THE STRUCTURE OF THE NUMBER SYSTEM

by

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My project entitled " The Structure Of The Number System " is more concerned with analysis and study of the literature available on the subject. It took me almost half a year to complete and to be made into a report. Along the way I faced a lot of hardship and obstacles. I scratched my head when I encountered abstract concepts which were hard to crack. There were bitter sweet moments as I overcame all obstacles with patience, determination and commitment. But, I couldn't have done it without the assistance and guidance given by my project supervisor Dr. Lee See Keong. I take this opportunity to express from the bottom of my heart, my sincere gratitude and appreciation to Dr. Lee See Keong. My gratitude also goes to others who had assisted me by giving me relevant information needed. One such person I would like to mention here is Pn.Faridah from the School of Mathematical Science.

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CONTENTS

STRUKTUR BAGI SISTEM NOMBOR

ABSTRAK

Matematik selalunya dihubungkaitkan dengan sistem nombor. Sistem nombor adalah satu set nombor yang bercirikan operasi arithmetik seperti operasi penambahan atau pendaraban. Secara kasarnya, sistem nombor yang ringkas dalam analisis nyata bermula dengan nombor jati. Pembentukan nombor jati akan dibincang dari segi pendekatan teori. Sehubungan dengan itu, perbincangan kertas kerja ini akan bermula dengan konsep asas teori set dan ciri-ciri yang berkaitan. Di samping itu, sejarah ringkas tentang nombor juga akan diimbas. Dari segi pembentukan nombor pula, perbincangan bertitik tolak dari nombor jati, diikuti dengan nombor integer, nombor nisbah dan akhirnya kepada nombor nyata .

ABSTRACT

Mathematics deals with a variety of number systems. A number system is a set of numbers, together with one or more operations, such as addition or multiplication. The simplest number system in real analysis is the natural numbers. The set of natural numbers is constructed from the viewpoint of set theoretical approach. In this paper, the discussion start with the basic set theory and the relevant properties. The brief history of numbers will then be revealed. The construction of numbers will be started with natural numbers, and then extended to the integers, rational numbers and finally the real numbers.

LIST OF SYMBOLS

Glossary of symbols

{ } curly brackets

$=$ is equal to

$\neq$ is not equal to

A = B A is equal to B (A and B have exactly the same elements)

$\Leftrightarrow$ if and only if

$\forall$ x for every set x

$\in$ is an element of (membership relation)

$\notin$ is not an element of

$\subseteq$ a subset of (inclusion relation)

$\subset$ a proper subset of

$\Rightarrow$ implies (" if __ , then __ ")

Ø empty set

$\cap$ the intersection

$\cup$ the union

or or (in the sense " of the other or *both* ")

$\exists$ x there exists a set x such that

:= is defined as

$\ni$ such that

CHAPTER 1

A BRIEF HISTORY OF THE NUMBERS

1.1 Preliminaries

A number is an abstract idea used in counting and measuring object. In addition to this, numerals (a symbol which represents a number) also often are used for labels (telephone numbers) , for ordering (serial numbers) , and for codes (ISBNs) .

Numbers can be classified into *sets* (a collection of things or objects) which form number systems. The common number systems are natural numbers, integers , rational numbers and real numbers. Each of these number systems is a *proper subset* (is included in) of the next number system, namely the set of natural numbers is included in the set of integers and so forth.

1.2 Historical Development

1.2.1 The First Numbers

It is believed that the first known use of numbers dates back to around 30000 BC. During that time, bones and other artifacts have been used with marks cut into them and considered as tally marks. These tally marks have been suggested to be used for counting elapsed time, such as numbers of days, or keeping records of amounts.

Tally numbers, sometimes regarded as counting numbers are called natural numbers, probably because they occurred to man almost naturally.

Tallying systems have no concept of place-value, which limit its representation of large numbers. It is often considered that this is the first kind of abstract system to be considered as a numeral system.

History of the natural numbers and number zero

If we go back to the historical viewpoint, we know that the first most advance in mathematics for abstraction was the use of numerals to represent numbers. This allows the number system to be developed for recording the larger numbers. For instance, the Babylonians have developed a powerful place-value system based essentially on the numerals for 1 and 10. The ancient Egyptians had a system of numerals with different hieroglyphs (a picture-like sign) for 1 , 10 , and all the powers of 10 up to one million.

The use of zero as a number can be seen in many ancient Indian texts. They used a Sanskrit word *Shunya* to refer to the concept of void, which means the number zero.

By 130, Ptolemy, influenced by Hipparchus and the Babylonians, was using a symbol for zero (a small circle with a long overbar). This Hellenistic zero was the first documented use of a true zero in the Old World. Another early documented use of the zero was done by Brahmagupta (Indian mathematician) dated around 628. In that time, the zero was treated as a number and arithmetic operations involving it, including division.

In the nineteeth century, a set-theoretical definition of natural numbers was developed , which included zero (corresponding to the empty set) as a natural number. This convention is followed by set theorists, logicians, and computer scientists. Other mathematicians, as number theorists, more often follow the older tradition which does not include zero to be a natural number.

The issue whether zero should be included has been going on for hundreds of years, and there is no general agreement until today. Therefore, 0, 1 ,2, 3, ... are often referred as nonnegative integers for whole numbers while 1 , 2 ,3 ,... are called positive integers.

Natural numbers, can be used for two purposes. They are, firstly to describe the position of an element in an ordered sequence, which leads to the concept of ordinal number . An ordinal number is a number showing position or order in a set , as what we usually say first, second, third , etc. Secondly, to specify the size of a finite set, which is generalized by the concept of cardinal number. A cardinal number refers as one of the numbers 1, 2, 3, etc.

1.2.2 Negative Numbers

The earliest known mention of the abstract concept of negative numbers in the East is possibly as early as between 100 BC – 50 BC. Based on the Chinese ancient writings " Nine Chapters On the Mathematical Art" (Jiu-zhang Suanshu) contains methods for finding the areas of figures involving negative notion, red rods were used to denote positive coefficients and black for negative. This is probably the earliest known mention of negative numbers in the East.

In the west, specifically Europe, the first reference involving negative was in the third century in Greece. In the book of Arithmetica, Diophantus referred to the equation equivalent to $4x + 20 = 0$ (the solution would be negative) , saying that the equation gave an absurd result.

During the seventh century, negative numbers were used in India to represent debts. Indian mathematician Brahmagupta, used negative numbers to produce the general form of the quadratic formula that remains in use today.

As recently as the eighteenth century, the Swiss mathematician Leonhard Euler believed that negative numbers were greater than infinity, and it was common practice to ignore any negative results returned by equations on the assumption that they were meaningless.

1.2.3 Rational Numbers

The concept of rational numbers is likely dated to prehistoric times. Ancient Egyptian mathematical texts describe how to convert general fractions into their special notation as rational numbers. Classical Greek and Indian mathematicians made studies of the theory of rational numbers as part of the general study of number theory, as in Euclid's Elements dating to roughly 300BC. Among the Indian texts, the most relevant is the Sthananga Sutra, which also covers number theory as part of a general study of mathematics.

1.2.4 Irrational and Real Numbers

The earliest known use of irrational numbers was in the Indian Sulba Sutras composed between 800 – 500 BC. The first evidence of the existence of irrational numbers is usually attributed to Pythagoras, who produced a proof of the irrationality of the square root of 2.

Irrationals can be subdivided into algebraic irrationals and transcendentals.

A real number a is said to be *algebraic* if it is an algebraic equation of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ where the coefficients a_i are integers (positive and negative whole numbers including zero), $a_0 \neq 0$, and n is a natural number.

This type of equation is commonly called a polynomial equation. For instance, $\sqrt{2}$ is algebraically irrational because it may be considered as one of the solutions of the equation $x^2 - 2 = 0$. Every rational number is obviously algebraic since any number of the form $\frac{a}{b}$ may be considered a root of the equation $bx - a = 0$.

Any irrational number which is not algebraic is called a *transcendental.* One of the transcendentals is π, the number indicating the ratio between the circumference of a circle and its diameter. Others are most of the logarithms, the majority of the values of the trigonometric functions and the interesting number that is called *e*.

The first results concerning transcendental numbers were Lambert's (1761) proof that π cannot be rational, and also that e^n is irrational if n is rational (unless n = 0). The constant e was first referred to in Napier's (1618) work on logarithms. Legendre extended this proof to show that π is not the square root of a rational number. The search for roots of quintic and higher degree equations was an important development. The Abel- Ruffin theorem showed that they could not be solved by radicals (formula involving only arithmetical operations and roots). Therefore, it was necessary to consider the wider set of algebraic numbers (all solutions to polynomial equations).

Even the set of algebraic numbers was not sufficient and the full set of real numbers includes transcendental numbers, the existence of which was first established by Liouville (1844 – 1851) .Hermite proved in 1873 that *e* is transcendental and Lindemann proved in 1882 that π is transcendental. Later Cantor showed that the set of all real numbers is uncountably infinite but the set of all algebraic numbers is countably infinite, so there is an uncountably infinite number of transcendental numbers.

(Reference from *Wikipedia, The Free Encyclopedia*)

CHAPTER 2

BASIC CONCEPTS AND NOTATIONS OF SET THEORY

2.1 Preliminaries

In mathematics, generally everything can be said in the form of a set. To know more about a set, we should know the properties of a set . The different objects as being bound together by some common property will form a set of objects having that property. By formal language, a set is a collection of a whole of definite, distinct objects. The objects are called the elements of the set. Therefore, before we go further to the sets of natural numbers, integers, rational numbers, and real numbers, it is better to deal with the most fundamental items of sets and the notations being used.

2.2 Set Theory

A *set* is a collection of objects . The objects in a set are called *elements* of the set. These elements are said to belong to the set. By using axiomatic approach, we can denote the set of all objects as follows :

Consider any property of a set as P(x). We use { x : P(x) } to mean the set of all objects x such that P(x) is true.

If , a set consists of a finite number of elements $a_1 , a_2 , \dots , a_n$, then we denote it by $\{ a_1 , a_2 , \dots , a_n \}$. In particular case, {a} is a set whose only element is *a*, and { b , c } is a set whose only elements are *b* and *c*. Some examples of a set are given below.

1. $\{ a, b \} = \{ x : x = a \text{ or } x = b \}$.
2. $\{ c, c \} = \{ c \}$.
3. $\{ x : x \text{ is an integer and } x^2 = 1 \} = \{ -1, 1 \}$.

2.2.1 Membership, Equality and Inclusion of Sets

(a) Membership

The principal relation in set theory is the relation of

" is an element of "

and this is a primitive (undefined) notation.

If x belongs to A, or x is an element of A, we write

$x \in A$ (The symbol $\in$ means *belongs to*)

Example : $2 \in \{ 1, 2, 3 \}$.

Meaning : 2 is an element of the set { 1 , 2 , 3 }.

If x does *not* belong to A , or x is not an element of A , then it is written as

$x \notin A$ (The symbol $\notin$ means *not* belong to)

Example : $2 \notin \{ 3, 4, 5 \}$.

Meaning : 2 is *not* an element of the set { 3, 4, 5 }.

(b) Equality

Two sets A and B are said to be *equal if and only if they contain the same elements*, and in notation it is written as

$$A = B \leftrightarrow \forall x \, (x \in A \Leftrightarrow x \in B)$$

(The symbol $\leftrightarrow$ means *equivalent*, $\forall$ means *for every* , $\Leftrightarrow$ stands for *if and only if*)

In words : Set A is *equal to* set B if, for every element x , x is in A if and only if x is in B .

(c) Inclusion of Sets

Let A and B be sets. We say that A is included in B if and only if every member of A is also a member of B .

In symbols , $A \subseteq B \leftrightarrow \forall x \ (x \in A \Rightarrow x \in B)$. (The symbol $\subseteq$ means *is a subset of* or *is included in,* $\Rightarrow$ stands for *implies that*)

In words : A *is included in* B if , for every element x in A , x is also in B . We say that A is a *subset* of B, that is , A is included in B.

Example : $\{ 2 \} \subseteq \{ 2, 3, 4 \}$.

Meaning : A set { 2 } is a subset of { 2, 3, 4 } i.e., element of { 2 } is also an element of { 2 , 3 , 4 }.

A is a *proper subset* of *B* if and only if $A \subseteq B$ and $A \neq B$.

In symbol, we write : $A \subset B$.

In words : B contains at least one element that does not belong to A.

Example : $\{ 1 , \{ 3 \} \} \subset \{ 1 , \{ 3 \} , 5 \}$.

2.2.2 The Empty Set

The set with no elements is called the *empty set* or null set, and is denoted as ø.

2.2.3 Union and Intersection

(a) Intersection

Let A and B be sets. Then $A \cap B$ denotes the set of all members which belong to both A and B. The *intersection* of A and B is symbolized by

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$.

In words : The set of elements each of which belongs to set A and B.

Example : $\{ 1, 2, 3 \} \cap \{ 2, 3, 4 \} = \{ 2, 3 \}$.

Example : Consider the sets

$$A = \{ 2, 4, 6, 8 \}, \quad B = \{ 1, 3, 5, 7 \}.$$

Then, $A \cap B = \varnothing$.

We notice that set A and set B have no common elements, i.e., the elements of set A are entirely different from the elements of other set B. We say these sets are *disjoint*. The intersection of two disjoint sets is the *empty set*.

(b) Union

$A \cup B$ called the *union* of A and B, denotes the set of all elements which belong to A or B or both. The union of two sets A and B is symbolized by

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$.

In words : The set of all elements each of which belongs to at least one of the two sets.

Example : $\{ 1, 2, 3 \} \cup \{ 2, 4, 6 \} = \{ 1, 2, 3, 4, 6 \}$.

2.3 Relations and Functions

2.3.1 Ordered Pairs

The *unordered pair* $\{ a, b \}$ is a set whose elements are exactly *a* and *b*.

The *ordered pair* of *a* and *b* is denoted by (a, b), with *a* is the first coordinate, *b* is the second coordinate of (a, b).

In mathematics, we can form the ordered pair (a, b) of two given objects *a* and *b* . The order of the objects in (a, b) is important. Here, ***a*** is the first coordinate and ***b*** is the second coordinate. Therefore, the point represented by $(1, 2)$ is different from the point represented by (2, 1) in a plane as shown in *Figure 1.1* below

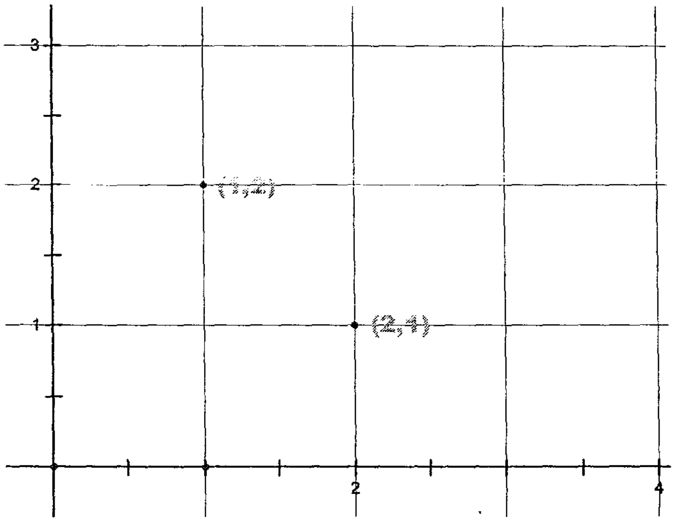

Figure 1.1 The points for ordered pairs (1 , 2) and (2 , 1)

2.3.2 Cartesian Product

Suppose that we have two sets A and B, and we form ordered pairs (x , y) with x $\in$ A and y $\in$ B. The collection of *all* such pairs is called the *Cartesian product* of the sets A and B. This can be denoted as

$$A \times B = \{ (x, y) : x \in A \ \& \ y \in B \}.$$

In words : The set of all ordered pairs whose first coordinate is from A and whose second coordinate is from B.

Example : $\{1, 2\} \times \{3, 4\} = \{(1,3), (1,4), (2,3), (2,4)\}$.

2.3.3 Relations

Before explaining what a *relation* is, let us look at the following example. When we look at the ordering relation < on the set { 2, 3 , 4 }, we might say that < relates each number to each of the larger numbers. Thus, 3 < 4 , means < relates 3 to 4. Pictorially we can represent this by drawing an arrow from 3 to 4 . Altogether we get three arrows in this relation as shown in *Figure 1.2.*

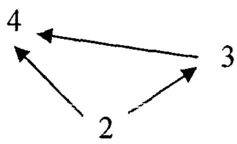

Figure 1.2 The ordering relation < on { 2 , 3 , 4 }

Now, a question is what **set** can be used to decode this ordering relation ? In fact, we can take the ordered pairs (2 , 3) , (2 , 4) , and (3 , 4) in place of the arrows. Therefore, the set of these pairs is

$$R = \{ (2,3), (2,4), (3,4) \}.$$

which completely captures the information in *Figure 1.3.*

Definition A *relation* is a set of ordered pairs.

Definition A set R is called a ***binary relation*** if all elements of R are ordered pairs, i.e., if for any $z \in R$, there exist x and y such that $z = \{ x , y \}$.

Example :

The relation R_1 is the set { z : there exist positive integers m and n such that $z = (m , n)$ and m divides n }. Then, elements of R_1 are ordered pairs

$$(1,1), (1,2), (1,3), \ldots$$
$$(2,2), (2,4), (2,6), \ldots$$
$$(3,3), (3,6), (3,9), \ldots$$
$$\ldots$$

Definition : Let R be a binary relation. We define the *domain* of R (dom R) , the *range* of R (ran R) as follows.

By the *domain* of R , we mean the set of *all objects x such that* (x , y) $\in$ *R* (x is in *relation R* with y) for some y . Then,

$$\text{domR} = \{ x : \exists y \ (x, y) \in R \}.$$

that is, *domR* is the set of *first coordinates* of all ordered pairs in R.

For the *range* of R , we mean the set of *all objects y such that* (x , y) $\in$ *R* for some x.

$$\text{ranR} = \{ y : \exists x \ (x, y) \in R \}.$$

that is, ranR is the set of *second coordinates* of all ordered pairs in R.

For example , let $\mathfrak{R}$ be the set of all real numbers and suppose that R $\subseteq$ $\mathfrak{R} \times \mathfrak{R}$. Then R is a subset of the coordinate plane (horizontal and vertical axis)(*Figure 1.3*). The projection of R onto the horizontal axis is domR , and the projection onto the vertical axis is ranR.

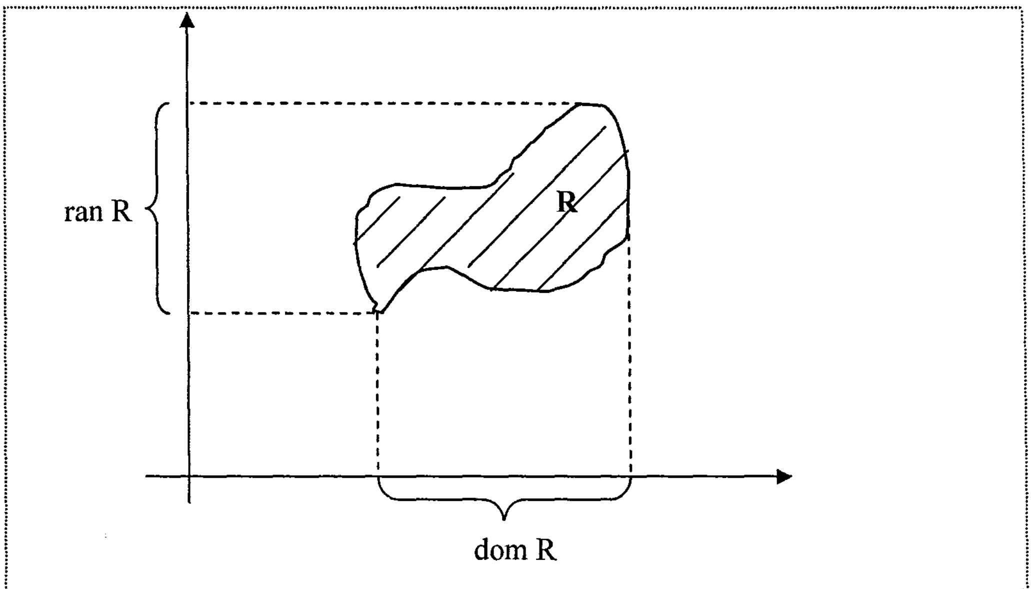

Figure 1.3 A relation as a subset of the plane

2.3.4 Equivalence Relations

Definition Let R be a binary relation in a set A.

(a) R is called *reflexive* on A , if and only if for all $a \in A$, aRa.

(b) R is called *symmetric* in A , if, for all $a, b \in A$, aRb implies bRa.

(c) R is called *transitive* in A, if , for all $a , b ,c \in A$, aRb and bRc imply aRc.

A relation R is said to be an *equivalence relation* on a set A if and only if R is reflexive, symmetric, and transitive in A .

Example : Define the *relation* R on the set $\mathbb{Z}$ of integers as follows.

xRy if $x - y$ is divisible by 2 (their difference is even)

Verification : In order to prove it to be an equivalence relation, we need to show that R is reflexive , symmetric and transitive on $\mathbb{Z}$.

Now , let $x , y, z \in \mathbb{Z}$

(i) Suppose xRy, if x-y is divisible by 2. Since x-x = 0 is divisible by 2, therefore, the relation R is reflexive.

(ii) Suppose xRy, and $x - y$ is divisible by 2. Then $y - x = - (x - y)$ also divisible by 2. This means that yRx is also true, it follows that R is symmetric in Z.

(iii) Suppose xRy and yRz divisible by 2, i.e. $x - y$ and $y - z$ are divisible by 2. We know that $x - z = (x - y) + (y - z)$. This means that $x - z$ is also divisible by 2. Thus, R is transitive since xRy and yRz imply that xRz.

Definition : Let R be an equivalence relation on A and let $a \in A$. The equivalence class of *a* modulo *R* is the set

$$[a]_R = \{ x \in A : xRa \}.$$

Example: Let identity relation $I_A = \{ (a,b) : a, b \in A \text{ and } a = b \}$.

For any a in A, $[a] = \{ c : c \in A \text{ and } a = c \}$.

Hence, $[a] = \{ a \}$. Thus, the equivalence classes are the singletons $\{ a \}$, where $a \in A$.

2.3.5 Operations

Let A be a set.

Definition By a *singulary operation on A* we mean a function F from A into A. Thus, F is a singulary operation on A if and only if $F : A \to A$. In other words, the values of F and its arguments must belong to A.

Example : Let F(x) = 2x for all integers x. F is a singulary operation on the set of integers, since, for every integer x, 2x is also an integer.

Definition By a *binary operation on A* we mean a function from A x A into A. Thus, F is a binary operation on A if and only if $F : A \times A \to A$.

Example : Let F(x,y) = x + y for all positive integers x and y. Then F is binary operation on the set of positive integers.

CHAPTER 3

NATURAL NUMBERS, $\mathcal{N}$

3.1 Preliminaries

In mathematics, a natural number can mean either an element of the set { 1 , 2 , 3 , ... } (the ordinary counting numbers) or an element of the set { 0 , 1 , 2 , 3 , ... } (the non-negative integers). Some books exclude zero from the natural numbers and use the term whole numbers, denoted by $\mathcal{W}$, for the set of non-negative integers. Set theorists often denote *the set of all natural numbers* by a lower-case Greek letter omega ω . When this notation is used , *zero* is explicitly included as a natural number.

3.2 Definition of Natural Numbers

(a) By Peano Axiom

Before we talk about the Peano Axiom, may be we need to know what does it mean by *axiom*. Axioms are the general truth statements that are fundamental and free of contradiction.

Examples : 1. $a = a$, is always true.

2. If $a = b$, then we cannot have $a \neq b$.

The set of natural numbers, $\mathcal{N}$ can be defined using the Peano Axioms.

Firstly, let us assume α be any symbol, then a set $\mathcal{N}$ satisfying the following axioms will be called the set of Natural Numbers. The elements of the set $\mathcal{N}$ are called natural numbers. The five Peano Axioms are as follows.

Axiom 1 : α is a natural number.

Axiom 2 : If x is a natural number, then its successor, denoted as $S(x)$ is a natural number too.

Axiom 3 : For any natural number x, the successor $S(x) \neq \alpha$.

Axiom 4 : For any two natural numbers x and y, if $S(x) = S(y)$, then $x = y$. (Different natural numbers have different successors.)

Axiom 5 : Suppose $M \subset \mathcal{N}$ satisfies the following

(a) $\alpha \in M$,

(b) For any $S(x) \in M$ whenever $x \in M$,

Then, we have $M = \mathcal{N}$.

Consider again $\alpha \in \mathcal{N}$ and the set M (subset of $\mathcal{N}$).

By Peano Axioms, note that we have

$$\alpha \in M \quad (\text{ Axiom } 1)$$

In fact, if $x = S(\alpha) \in M$ (Axiom 2), $S(\alpha) \neq \alpha$ (Axiom 3)

then, $S(x) = S(S(\alpha)) \in M$ (Axiom 2), $S(S(\alpha)) \neq \alpha$ (Axiom 3)

$$\vdots$$

By continuous operation like this, we will have

$$M = \{ \alpha, S(\alpha), S(S(\alpha)), S(S(S(\alpha))), \ldots \}.$$

By Axiom 5, $M = \mathcal{N}$.

That is, $\mathcal{N} = \{ \alpha, S(\alpha), S(S(\alpha)), S(S(S(\alpha))), \ldots \}$.

If we rename it with the known set $\{1, 2, 3, \ldots\}$, namely α corresponding to 1, $S(\alpha)$ to 2, $S(S(\alpha))$ to 3, … and so on, then we have the set of all natural numbers that we are familiar with.

(b) By Set Theory Approach (zero is explicitly included)

From the set-theoretical point of view, the natural numbers can be defined as follows (constructing the natural numbers in terms of sets) :

We define $0 := \{\ \}$, set with zero elements. We also define successor of x as $S(x) = x \cup \{x\}$ for every set x, S is called the *successor function.*

Definition A set $\boldsymbol{B}$ is *inductive* if

(a) $0 \in \boldsymbol{B}$,

(b) if $x \in \boldsymbol{B}$, then $S(x) \in \boldsymbol{B}$.

The Axiom of Infinity : *An inductive set exists.*

By *Axiom of infinity*, there exists a set containing 0 and containing the successor of each of its elements. If the axiom of infinity holds, then the set of all natural numbers exists. If the set of natural numbers exists, then it satisfies the Peano axioms.

Hence, the *natural numbers* for each can be expressed as the set of natural numbers less than it, namely

$0 = \{\ \}$

$1 = \{0\} = \{\{\ \}\}$ ---------(*)

$2 = \{0, 1\} = \{0, \{0\}\} = \{\{\ \}, \{\{\ \}\}\}$

$3 = \{0, 1, 2\} = \{0, \{0\}, \{0, \{0\}\}\} = \{\{\ \}, \{\{\ \}\}, \{\{\ \}, \{\{\ \}\}\}\}$

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.

.

$n = \{ 0, 1, 2, \ldots, n\text{-}2, n\text{-}1 \} = \{ 0, 1, 2, \ldots, n\text{-}2 \} \cup \{ n-1 \} = (n-1) \cup \{ n-1 \}$

and so on.

Remark : From (*), since $S(x) = x \cup \{ x \}$

$$1 = S(0) = 0 \cup \{ 0 \} = \{ 0 \}$$

$$= \{ \} \cup \{\{ \}\} = \{\{ \}\}.$$

Under this definition, there are exactly n elements in the set. A set n is **a natural number** means that it is *either 0 (empty) or a successor*, and each of its elements is either 0 or the successor of another of its elements.

3.3 Arithmetic Properties Of Natural Numbers

In mathematics , certain procedures that when input one or more numbers and output a number are called *numerical operations. Singulary operation* is one that when input a single number will output a single number too. For example, the successor operation can be represented as $S(x) = x + 1$, that is the successor of 3 is 4.

The more common operations we perform are *binary operations*, which *input two numbers will output a single number*. Example of binary operations included four basic operations such as addition, subtraction, multiplication and division. The study of such numerical operations is called *arithmetic*.

The five fundamental laws of arithmetic are :

1. ***THE COMMUTATIVE LAW :***

The order of elements is unimportant in addition and multiplication.

That is, $a + b = b + a$ (1)

$$a \cdot b = b \cdot a \qquad (2)$$

2. ***THE ASSOCIATIVE LAW:***

The grouping of elements in addition or multiplication may be taken in any manner.

So, $$a + b + c = (a + b) + c = a + (b + c) \qquad (3)$$

$$a \cdot b \cdot c = (a \cdot b)c = a(b \cdot c) \qquad (4)$$

3. ***THE DISTRIBUTIVE LAW:***

This will combine both addition and multiplication as follows.

$$a \cdot (b + c) = a \cdot b + a \cdot c \qquad (5)$$

By using Peano Axiom, addition and multiplication operations on $\mathcal{N}$ can be defined as below.

3.3.1 Addition (+) (Addition of Natural Numbers)

Definition :

For any $x, y \in \mathcal{N}$

A 1 : $x + \alpha = S(x)$

A 2 : $x + S(y) = S(x + y)$

In the natural number system, we show that the addition operation has both of the associative and commutative properties.

(a) Addition in $\mathcal{N}$ is associative

Associative Law : $\mathbf{x + (y + z) = (x + y) + z}$.

Proof :

For any $x, y, z \in \mathcal{N}$, we want to prove $x + (y + z) = (x + y) + z$

Choose $y, z \in \mathcal{N}$ and fix them.

Suppose $P = \{ x \in \mathcal{N}: (y+z)+x = y+(z+x) \}$. We want to show that $P = \mathcal{N}$.

Since $(y+z)+\alpha = S(y+z)$

$$= y + S(z)$$

$$= y+(z+\alpha)$$, then $\alpha \in P$.

Next, if $x \in P$, then $(y+z)+x = y+(z+x)$ for any y, z in $\mathcal{N}$.

We have $(y+z)+S(x) = S((y+z)+x)$

$$= S(y+(z+x))$$

$$= y + S(z+x)$$

$$= y+(z+S(x)),$$

this implies that $S(x) \in P$.

By Peano Axiom 5, we have $P = \mathcal{N}$.

Therefore, for any $x, y, z \in \mathcal{N}$

$$x+(y+z) = (x+y)+z$$

(b) Addition in $\mathcal{N}$ is commutative

Commutative Law : $x + y = y + x$.

Proof :

First show that $y + \alpha = \alpha + y$ for any $y \in \mathcal{N}$

Let $Y = \{ y \in \mathcal{N} : y + \alpha = \alpha + y \}$

Then $\alpha \in Y$ since $\alpha + \alpha = \alpha + \alpha$

If $y \in Y$, then $y + \alpha = \alpha + y$ for any $y \in \mathcal{N}$

So, $S(y) + \alpha = (y+\alpha)+\alpha$

$$= (\alpha + y) + \alpha$$

$$= \alpha + (\alpha + y)$$

$= \alpha + (y + \alpha)$

$= \alpha + S(y).$

Hence $S(y) \in Y$

By Peano Axiom 5, we have $Y = \mathcal{N}$

Therefore, for any $y \in \mathcal{N}$

$y + \alpha = \alpha + y.$

Next, we show that $x + y = y + x.$

Fix $y \in \mathcal{N}$.

Suppose $M = \{ x \in \mathcal{N}: x + y = y + x \}$. We want to show that $M = \mathcal{N}$.

Let $\alpha \in M$, $y + \alpha = \alpha + y$ for $y \in \mathcal{N}$

If $x \in M$, then $y + S(x) = S(y + x)$

$= S(x + y)$

$= x + S(y)$

$= x + (y + \alpha)$

$= x + (\alpha + y)$

$= (x + \alpha) + y$

$= S(x) + y.$

So $S(y) \in M$, and, by Peano Axiom 5 $M = \mathcal{N}$,

So, for any $x, y \in \mathcal{N}$,

$x + y = y + x.$

Examples : Standard definitions : $2 = S(1)$, $3 = S(2)$, $4 = S(3)$, $5 = S(4)$, etc .

1. $1 + 3 = \alpha + S(2)$ (By $\alpha = 1$)

$= S(\alpha + 2)$ (By A2)

$= S(2+\alpha)$ (By Commutative Law)

$= S(S(2))$ (By A1)

$= S(3)$ (By definition of "3")

$= 4$ (By definition of "4")

2. $2+2 = 2 + S(1)$ (By definition of "2")

$= S(2+1)$ (By A2)

$= S(2+\alpha)$ (By $\alpha = 1$)

$= S(S(2))$ (By A1)

$= S(3)$ (By definition of "3")

$= 4$ (By definition of "4")

The Order Relation

By order concept, there are three possibilities in comparison of two natural numbers a and b :

1. There exists a natural number *c* such that $a = b + c$, in which case we say that *a* is greater than *b*, and we write $a > b$.
2. The case $a = b$
3. There exists a natural number *d* such that $a + d = b$. This means that *a* is smaller than *b*, and would be written as $a < b$.

3.3.2 Multiplication (·) (Multiplication of Natural Numbers)

Definition :

For any $x, y \in \mathcal{N}$

M1 : $x \cdot \alpha = x$

M2 : $x \cdot S(y) = (x \cdot y) + x$

For multiplication of Natural Numbers, we can show that the operation of $\mathcal{N}$ has associative , commutative and distributive properties.

(a) Multiplication in $\mathcal{N}$ is distributive

Distributive Law : $(y + z) \cdot x = y \cdot x + z \cdot x$. (right - distributive)

We can show that $(y + z) \cdot x = y \cdot x + z \cdot x$

Proof : Choose y, z in $\mathcal{N}$ and fix them.

Let $P = \{x \in \mathcal{N} : (y + z) \cdot x = y \cdot x + z \cdot x\}$

We want to show $P = \mathcal{N}$

1. To show $\alpha \in P$

since $(y + z) \cdot \alpha = y + z$

$= y \cdot \alpha + z \cdot \alpha$,

so, $\alpha \in P$.

2. To show $x \in P$ implies that $S(x) \in P$

If $x \in P$, then $(y + z) \cdot x = y \cdot x + z \cdot x$.

Thus, $(y + z) \cdot S(x) = (y + z) \cdot x + (y + z)$

$= (y \cdot x + z \cdot x) + y + z$

$= (y \cdot x + y) + (z \cdot x + z)$

$= y \cdot S(x) + z \cdot S(x)$.

So, $S(x) \in P$.

By Peano Axiom 5, $P = \mathcal{N}$

Therefore, for any x, y, z $\in \mathcal{N}$

$$(y+z)\cdot x = y\cdot x + z\cdot x$$

(b) Multiplication in $\mathcal{N}$ is associative

Associative Law : $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.

We want to show that $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ for any $x, y, z \in \mathcal{N}$

Proof: Fix $y, z \in \mathcal{N}$.

Let $P = \{x \in \mathcal{N} : (x\cdot y)\cdot z = x\cdot(y\cdot z)\}$

We want to show that $P = \mathcal{N}$

1. To show $\alpha \in P$, $(y\cdot z)\cdot\alpha = y\cdot z$

$$= y\cdot(z\cdot\alpha)$$

So, $\alpha \in P$.

2. To show that $x \in P$ implies that $S(x) \in P$

If $x \in P$, then $(y\cdot z)\cdot S(x) = (y\cdot z)\cdot x + y\cdot z$

$$= y\cdot(z\cdot x) + y\cdot z$$

$$= y\cdot(z\cdot x + z)$$

$$= y\cdot(z\cdot S(x)).$$

Thus, $S(x) \in P$.

By Peano Axiom 5, $P = \mathcal{N}$.

Therefore, for any $x, y, z \in \mathcal{N}$

$$(x\cdot y)\cdot z = x\cdot(y\cdot z).$$

(c) Multiplication in $\mathcal{N}$ is commutative

Commutative Law : $x\cdot y = y\cdot x$.

We first show that $y\cdot\alpha = \alpha\cdot y$ for any $y \in \mathcal{N}$

Let $Y = \{y \in \mathcal{N} : y\cdot\alpha = \alpha\cdot y\}$

Then $\alpha \in Y$ since $\alpha\cdot\alpha = \alpha\cdot\alpha$