[Ref.2]. Such information about the characteristics of norms will provide a useful guideline on the suitability of each different type of norm to be used in shape optimization analysis.

Based on the above reason and the fact that no previous study has been attempted to date, the present research has been carried out with the objective of studying the characteristic of different types of vector and matrix norms which exist in the literature[Ref.3]. The information obtained from this preliminary step could pave the way for more detail studies into understanding the characteristic of norms and their suitabilities in shape optimization analysis.

This paper consists of five sections. Section One introduces the background to the study as well as the research objective. This is then followed by Section Two which describes the basic equations used in the study. Section Three shows the numerical examples analysed. Discussion to the numerical results obtained are given in Section Four. Section Five concludes the paper.

## BASIC EQUATIONS

The type of problem treated in this study is shape optimization with maximization of stiffness as the optimization criteria. As mentioned in the previous section, definition of maxima or minima of vector and matrix quantities are necessary in the analysis of maximum stiffness

problem. In this section, the basic equations used in this study will be first described. This is then followed by explanation on the types of norm used in this study.

### Formulation for maximum stiffness problem

As mentioned in the previous section, the basic equation for maximum stiffness problem for a multiple d.o.f. structure is given by the following equations :

$$d = K^{-1}(x) f \quad \dots(6)$$

and maximum stiffness problem itself could be represented as follows : for a prescribed external load vector  $f$ ,

[ find the shape  $x$  which maximizes stiffness matrix  $K$  ] or equivalently  
[ find the shape  $x$  which minimizes displacement vector  $d$  ].

Since  $d$  and  $K$  are vector and matrix, respectively, the quantity called 'norm' is necessary in the evaluation of their 'size'.

### Norms

Definition of vector and matrix norms are given as follows, respectively :

vector norm :

$$\|a\| \geq 0, \|a\| = 0 \rightarrow a = \theta, \quad \|\alpha a\| = |\alpha| \|a\|, \quad \|a + b\| \leq \|a\| + \|b\| \quad \dots(7)$$

matrix norm :

$$\|A\| \geq 0, \|A\| = 0 \rightarrow A = \theta, \quad \|\alpha A\| = |\alpha| \|A\|, \quad \|A + B\| \leq \|A\| + \|B\| \quad \dots(8)$$

$$\|AB\| \leq \|A\| \|B\| \quad (\text{multiplicative norm})$$

where  $a$  : arbitrary vector of size  $n$ ;  $\alpha$  : arbitrary scalar;  $A, B$  : arbitrary matrix of size  $m \times n$  and  $\theta$  : null vector (or matrix).

Three vector norms which satisfy Eq.(7) and studied in this paper are given as follows :

$$\|a\|_1 = \sum_{i=1}^n |a_i| \quad : \text{absolute value norm}$$

$$\|a\|_2 = \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2} \quad : \text{Euclidean norm} \quad \dots(9)$$

$$\|a\|_\infty = \max(|a_i|) \quad : \text{maximum value norm}$$

For the case of matrix norm, five matrix norms that satisfy Eq.(8) and studied in this paper are given as follows :

$$\|A\|_M = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| \quad : \text{absolute value norm}$$

$$\|A\|_R = \max \left( \sum_{j=1}^n |a_{ij}| \right) \quad : \text{maximum row norm}$$

$$\|A\|_c = \max \left( \sum_{i=1}^m |a_{ij}| \right) \quad : \text{maximum column norm} \quad \dots(10)$$

$$\|A\|_E = \left( \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \quad : \text{Euclidean norm}$$

$$\|A\|_S = \sqrt{\mu_{\max}} \quad : \text{spectral norm}$$

where  $\mu_{\max}$  is maximum eigenvalue of matrix product  $A^T A$ .

### NUMERICAL EXAMPLES

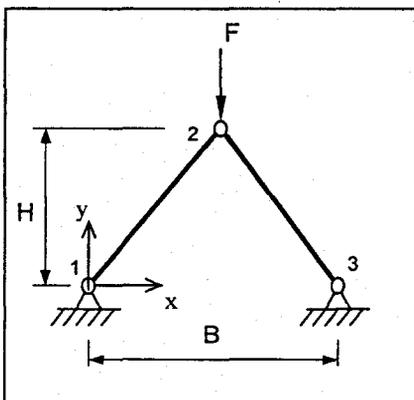
Three numerical examples have been analysed. The first two examples are 2-d.o.f. problems and the third one is a 6-d.o.f. problem. Procedures of analysis are as follows :

- Assume initial shape  $x_{ini}$  and obtain stiffness matrix  $K$  and its inverse  $K^{-1}$ .
- Obtain displacement vector by solving the basic equation  $d=K^{-1}f$ .
- Compute norm of vector  $d$ (Eq.(9)) and norms of matrix  $K$  and  $K^{-1}$ (Eq.(10)).
- Assume new shape  $x$  and obtain corresponding stiffness matrix  $K$  and  $K^{-1}$ .
- Repeat step (b) to (d) over a predetermined limit of shape  $x$ .

Values of vector and matrix norms obtained from the above calculation are then plotted against shape  $x$  in order to identify shape corresponding to maxima or minima on the plots. It is assumed that minima on plots of  $\|d\|$  versus  $x$  correspond to shape with maximum stiffness. As for plots of  $\|K\|$  and  $\|K^{-1}\|$  versus  $x$ , it is first assumed that maxima and minima on plots correspond to shape with maximum stiffness, respectively. This latter assumption is a simple extension from relation between  $K$  and  $K^{-1}$  in a single d.o.f. problem. In the figures shown in the following sections, the following symbols have been used to denote the three different vector and five different matrix norms studied :

- vector norm :  $\text{norm}(l,inf)$ =maximum value norm,  $\text{norm}(l,1)$ = absolute value norm and  $\text{norm}(l,2)$ =Euclidean norm
- matrix norm :  $\text{norm}(A,1)$ =maximum row norm(=maximum column norm since  $K^T=K$ ),  $\text{norm}(A,2)$ =spectral norm,  $\text{norm}(L,1)$ =absolute value norm,  $\text{norm}(L,2)$ =Euclidean norm

### Numerical example 1



Numerical example 1 is a simple 2 d.o.f. truss structure with two members as shown in Fig.1. The example is loaded by a vertically downward load of  $F=10$  kN. Data of analysis for the example are as follows :  $EA=160 \times 10^3$  kN,  $B=400$  cm and  $H=200$  cm.  $X$ -coordinate of node 2 of the example has been selected as the design variable and assigned values ranging from  $x=0$  cm to  $x=400$  cm. The results of analysis are as shown in Fig.2 to 4.

Fig.1 Numerical example 1

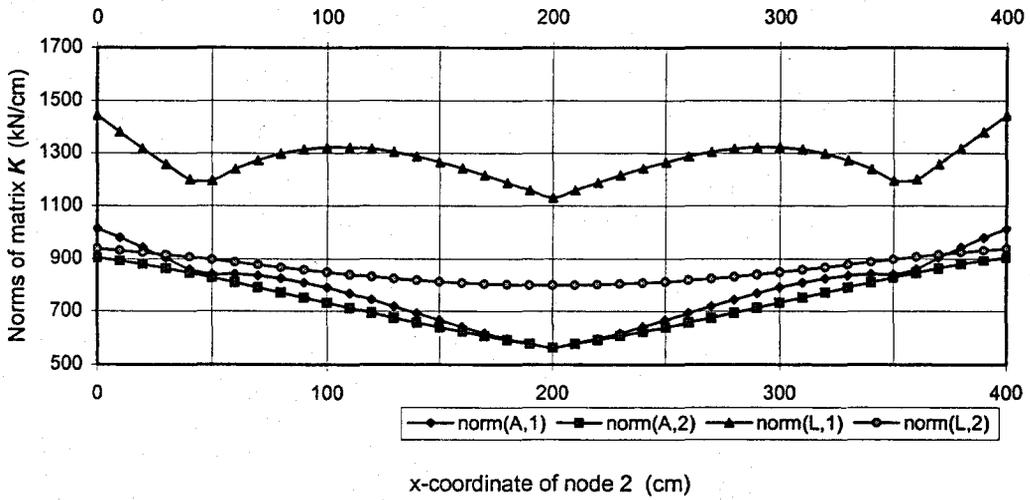


Fig.2 Variation of norm of  $K$  versus x-coordinate of node 2 ( $H/B=0.5$ )

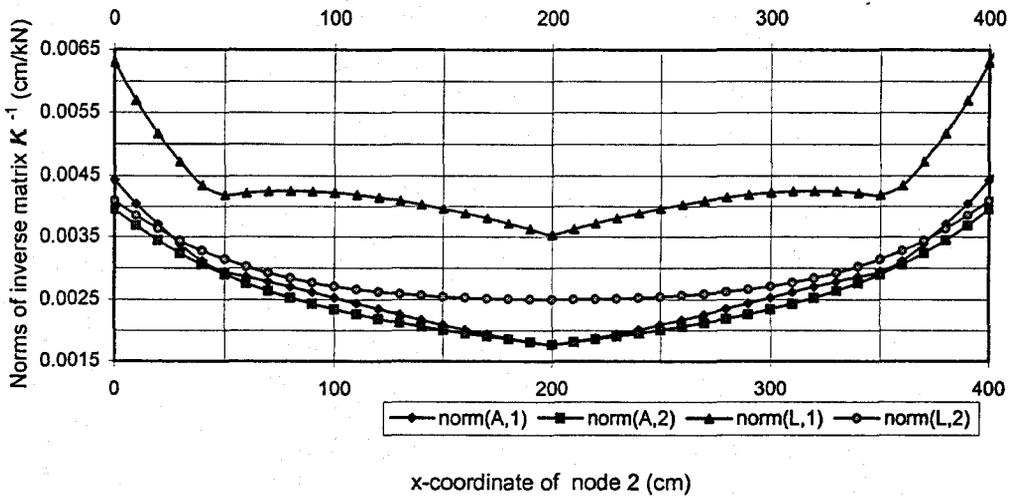


Fig.3 Variation of norm of inverse matrix  $K^{-1}$  versus x-coordinate of node 2 ( $H/B=0.5$ )

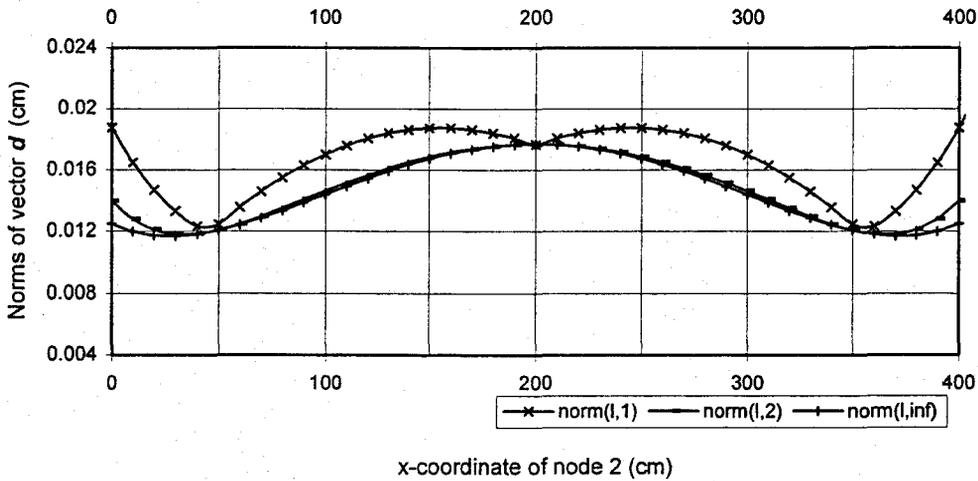
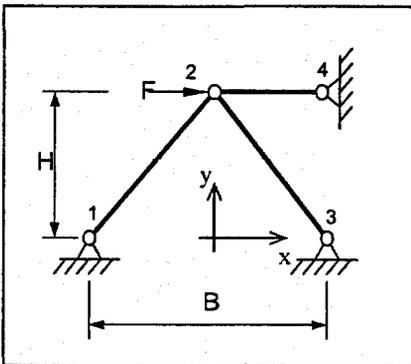


Fig.4 Variation of norm of vector  $d$  versus x-coordinate of node 2 ( $H/B=0.5$ )

**Numerical example 2**



Numerical example 2 is also a simple 2 d.o.f. truss structure with three members as shown in Fig.5. The example is loaded by a horizontal load of  $F=10$  kN. Data of analysis for the example are as follows :  $EA=160 \times 10^3$  kN,  $B=400$  cm and  $H=200$  cm. Y-coordinate of node 4 of the example has been selected as the design variable and assigned values ranging from  $y=170$  cm to  $y=230$  cm. The results of analysis are as shown in Fig.6 to 8.

Fig.5 Numerical example 2

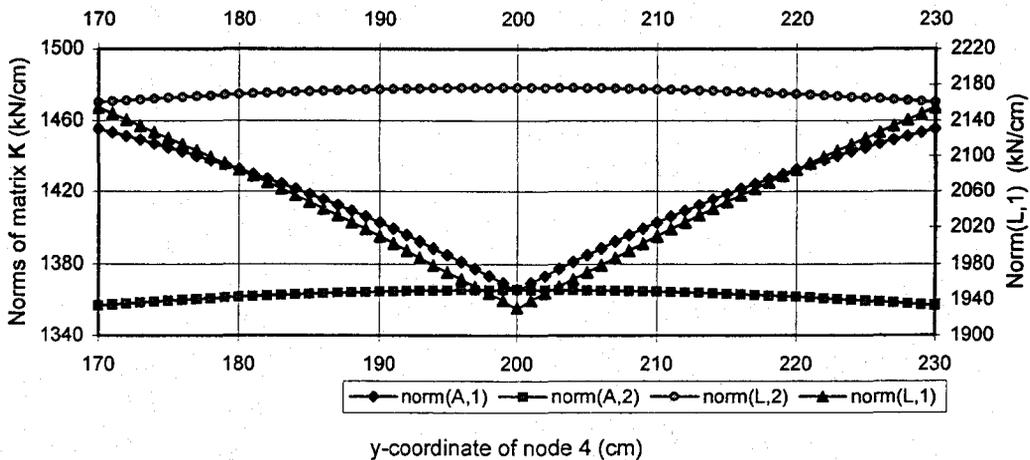


Fig.6 Variation of norm of matrix  $K$  versus y-coordinate of node 4 ( $H/B=0.5$ )

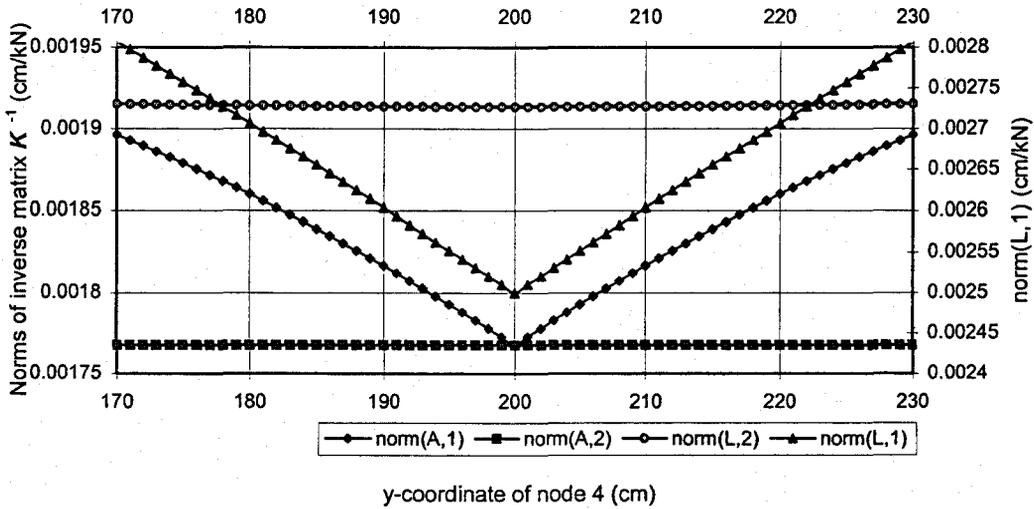


Fig.7 Variation of norm of inverse matrix  $K^{-1}$  versus y-coordinate of node 4 ( $H/B=0.5$ )

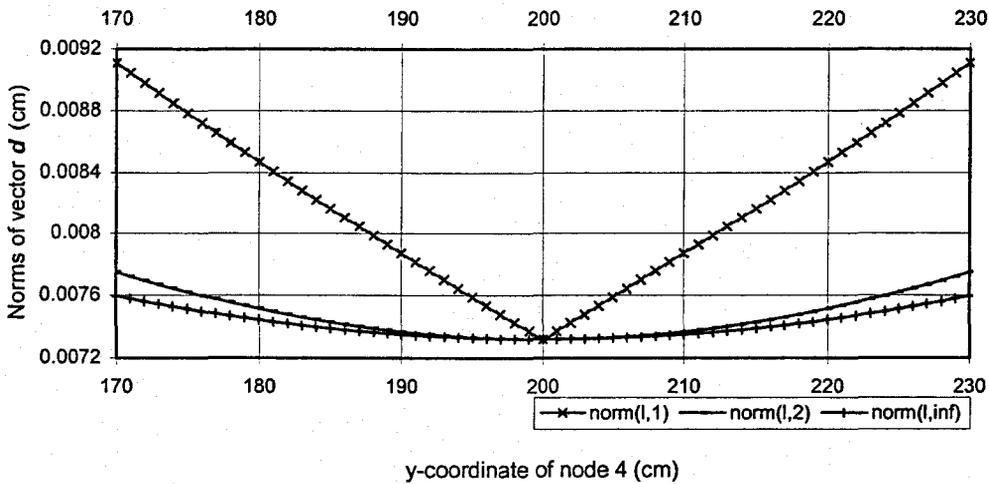


Fig.8 Variation of norm of vector  $d$  versus y-coordinate of node 4 ( $H/B=0.5$ )

### Numerical example 3

The third numerical example is a 6 d.o.f. link structure with four links supported at the three free joints by three linear spring as shown in Fig.9. Three vertically downward loads each of  $0.7071F$ ,  $F$  and  $0.7071F$  in magnitude where  $F=20\text{kN}$  are applied to the example. Other analysis data for this third example are as follows :  $EA=200 \times 10^6 \text{ kN}$ ,  $L=10\text{m}$ ,  $H=4\text{m}$  and  $k=0.8EA/L$  where  $k$ =stiffness of spring. Y-coordinates of node 2 and 4 have been selected as design variables and assigned values ranging from  $y=0.828\text{m}$  to  $y=4.828\text{m}$ . In this example, only  $\text{norm}(l,1)$  and  $\text{norm}(l,2)$  for vector and  $\text{norm}(A,2)$  and  $\text{norm}(L,2)$  for matrix have been considered. Results of analysis are shown in Fig.10 to 12.

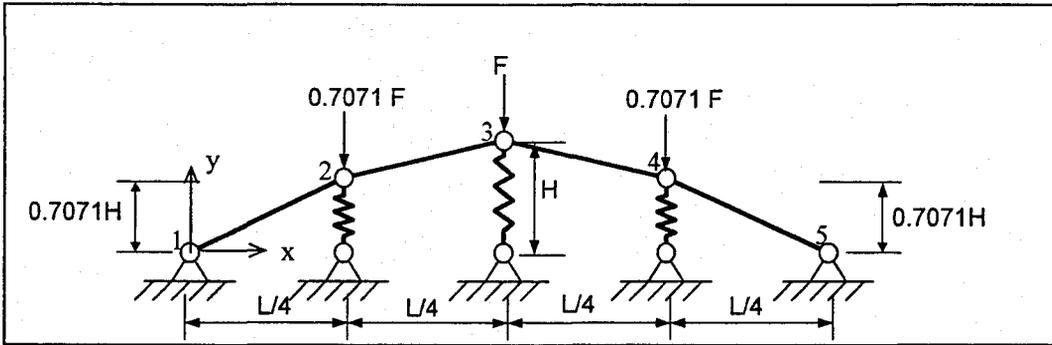


Fig.9 Numerical example 3

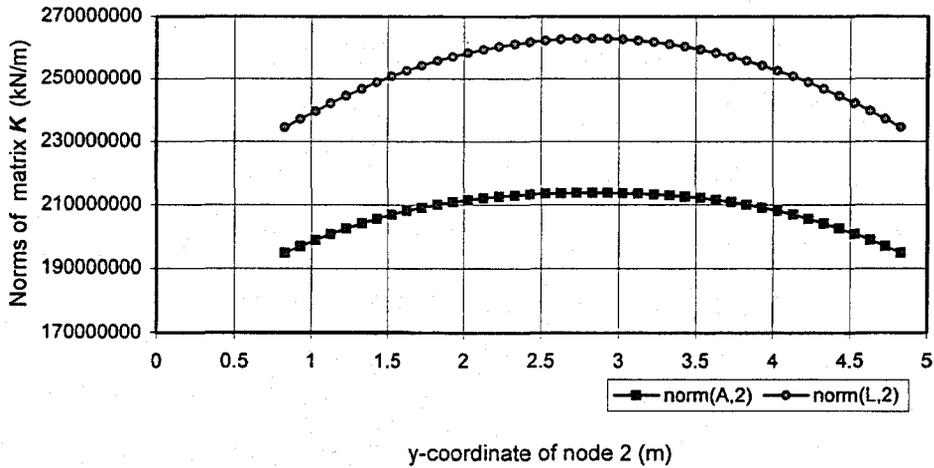


Fig.10 Variation of norm of  $K$  versus y-coordinate of node 2

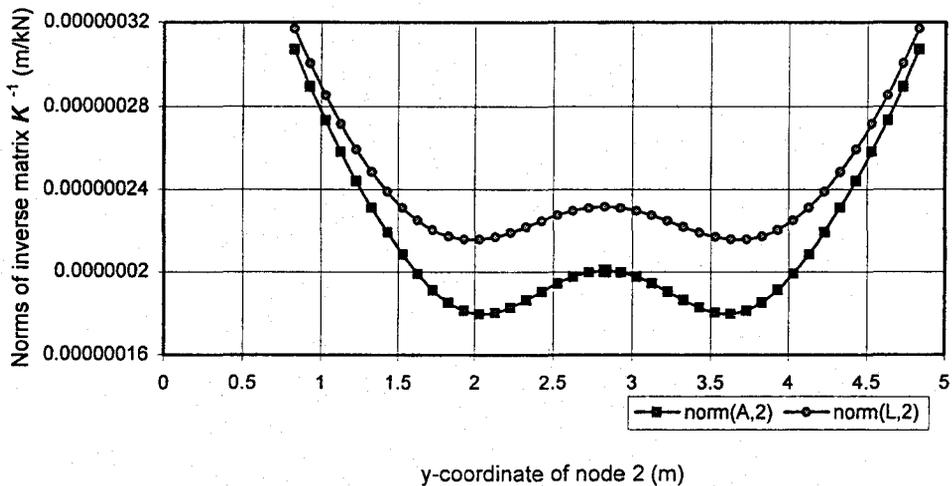


Fig.11 Variation of norm of  $K^{-1}$  versus y-coordinate of node 2

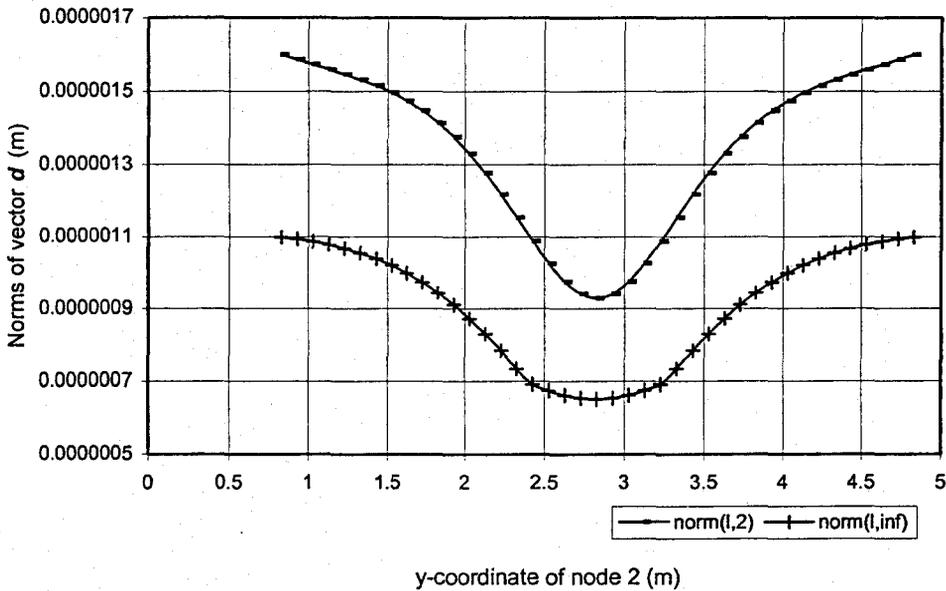


Fig.12 Variation of norm of  $d$  versus y-coordinate of node 2

## DISCUSSION

In terms of existence of maxima/minima on plots, it can be seen from results for all three numerical examples that analysis using vector norms (Figs.4, 8 and 12) shows better consistency than that using matrix norm. The assumption that maxima and minima on plot of norm of stiffness matrix  $K$  and norm of inverse stiffness matrix  $K^{-1}$  versus shape, respectively, correspond to shapes with maximum stiffness, is not generally true. Although comparison between Fig.10 and 11 shows the correspondence, the same correspondence are not observed either in the comparisons between Fig.2 and 3 or between Fig.6 and 7. From results of numerical example 2, it is observed that Euclidean and spectral matrix norms do not show any change in stiffness with change in shape of structure (Fig.7). Such trend might cause difficulty in determining the shape with maximum stiffness during optimization analysis. This characteristic of constant values of Euclidean and spectral matrix norms might be problem dependent and will need further study.

## CONCLUSIONS

Characteristics of three vector and five matrix norms in shape optimization analysis have been studied by means of analysis carried out on three numerical examples of simple truss structure. From this preliminary study, the following conclusions could be drawn :

- (a) Vector norm is more suitable to be used than matrix norm. Amongst the three types of vector norms studied, Euclidean norm is the most suitable to be used.
- (b) In a general multiple d.o.f. problem, the reciprocal relation between norm of stiffness matrix  $K$  and its corresponding inverse  $K^{-1}$  is not observed.

Aspects that require further study are as follows :

- (a) Analysis using truss structures with higher d.o.f. and other types of structures such as shells
- (b) Average or individual characteristic of different types of norms

(c) Computational efficiency of different types of norms

#### **ACKNOWLEDGEMENT**

The authors acknowledge the research grant provided by Universiti Sains Malaysia that has resulted in this conference paper.

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