PLANAR GRAPHS AND COLORING

by

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The last thing one discovers in writing a book is what to put first.

Blaise Pascal

Thank You !

A very special and endless thanks to God for creating Mathematician such as L. Euler to this world, without his research the art and beauty of graph theory would not be discovered and most of the real world problems will remain unsolved.

My valueless appreciation to my supervisor, Professor K.G. Subramaniam of School of Mathematical Sciences, Universiti Sains Malaysia, for his unwavering support, motivation and sincere guidance with his pleasant responds throughout the completion of my project.

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Finally, my heart felt thanks to my beloved children, husband and mum for their sacrifice and tremendous support throughout the writing of this project.

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GRAF PLANAR DAN PEWARNAAN

ABSTRAK

Objektif utama projek ini adalah untuk membincangkan pentingnya Graf Planar dan Pewarnaan. Dalam menyelesaikan masalah sebenar dunia, graf atau rangkaian yang lebih maju dan canggih diperlukan. Disini kita kemukakan konsep-konsep graf planar dan pewarnaan untuk membolehkan pemodelan graf yang lebih efisien.

Walaubagaimanapun, pada permulaan laporan projek ini, kita telah memperkenalkan konsep - konsep asas dan beberapa teorem berkaitan graf yang penting bagi membina asas dalam memahami bab-bab seterusnya.

Dalam projek ini, selain memberikan pendedahan tentang definisi graf planar dan contoh-contoh berkaitan, pengenalan kepada graf planar dan juga bukan planar telah dinyatakan dengan contoh –contoh yang mudah. Formula Euler dan Teorem Kuratowski digunakan untuk menunjukkan K_5 and $K_{3,3}$ adalah graf bukan planar.

Salah sejenis graf yang istimewa yang dikenali sebagai, 'pokok' juga dikatakan graf planar. Oleh itu, konsep-konsep , teorem – teorem dan ciri - ciri berkaitan pokok juga dibincangkan. Pokok juga mempunyai pelbagai aplikasi dalam kajian operasi, rangkaian dan sebagainya, dengan ini bab ini diakhiri dengan membincangkan masalah dari salah satu permodelan iaitu pokok penjana minimum.

Dalam pewarnaan graf kita telah memberikan sejarah tentang permasalahan empat – warna. Kemudian, kita juga telah memberikan teorem - teorem tentang pewarnaan bucu dalam graf planar dan dalam akhir bab ini, aplikasi tentang pewarnaan graf dalam menyelesaikan masalah umum seperti mengatur jadual peperiksaan supaya mengelakkan konflik dan menyimpan bahan kimia untuk mengelakkannya daripada bertindak balas, juga telah dibincangkan.

Pada akhir projek ini, rumusan serta cadangan projek lanjutan diberikan.

ABSTRACT

The key aim of this project is to discuss the importance of Planar Graphs and Coloring. In modeling the real-world problems, more complex and advance structures are needed. Here come the notions of planar graph and the concept of coloring to make the modeling by graphs more efficient.

However at the beginning of this project report we introduce some basic concepts and certain properties of graphs to develop the foundation in understanding the subsequent chapters.

In this project, besides providing an exposure to the definition of planar graphs, and relevant examples, identification of planarity and nonplanarity are described with simple examples. Then Euler's formula and Kuratowski's theorem are used in showing the nonplanarity of K_5 and $K_{3,3}$.

Nevertheless, a special kind of graphs called, 'trees' are also planar graphs. So the notions relating to trees, properties and characterizations of trees are discussed. Trees are known to have wide applications in operation research, networking and so on. We describe the problem of minimum spanning tree model to end this chapter.

In graph coloring we have addressed the history behind the four – color problem. Then, we have included theorems on vertex coloring in planar graphs and in the end of this discussion, applications of graph coloring in solving common

problems such as examination scheduling to avoid conflicts and storage of chemicals to prevent adverse interactions are described.

Finally the project ends with a simple summary of findings and suggestion for future work.

CHAPTER 1

ORIGIN AND BASIC NOTIONS OF GRAPHS

The basic ideas of graph theory were introduced by Leonhard Euler in 1736, a Swiss mathematician, while he was solving the now famous K¨onigsberg bridge problem. The city of K¨onigsberg (now called Kaliningrad) was divided into four parts by the Pregel river, with seven bridges connecting the parts. It is said that residents spent their Sunday afternoons trying to find a way to walk around the city crossing each bridge exactly once and returning to where they started.

Euler was able to solve this problem by constructing a graph of the city and investigating the features of this graph. (Dickson. A., 2006a).

Figure 1.1 Leonhard Euler (1707- 1783).

1.1 Introduction

1.1.1 Motivation and Background

In the past 50 years, graph theory has had many practical applications in various disciplines, including operational research, biology, chemistry, computer science, economics, engineering, informatics, linguistics, mathematics, medicine, social science and etc. Graphs are excellent modeling tools and mathematical abstraction that is useful for solving many kinds of problems. (Agnarsson, G. & Greenlaw, R., 2007).

In the school of Mathematical Sciences* only a few projects related to graph theory have been done in the past 3 years (2006 – 2009). In fact, the topics that were dealt with in those projects are bipartite graphs, dominations in graphs, Euler and Hamilton graphs. Motivated by these reasons, the current project is on an important area of graph theory.

In this project we discuss certain special kind of graphs, called planar graphs and in particular trees and application of the concept of coloring of graphs to certain real – life problem.

1.1.2 Objectives

The objectives of this project are as follows :

- To develop an understanding in basic concepts and certain properties of graph.

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* I thank the School of Mathematical Sciences who allowed me to peruse the MGM599 project reports for the years 2006 – 2009.

- To introduce planar graphs and identify planarity and nonplanarity.
- To show the use of Euler's formula and Kuratowski's theorem to test the planarity of graphs.
- To discuss definitions, properties, characterizations and applications of trees.
- To reveal the origin of the four color problem.
- To elaborate theorems of chromatic number with the reference to Brooks' theorem.
- To provide several examples in the applications of graph coloring.

1.1.3 Overview

In chapter one, we first describe the meaning of graph. We then introduce some basic definitions and theorems of graphs. Then, we end this chapter by describing certain common families of graphs.

Chapter two presents the main part of the project which is about the planar graphs, one of the important subclasses of graphs. This chapter discusses the notion of a planar graph with brief examples of planarity and nonplanarity. Some properties related with number of vertices, edges and faces of plane graph are introduced with reference to Euler's formula and Kuratowski's theorem to test the planarity of a graph. Finally this chapter ends with concept of duality.

Chapter three discusses the 'trees' which are also a special kind of planar graphs. Some definitions of trees which are useful in understanding the concepts are also published. Then, the properties and characterizations of trees are explained. Finally, we reveal the application of trees with one of the models, minimal spanning tree.

Chapter four is the most attractive chapter with colorings of graphs. In this chapter we will see the origin of four – color problem with some flying letters, the theorems of chromatic number with the supporting proof of Brooks' theorem and lastly with some examples of applications of graph coloring in real world problems.

Chapter five is concerned with overall conclusion of this project and it also discusses suggestions for future study.

1.2 Basic Definition of Graphs

A graph is a collection, or set of very simple objects, namely a set of line segments terminated by dots as depicted by Figure 1.2(a) and 1.2(b). These line segments and dots are the sole objects of concern in the graph and have no properties other than their visual objectivity. No line length, or curvature, or point content or position of line segments is considered significant. The graphs of Figure 1.2 , have the same dot and line segment content and so are the same graph.

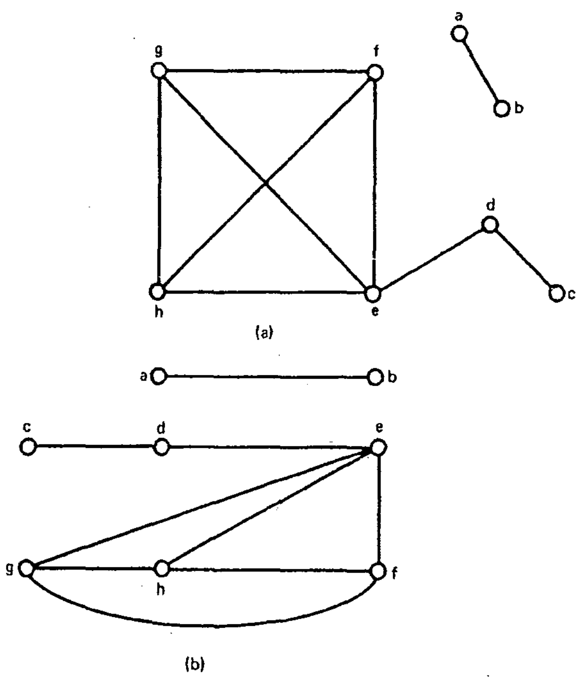

Figure 1.2 Illustration of graphs.

These objects of graph theory are so simple that on the basis presented here they seem to have no properties of their own. It is a study of the manner in which these line segments and dots can be inter-related which constitutes graph theory. (Maxwell, L.M. & Reed, M.B., 1971).

The Graph, Its Elements (or Edges) and Vertices

The words element or edge and vertex are used to denote line segment and dot, based on the work of Maxwell, L.M. & Reed, M.B. (1971).

Definition 1.2.1 Vertex. A vertex is called a dot, a point or a node. A vertex is the only significant joining of line segments (elements). Vertices are illustrated either as small circles or solid dots (Figure 1.3). The intersections of line segments at *(h)* and *(i)*, in Figure 1.3(c), are not vertices.

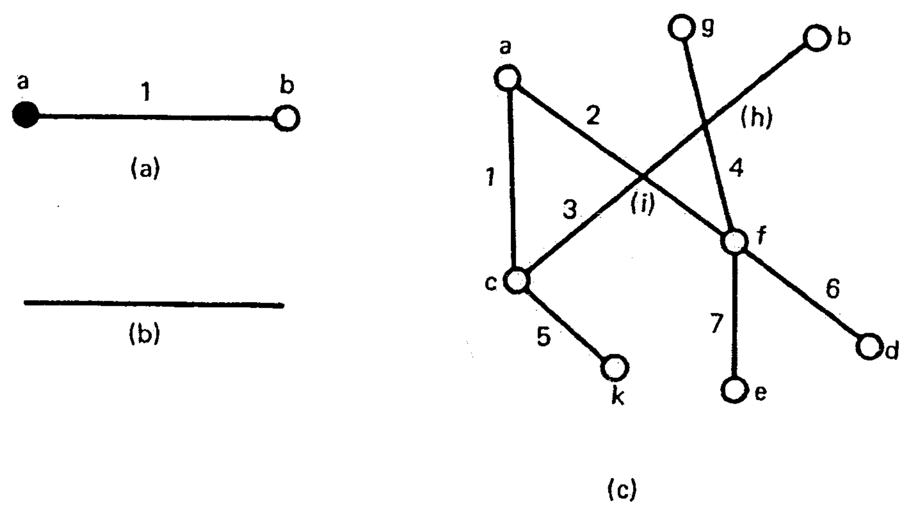

Figure 1.3 Illustration of vertices and elements (or edges).

Definition 1.2.2 Element. An element is a line segment and its vertices, always one on each end of the line segment. For example, Figure 1.3(a) is a correct illustration of an element. Notice that vertices a and b are included as part of the element. Figure 1.3(b) is not a correct illustration of an element because a line segment may not be disassociated from its vertices. Elements 1 and 2, in Figure 1.3(c), have a common vertex, a, while elements 2 and 3 do not. We also call an element as a line, a link, an arc or more commonly an ***edge***. A graph is thus defined as follows :

Definition 1.2.3 Graph. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of vertices or (nodes or points) and $E(G)$ is a finite set of distinct unordered pairs distinct elements of $V(G)$ called edges (lines). We call $V(G)$ the ***vertex-set*** of G and $E(G)$ the ***edge-set*** of G; when there is no possibility of confusion, these are sometimes abbreviated to V and E, respectively. Figure 1.4 represents the simple graph G whose vertex-set $V(G)$ is the set $\{u, v, w, z\}$ and whose edge-set $E(G)$ consists of the pairs $\{u, v\}$, $\{v, w\}$, $\{u, w\}$ and $\{w, z\}$. The edge $\{v, w\}$ is said to ***join*** the vertices v and w, and will usually be abbreviated to vw.

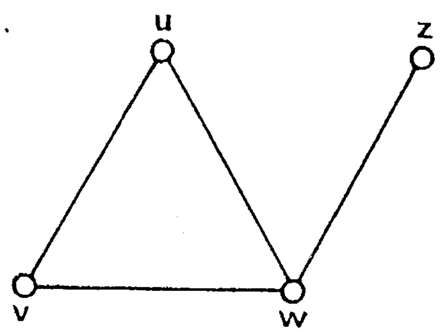

Figure 1.4 The simple graph G.

Various vertex and edge connection features are given by next eight definitions:

Definition 1.2.4 Incidence. An edge is incident to a vertex, and a vertex is incident to an edge if the vertex is a vertex of the edge. In Figure 1.5(a) edge 1,2 and 3 are incident to vertex *a* and vertex *a* is incident to edges 1,2 and 3.

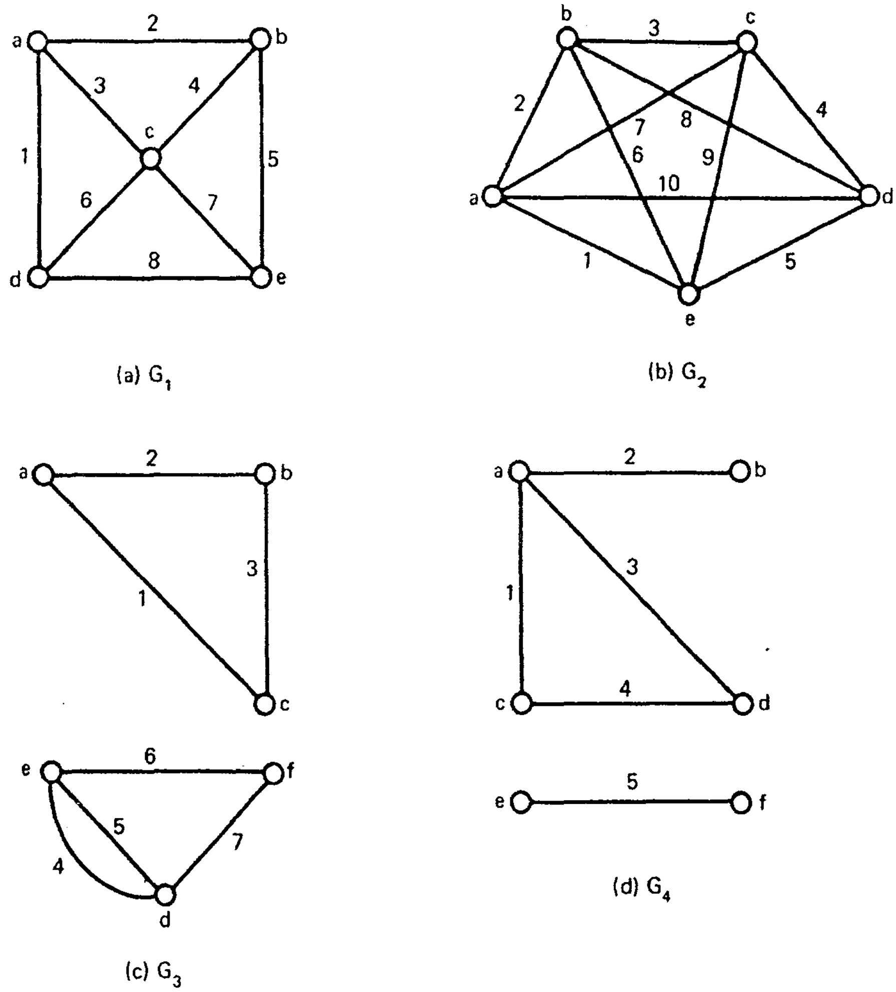

Figure 1.5 Four graphs G_1, G_2, G_3 and G_4.

Definition 1.2.5 Degree of vertex. The degree of a vertex is the number of edges incident to the vertex. Vertices *a, b, d* and *e* of Figure 1.5(a) are of degree three. All vertices of Figure 1.5(b) are of degree four.

Definition 1.2.6 Adjacent (incident) edges. Two edges are adjacent (incident) if the edges are incident to the same vertex.

Definition 1.2.7 Adjacent vertices. Two vertices are adjacent if the vertices are incident to the same edge. Edges 2 and 3 in Figure 1.5(c) are adjacent. In the same graph vertices *a* and *c* are adjacent. In Figure 1.5(b) each vertex is adjacent to every other vertex in the graph.

Definition 1.2.8 End vertex. An end vertex is a vertex of a degree one. Vertices *b*, *e*, and *f* of Figure 1.5(d) are end vertices.

Definition 1.2.9 End edge. An end edge is an edge, incident to at least one end vertex. Edge 2 and 5 in Figure 1.5(d) are end edges.

Definition 1.2.10 Interior vertex. A vertex of degree greater than unity is an interior vertex. All vertices of the graph of Figure 1.5(b) are interior vertices.

Definition 1.2.11 Interior edge. If both vertices of an edge are of degree greater than unity, the edge is an interior edge. Edges 1, 3 and 4 of the graph of Figure 1.5(d) are interior edges.

1.3 Basic Theorems of Graphs

There are some properties of graphs which are dependent only on the definitions of this chapter. We can consider the following theorems, interesting graph properties. (Maxwell, L.M. & Reed, M.B., 1971).

Theorem 1.3.1. In any graph G (e,v) there are an even number of vertices of odd degree.

Proof : Let m_i be the number of vertices of degree i in G for $i = 1,2,....d,$ where d is the vertex of highest degree in G. Counting the number of edges incident to each vertex and summing over all vertices of G each edge of G is counted twice thus: $2e = 1m_1 + 2m_2 + 3m_3 + ...+ dm_d.$ Solving for the total number of vertices of odd degree, $m_1 + m_3 + ... = 2e - 2m_2 - 2m_3 - 4m_4 - 4m_5 - ....,$ *which is an even number.*

Theorem 1.3.2. If a graph, G(e,v), contains no end edges then e = v.

Proof : Let the number of vertices of degree i in G be denoted m_i. By the argument in the proof of Theorem 1.2.1,

$$2e = 2m_2 + 3m_3 + ... + dm_d,$$

where, $m_2 + m_3 + ... + m_d = v,$

then, $e = m_2 + (3/2)m_3 + (4/2)m_4 + ... + (d/2m)m_d = v$

1.4 Additional Definitions

Before we look into the families of graphs we need to know additional definitions in graph theory which are useful subsequently. The descriptions are based on the work of Trudeau, R.J. (1976), White, A.T. (1973) and Wilson, R.J. (1985).

If in the definition of a graph, we remove the restriction that the edge must be distinct, and then the resulting object is called a ***multigraph,*** see Figure 1.6(a) two or more edges joining the same pair of vertices are then called ***multiple edges***. If M is a multigraph, its ***underlying graph*** is the graph obtained by replacing each set of multiple edges by a single edge; for example the underlying graph of the multigraph in Figure 1.6(a) is the graph in Figure 1.4.

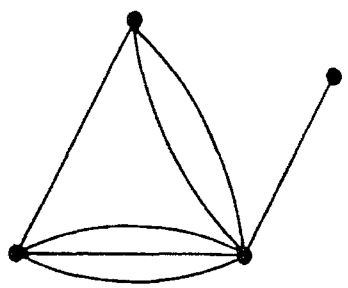

Figure 1.6(a) A multigraph.

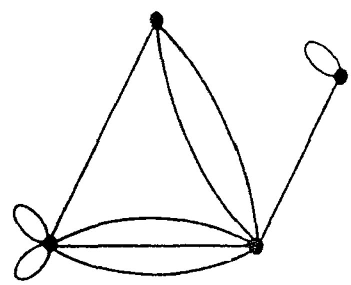

Figure 1.6(b) A general graph.

If we also remove the restriction that the edges must join distinct vertices, and allow the existence of ***loops***, which means edges joining vertices to themselves then the resulting object is called a ***general graph***, or ***pseudograph*** in Figure 1.6(b).

A graph in which one vertex is distinguished from the rest is called a ***rooted graph***. The distinguished vertex is called the ***root-vertex***, or simply the ***root*** and is often indicated by a small square which is shown in Figure 1.6(c). A ***labeled graph*** of order p is a graph whose vertices have been assigned the numbers $1,2,\ldots p$ in such a way that no two vertices are assigned the same number, this is shown in Figure 1.6(d).

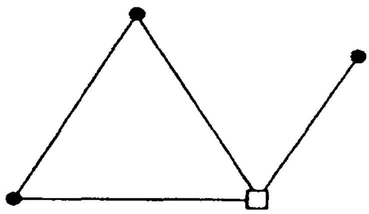

Figure 1.6(c) A rooted graph.

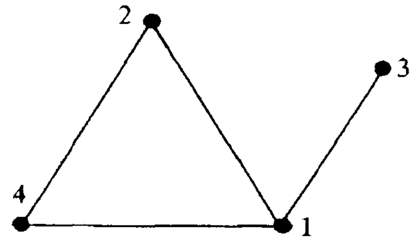

Figure 1.6(d) A labeled graph.

A ***subgraph*** of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is called a ***spanning subgraph*** of G.

Two graphs G_1 and G_2 are ***isomorphic*** if there is a one-one correspondence between the vertices of G_1 and those G_2 with the property that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

Thus the two graphs shown in Figure 1.7 are isomorphic under the correspondence $u \leftrightarrow l$, $v \leftrightarrow m$, $w \leftrightarrow n$, $x \leftrightarrow p$, $y \leftrightarrow q$, $z \leftrightarrow r$. (There are only six vertices – the other points at which edges cross are not vertices).

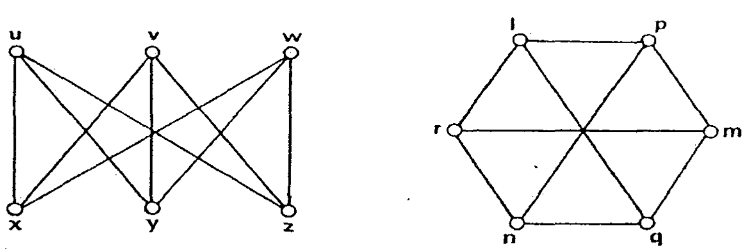

Figure 1.7 Two isomorphic graphs.

The street or roads in a town can be modeled by a graph with the street – corners being vertices and streets as edges. The ***directed graph*** or ***digraph*** arises out of a question, "What happens if all of the roads are one-way streets?" An example of a digraph is given in Figure 1.8, the directions of the one-way streets being indicated by arrows.

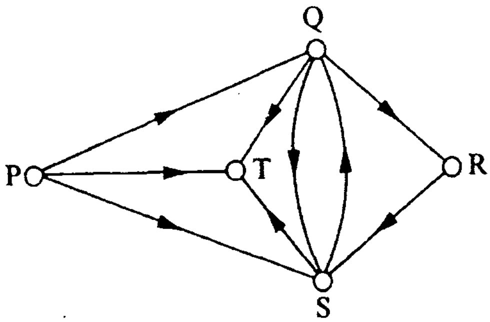

Figure 1.8 A directed graph.

Much of graph theory involves the study of ***walks*** of various kinds, a walk is sequence of vertices and edges of graph, G of the form : $\{ v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4, \ldots, v_{n-1}, (v_{n-1}, v_n), v_n \}$. The sequence begins and ends with the vertices immediately preceding and succeeding it. A walk is termed closed if $v_1 = v_n$, and open otherwise. A walk is termed a ***trail*** if all the edges are distinct.

A ***path*** is a sequence of links that are traveled in the same direction. It is a connected sequence of edges (connecting vertices) in a graph, that does not contain repeated edges and the length of the path is the number of edges traversed. Example in Figure 1.8, $P \rightarrow Q \rightarrow R$ is a path of length two. A ***circuit*** or a ***cycle*** is a path which ends at the vertex, where it begins. For this reason, a path of length three, in Figure 1.8, where $Q \rightarrow R \rightarrow S \rightarrow Q$ is a circuit.

A graph is said to be ***connected*** if every pair of vertices is joined by a walk. Otherwise a graph is said to be ***disconnected.***

A ***contraction*** of a graph can be defined to be any graph which results from *G* after a succession of such edge contraction. Figure 1.9(a) is the original graph. Figure 1.9(b) is a graph denote *G\e* the graph obtained by taking an edge *e,* and "contracting" it or in other word, removing *e* and identifying its ends *v* and *w* in such a way that the resulting vertex is incident to those edges (other than *e*) which were originally incident to *v* or *w*. From here we can say that *G* is ***contractible*** to *G\e.*

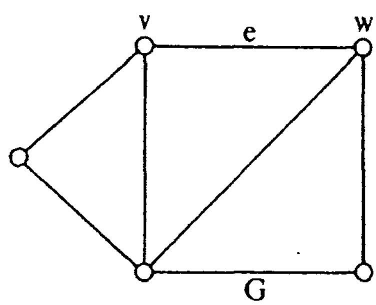

Figure 1.9(a) Graph, G.

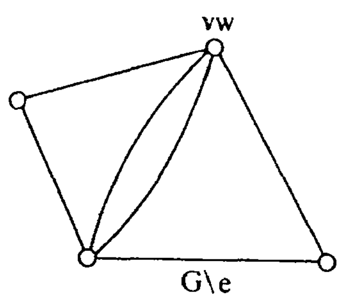

Figure 1.9(b) Graph, G\e.

Any graph which can be redrawn in a way without edge crossings is called a ***planar graph.*** We shall discuss briefly about planar graph in our next chapter, which will be our most important discussion to provide useful information not only for that chapter itself but also in developing the subsequent chapters.

1.5 Common Families of Graphs

In this section we shall define some important types of graphs. Sometimes, simple graphs are adequate; other times, non-simple graphs are needed. The knowledge of understanding the families of graphs might be useful in modeling the real world problems

Complete Graphs

A complete graph is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on *n* vertices is denoted K_n. Examples of complete graphs on one, two, three, four and five vertices are shown in Figure 1.10 (Gross, J.L. & Yellen, J., 2006).

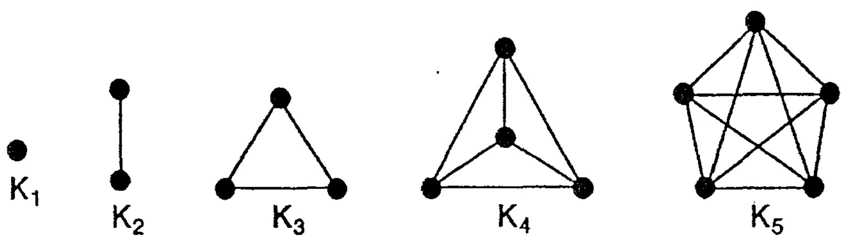

Figure 1.10 The first five complete graphs.

Null graphs

A graph whose edge-set is empty is called a null graph (or totally disconnected graph). We shall denote the null graph on *n* vertices by N_n, N_4 is shown in Figure 1.11. In a null graph, every vertex is isolated. Null graphs are not very interesting. (Muhammad, R.B., 2005a).

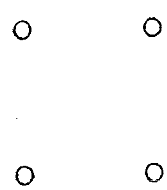

Figure 1.11 The null graph.

Bipartite Graphs

Suppose that the vertex-set of a graph G can be divided into two disjoint sets V_1 and V_2, in such a way that every edge of G joins a vertex of V_1 to a vertex of V_2, this is shown in Figure 1.12(a).

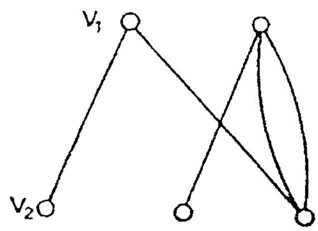

Figure 1.12(a) A bipartite graph.

G is the said to be a ***bipartite graph*** [sometimes denoted by $G(V_1,V_2)$, if we wish to specify the two sets involved]. An alternative way of thinking of a bipartite graph is in terms of coloring its vertices with two colors, say red and blue – a graph is bipartite if we can color each vertex red or blue in such a way that every edge has red end and a blue end.

It is worth emphasizing that in a bipartite graph $G(V_1,V_2)$, it is not necessarily true that every vertex of V_1 is joined to every vertex of V_2; if however this happens, and if G is simple, then G is called a ***complete bipartite graph***, usually denoted by $K_{m,n}$ where m and n are the numbers of vertices in V_1 and V_2 respectively. For an example, Figure 1.12(b) represents $K_{4,3}$. This is from the work of Wilson, R.J. (1985).

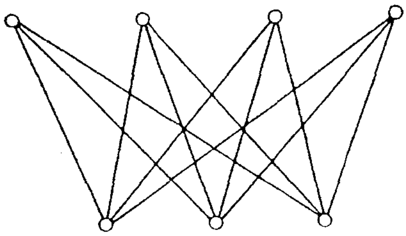

Figure 1.12(b) $K_{4,3}$, A bipartite graph.

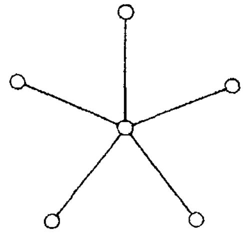

Figure 1.12(c) The star graph.

We have to take note that $K_{m,n}$ has $m + n$ vertices and mn edges. A complete bipartite graph of the form $K_{1,n}$ is called a ***star graph***, $K_{1,5}$ shown in Figure 1.12(c).

Regular Graphs

A regular graph whose vertices all have equal degree. A k-regular graph is a regular graph whose common degree is k. Of our special interest among the regular graphs are so-called ***Platonic graphs***, the graphs formed by vertices and edges of the five regular (platonic) solids – the Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron shown in Figure 1.13(a).

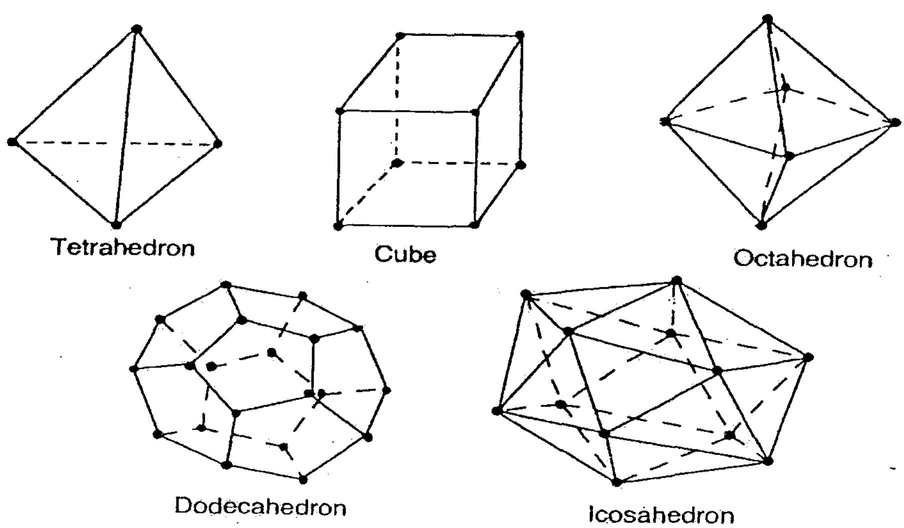

Figure 1.13(a) The five platonic graphs.

The ***Petersen graph*** is the 3-regular graph represented by the line drawing in Figure 1.13(b). It possesses a number of interesting graph-theoretic properties; the Petersen graph is frequently used both to illustrate established theorems and to test conjectures. (Gross, J.L. & Yellen, J., 2006).

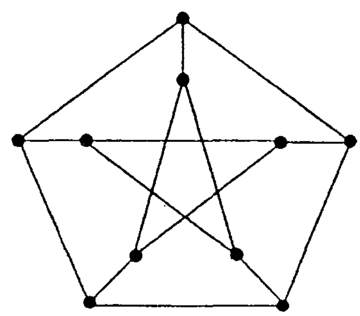

Figure 1.13(b)   The Petersen graph.

We hope that this introductory chapter has provided some useful information and knowledge about graph theory such as preliminary definitions, interpretation and examples which will be useful in setting the stage and describing some of the things which lie ahead and might be very useful in understanding the subsequent chapters.

CHAPTER 2

PLANAR GRAPHS

The objective of this chapter is to provide in detail one of the important components from graph theory, namely, planar graphs. Here in this chapter we will introduce the notion of a planar graph with examples of planar and nonplanar graphs. Euler's formula and a theorem related with number of vertices, edges and faces of a plane graph, Kuratowski's theorem of planarity and we end this chapter with a study of duality.

2.1 Introduction

A ***planar drawing*** of a graph is a drawing of the graph in the plane without edges-crossing. A graph is said to be ***planar*** if there exists a planar drawing of it. (Gross, J.L. & Yellen, J., 2006).

Four drawings of the complete graph K_4 are shown in Figure 2.1(a) and 2.1(b). The drawing on Figure 2.1(a) shows a graph drawn in the plane with edge-crossing. In redrawing the graph, we move the edges or the vertices in three different ways to eliminate the edges-crossing to form three ***plane drawings*** of K_4.

Figure 2.1(b) shows three plane drawings of K_4. From this, we can clearly state that K_4 is ***planar graph,*** since it can be drawn in plane without edges- crossing. (Muhammad, R. B., 2005b).

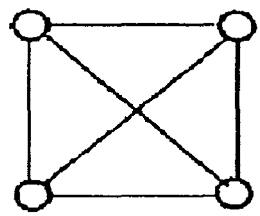

Figure 2.1(a) A nonplanar drawing of K_4.

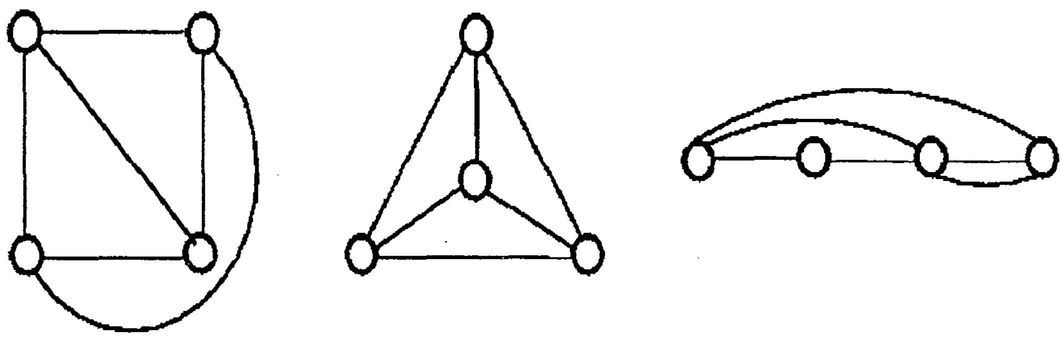

Figure 2.1(b) Three planar drawings of K_4.

The definition of planar graph can be defined more clearly with these two examples:

2.1.1 Example of Planar Graph

Example of bulletin board (Buckley, F. & Lewinter, M., 2003) in Figure 2.2(a). The diagram was posted on a bulletin board by using rubber bands and thumbtacks. Based on the arrangement the rubber bands are crossing each other and look messy. The president of the bulletin board is required to rearrange the rubber bands and thumbtacks in such a way to make the diagram neat without any rubber bands overlapping each other.

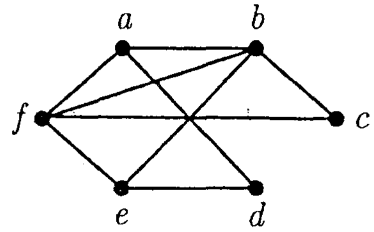

Figure 2.2(a) Diagram posted on a bulletin board.

We can model this problem in graph theoretic terms by representing vertices (V) are thumbtacks and the edges (E) are the rubber bands joining the thumbtacks.

V = [a, b, c, d, e, f]

E = [ab, ad, af, bc, be, bf, cf, de, ef]

Now, we have to think of a way in remodeling the diagram, where there is no intersection between any of the rubber bands. Is this possible? The suggestion given was to stretch a rubber band while leaving the pair of the thumbtacks joined by the rubber band in place or to move a thumbtack to a different location on the bulletin board while remaining the joined rubber band. It can be clearly shown that all the rubber bands can be arranged in such a way that no one rubber band intersect with any other. This is shown in Figure 2.2(b).

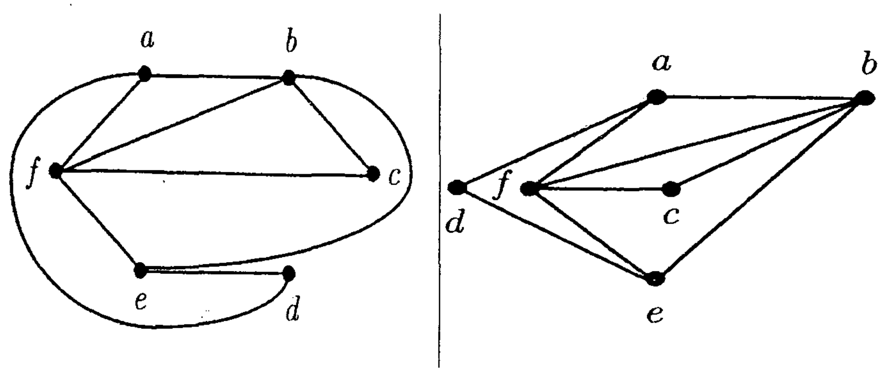

Figure 2.2(b) Two planar drawings of Figure 2.2(a).

From the graph theoretic terms, Figure 2.2(b) is the planar drawings of Figure 2.2(a). Hence, we can conclude that Figure 2.2(a) is a ***planar graph***.

2.1.2 Example of Nonplanar Graph

Example of Utilities Problem (Foulds, L.R., 1992), suppose there are three houses: h_1, h_2, and h_3, each of which connected by cables to the centre of three companies which supply television, telephone service and electricity; TV, TS and TE. A schematic diagram indicating the cable service required is given in Figure 2.3(a).

It is a requirement of all companies that the cables be laid underground in such a way that no cable crosses over the top of any other. The rational is to find a layout of the cables so that each house can be supplied with the three services from the three centers in such a way that no two cables intersect.

We can model this problem in graph theoretic terms by representing the six locations as the vertices (V) of a graph and the cables as the edges (E) of a graph directly joining the two vertices representing the locations which the cable directly connects.

V = [h_1, h_2, h_3, TV, TS and TE]

E = [h_1TV, h_1TS, h_1TE, h_2TV, h_2TS, h_2TE, h_3TV, h_3TS, h_3TE]

From the partial cable layout in Figure 2.3(b), it is impossible to supply h_3 with TV without cable intersection. Indeed, it can be easily shown that any eight of the nine required cables can be laid out without intersection, but it is impossible to layout all the nine cables without intersection. Here we failed to draw the planar drawing of it. From the graph theoretic terms, we can conclude that Figure 2.3(a) is a ***nonplanar graph***.

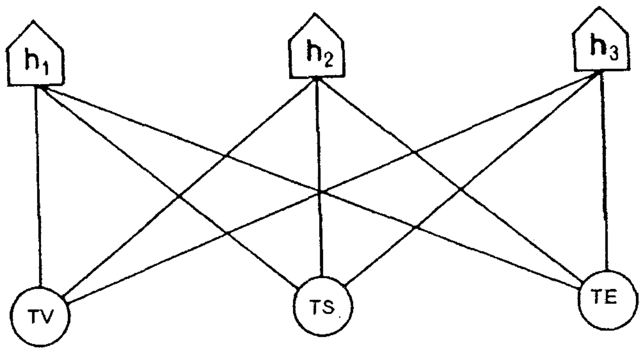

Figure 2.3(a) A schematic diagram indicating the cable service required.

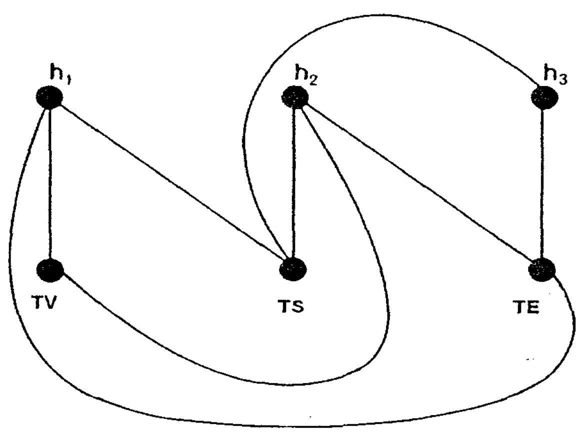

Figure 2.3(b) A partial cable layout.

2.2 Basic Theorem to the Nonplanarity of K_5 and $K_{3,3}$

K_5 is a complete graph on five vertices and $K_{3,3}$ is a complete bipartite graph on six vertices. K_5 is a unique nonplanar graph with the smallest number of vertices and $K_{3,3}$ is a unique nonplanar graph with the smallest number of edges. We are now interested in proving both of these unique graphs are nonplanar. (Gross, J.L. & Yellen, J., 2006).

Theorem 2.2.1 Every drawing of the complete graph K_5 in the plane (or sphere) contains at least one edge-crossing.

Proof : Label the vertices 0, 1, 2, 3, 4. By the ***Jordan Curve Theorem***, any drawing of the cycle (1, 2, 3, 4, 1) separates the plane into two regions. Consider the region with vertex 0 in its interior as the "inside" of the cycle. By the Jordan Curve Theorem, the edges joining vertex 0 to each of the vertices 1, 2, 3 and 4 must also lie entirely inside the cycle, as illustrated in Figure 2.4.

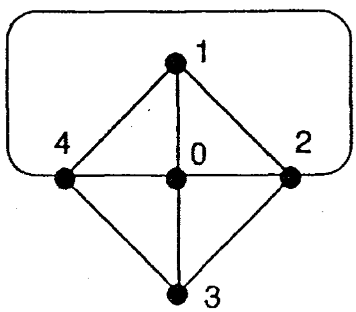

Figure 2.4 Drawing most of K_5 in the plane.

Moreover, each of the 3-cycles {0, 1, 2, 0}, {0, 2, 3, 0}, {0, 3, 4, 0} and {0, 4, 1, 0} also separates the plane and hence the edge (2, 4) must also lie to the exterior of the cycle {1, 2, 3, 4, 1}, as shown. It follows that the cycle formed by edges (2, 4), (4, 0) and (0, 2) separates vertices 1 and 3, again by the Jordan Curve Theorem. Thus, it is impossible to draw edge (1, 3) without crossing an edge of that cycle. So it is proven that the drawing of the K_5 in the plane contains at least one edge- crossing.

Theorem 2.2.2 Every drawing of the complete bipartite graph $K_{3,3}$ in the plane (or sphere) contains at least one edge-crossing.

Proof : Label the vertices of one partite set 0, 2, 4 and of the other 1, 3, 5. By the Jordan Curve Theorem, cycle {2, 3, 4, 5, 2} separates the plane into two regions, and as in the previous proof, we regard the region containing vertex 0 as the "inside" of the cycle. By the Jordan Curve Theorem, the edges joining vertex 0 to each of the vertices 3 and 5 lie entirely inside that cycle, and each of the cycle {0, 3, 2, 5, 0} and {0, 3, 4, 5, 0} separates the plane, as illustrated in Figure 2.5.

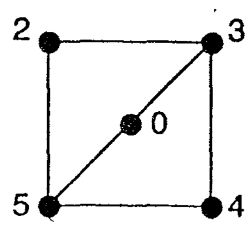

Figure 2.5 Drawing most of $K_{3,3}$ in the plane.

Thus, there are three regions: the exterior of cycles {2, 3, 4, 5, 2} and the inside of each of the other two cycles. It follows that no matter which region contains vertex 1, there must be some even-numbered vertex that is not in that region, and hence the edge from vertex 1 to that even-numbered vertex would have to cross some cycle edge.