BIFURCATION AND TRANSITION FOR ELECTRICALLY CHARGED MULTIMONOPOLE CHAINS IN SU(2) YANG-MILLS-HIGGS THEORY

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by

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LIST OF ABBREVIATIONS

ATLAS	A Toroidal LHC Apparatus
BPS	Bogomol'nyi-Prasad-Sommerfield
CERN	European Organization for Nuclear Research
CMS	Compact Muon Solenoid
FDM	Finite Difference Method
LHC	Large Hadron Collider
MAC	Monopole-Antimonopole Chain
MAP	Monopole-Antimonopole Pair
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Flavodynamics
YM	Yang-Mills
YMH	Yang-Mills-Higgs

LIST OF SYMBOLS

$\Gamma(x)$	Local transformation
δ^{ab}	Kronecker delta
ε^{abc}	Levi-Civita symbol
η	Electric charge parameter
$\Theta_{\mu\nu}$	Energy-momentum tensor
$\hat{\theta}_i$	Spatial spherical coordinate unit vectors
κ_μ	Magnetic current four-vector
λ	Higgs field strength
μ	Higgs field mass
μ_m	Magnetic dipole moment
ξ	Vacuum expectation value
ρ	Electric charge density
σ^a	Pauli matrices
τ_1	Profile function of r and θ
τ_2	Profile function of r and θ
$\hat{\phi}_i$	Spatial spherical coordinate unit vectors
Φ_1	Profile function of r and θ
Φ_2	Profile function of r and θ

Φ^a	Higgs field
$\hat{\Phi}^a$	Higgs field unit vector
$\hat{\Phi}^a_1$	Higgs field unit vector along first perpendicular direction
$\hat{\Phi}^a_2$	Higgs field unit vector along second perpendicular direction
ψ_1	Profile function of r and θ
ψ_2	Profile function of r and θ
ω	Frequency
$\mathfrak{A}$	Profile function of r and θ
A^a_μ	Gauge field
B_i	Abelian magnetic field
B^a_i	Non-Abelian magnetic field
∂_μ	Partial derivative
D_μ	Covariant derivative
D_z	Dipole separation
$\mathscr{D}_z$	Vortex-ring separation
D_ρ	Diameter of vortex-ring
d_z	Separation of poles from origin
d_ρ	Radius of Vortex-ring
E	Energy
$\mathscr{E}$	Energy density

$E_{Abelian}$	Abelian energy
E_i	Abelian electric field
E_i^a	Non-Abelian electric field
$\mathfrak{e}$	Electric charge
$F_{\mu\nu}$	Electromagnetic tensor
$F^a_{\mu\nu}$	Gauge field strength tensor
g	Gauge coupling constant
$\mathscr{J}$	Divergenceless current
$\mathfrak{J}_z$	Angular momentum
k_μ	topological current
$\mathscr{L}$	Lagrangian
$\mathfrak{m}$	magnetic charge
$\mathscr{M}_1$	Vacuum state
$\mathfrak{M}$	Topological magnetic charge
n	ϕ-winding number
$\hat{n}^a_r$	Isospin spherical coordinate unit vectors
$\hat{n}^a_\theta$	Isospin spherical coordinate unit vectors
$\hat{n}^a_\phi$	Isospin spherical coordinate unit vectors
$\hat{r}_i$	Spatial spherical coordinate unit vectors
R_1	Profile function of r and θ

R_2	Profile function of r and θ
$\mathscr{Q}$	Conserved electric charge
Q	Electric charge
$\bar{x}$	Compactified coordinate
X_1	Profile function of r and θ
X_3	Profile function of r and θ
X_4	Profile function of r and θ
$\mathscr{X}_{\nu}^{\mu}$	Pure translational variation
Y_1	Profile function of r and θ
Y_3	Profile function of r and θ
Y_4	Profile function of r and θ

PERCABANGAN DAN PERALIHAN BAGI RANTAIAN MULTIMONOKUTUB BERCAS ELEKTRIK DALAM TEORI SU(2) YANG-MILLS-HIGGS

ABSTRAK

Penyelesaian MAP dan MAC sebagai penyelesaian multimonokutub bersimetri paksian dengan tenaga terhingga dalam teori SU(2) Yang-Mills-Higgs (YMH) telah menerima perhatian yang meluas akhir-akhir ini. Dalam tesis ini, kebergantungan sifat-sifat fizikal dan geometri bagi penyelesaian MAP dan MAC yang bercas elektrik kepada angkatap gandingan diri Higgs λ telah dikaji. Bagi sistem MAC, kes dengan tiga dan empat kutub telah dipertimbangkan. Kajian ini merangkumi nombor winding ϕ bagi $n = 2, 3$ dan 4 untuk sistem-sistem MAP dan MAC empat-kutub. Bagi kes sistem MAC tiga-kutub, kajian dilanjutkan kepada nombor winding ϕ bagi $n = 5$. Dalam kes sistem MAP, kami telah menemui bifurkasi yang bertenaga lebih tinggi daripada tenaga bagi penyelesaian asas untuk nilai-nilai $n = 2, 3$ dan 4. Bagi kes $n = 3$ dan 4, penyelesaian bercabang tenaga tinggi mengalami peralihan daripada konfigurasi MAP kepada konfigurasi gelang-vorteks. Bagi kes $n = 2$, satu bifurkasi yang baru telah diperkenalkan dalam tesis ini. Sehingga kini, kedua-dua cabangan baru ini merupakan satu-satunya penyelesalan gelang vorteks bifurkasi tulen yang diketahui. Bagi sistem MAC tiga-kutub, terdapat hanya satu bifurkasi bagi setiap kes dengan $n = 3$ dan 4. Namun, terdapat dua bifurkasi bagi kes $n = 5$, lantaran lima cabang yang wujud bersama-sama pada nilai λ yang besar. Bagi sistem-sistem tersebut, peralihan telah dikesan dalam penyelesaian asas dan juga penyelesaian bercabang tenaga tinggi. Suatu titik gabungan telah juga ditemui, di mana penyelesaian asas bagi kes $n = 3$ bergabung dengan penyelesaian bercabang tenaga rendah bagi kes $n = 3$. Kedua-dua cabang tersebut berhenti pada titik tersebut dan tidak terus hidup pada nilai λ yang

lebih besar. Selain daripada itu, buat kali pertamanya, kami telah dapat mengesan peralihan di antara monokutub dan antimonokutub bagi kutub yang terletak pada pusat. Untuk sistem empat-kutub, satu struktur berbilang cabang telah ditemui untuk kes $n = 3$ dan 4. Untuk kes $n = 4$, terdapat empat bifurkasi yang teletak di sepanjang penyelesaian asas. Dijumpai juga suatu titik gabungan yang mana dua penyelesaian bercabang tenaga tinggi daripada dua bifurkasi yang berbeza bercantum bersama. Di sini, buat kali pertamanya, satu peralihan yang terletak di sepanjang penyelesaian bercabang tenaga rendah telah dikesan. Satu lagi hasil penemuan yang penting adalah bifurkasi baru yang diperkenalkan untuk kes $n = 3$. Ia tidak pernah dikesani oleh mana-mana kajian sebelum itu.

BIFURCATION AND TRANSITION FOR ELECTRICALLY CHARGED MULTIMONOPOLE CHAINS IN SU(2) YANG-MILLS-HIGGS THEORY

ABSTRACT

MAP and MAC solutions as axially symmetric multimonopole solutions with finite energy in SU(2) Yang-Mills-Higgs (YMH) theory, recently have caused a great amount of attention. In this thesis, the dependence of physical and geometrical properties of electrically charged MAP and MAC solutions in the Higgs self-coupling constant λ, is investigated. For MAC systems, the cases with three and four poles are considered here. The study includes ϕ-winding numbers of $n = 2,3$ and 4 for MAP and four-pole MAC systems. For the case of three-pole MAC systems, we extended the study to the ϕ-winding number of $n = 5$ as well. For the case of MAP systems, for each value of $n = 2,3$ and 4, we found a bifurcation with higher energy in comparison with the fundamental solution. For the cases with $n = 3$ and 4, the Higher Energy Branch (HEB) solution undergoes a transition from MAP configuration to vortex-ring configuration. For the case of $n = 2$ a new bifurcation is introduced in this thesis. The two new branches are the only known bifurcating purely vortex-ring solutions so far. For the three-pole MAC systems, there is only one bifurcation for each one of the cases with $n = 3$ and 4. However for the case of $n = 5$, there are two bifurcations and therefore five co-existing branches for large values of λ. For these systems transitions are detected along fundamental solutions as well as HEB solutions. There is also a joining point for which the fundamental solution of the case of $n = 3$ joins to the Lower Energy Branch (LEB) solution for the case of $n = 3$ and both branches stop at this point and do not survive for larger values of λ. Also, for the first time we have detected a transition between a monopole and antimonopole for the pole which is located at the centre. For four-pole systems, a multi-branch structure is found for the cases of $n = 3$ and 4. For the case of $n = 4$ there are four bifurcations along with the fundamental solution. Also there is a joining point for which two HEB solutions of two different bifurcations join to each other. Here for the first time, a transition along the one LEB solution is detected. As another important result, a new bifurcation is introduced for the case of $n = 3$ which was not detected with the previous studies.

CHAPTER 1

INTRODUCTION

1.1 A Brief Review of Previous Contributions

Classical field theories propose a rich collection of nonlinear solutions with finite energies, including different mathematical configurations. These topological objects which are not singular at any point, have caused a great amount of attention in recent decades. In fact, magnetic monopoles are some of the most interesting topological objects.

The idea of the duality of the electricity and magnetism (which will be expressed in the next chapter) has been the first motive for the physicists for monopole studies. This idea proposes that if we have a magnetic charge, Maxwell's equations will be symmetric under a transformation known as duality transformation. Such a theory always encounters an important question: Is this symmetric theory consistent with quantum mechanics or not? Dirac's monopole (1931) which is known as the first important monopole solution, was indeed an effort to find a monopole solution consistent with quantum mechanics. His brilliant solution however did not convince physicists about the existence of the magnetic monopoles and this was dealt with as a possibility before 't Hooft-Polyakov monopole.

The 't Hooft-Polyakov monopole solution which was proposed separately by Gerard 't Hooft (1974) and Alexander Polyakov (1974; 1975a; 1975b) was a natural consequence of spontaneous symmetry breaking of non-Abelian gauge theories. This new finding implies that, if non-Abelian gauge theories are correct and if the Higgs mechanism is what occurs in Nature (which is recently proven by Conseil Européen pour la Recherche Nucléaire, CERN (ATLAS, 2012; CMS, 2012) to be), then the existence of such magnetic charges is unavoidable.

The 't Hooft-Polyakov monopole, has caused increasing interest to the magnetic monopoles as a new field in theoretical physics studies. This brought large amount of studies on the topic and several different solutions including magnetic monopoles have been found since that time (we will discuss in detail about Dirac's monopole, 't Hooft-Polyakov monopole and some other important contributions in chapter3).

Among those solutions multimonopole solutions are those with more than a single magnetic pole. One of the recent multimonopole solutions is monopole-antimonopole chain (MAC) solution in Yang-Mills-Higgs (YMH) model. These solutions have axial symmetry and consist of two different configurations of magnetic charge: Node structure (or sometimes simply referred as MAC structure) and vortex-ring structure. These sorts of solutions were first introduced by a research group in Oldenburg University in (1999). Their first version included only one monopole and one antimonopole along the symmetry axis. They could successfully generalize their model to a chain of monopole-antimonopoles along the symmetry axis in (2003a; 2003b; 2004).

Immediately after this development, the study on these new solutions started by a group in Universiti Sains Malaysia (Teh and Wong, 2005a,b). Shortly later the Oldenburg group studied the behavior of these axially symmetric solutions with increasing value of Higgs self-coupling constant (Kunz et al., 2006). This study caused the new concepts of bifurcation and geometrical transitions to arise. Based on this study, for some given value of Higgs self-coupling constant we would have more than a single solution. Indeed, in some cases, at some critical value of Higgs self-coupling constant new solutions arise which are absent for smaller values of self-coupling constant.

On the other hand the research on the electrically charged axially symmetric multimonopole solutions (dyons) started by the study of (Hartmann et al., 2000). The group of Universiti Sains

Malaysia started to work on the dyons with axial symmetry at 2011 where they restricted the study to the physical and geometrical properties of solutions with Higgs self-coupling constant values of zero and one (Lim et al., 2012). However, this was just the first step in the study of the electrically charged multimonopole solutions with axial symmetry.

1.2 The Current Study: Objectives and Perspective

As the next step it was necessary to study the properties of the dyon solutions for larger values of Higgs self-coupling constant and study the properties of bifurcations and transitions in this new context. This is what we have tried to do within this thesis. Some of the major questions which this thesis tries to answer them are as follows:

- What kind of changes in physical and geometrical properties of the solutions would arise by adding the electric charge to the solutions?

- Does electric charge cause some new branch of solution appear which was absent in neutral case studies?

- How does the total electric charge of a solution changes with changing Higgs self-coupling constant?

- How is the electric charge distribution of the solutions? Does it behave like the magnetic charge distribution or not?

As we will see in this thesis, besides these objectives, this study has brought some other valuable results (like the transition between monopole and antimonopole) which were not expected in the beginning.

This study -apart from its unexpected interesting results- would be of great importance because it gives the ultimate picture of a YMH multimonopole. Also the results of this study can always reduce to the results of electrically neutral case by switching off the electric charge. This provides an important method for double checking of the results generated previously in neutral case with a new numerical method. [1]

The mathematical background of magnetic monopoles is reviewed in Chapter 2. We have tried to show the importance of symmetries and the invariance of the physical state under transformations in this chapter. In the first section we have had a review on Dirac's approach into his fundamental equation for which in order to make a relativistic framework, he introduces the concept of invariance under Lorentz transformations. In the next steps, we have tried to show that how the implications of special relativity make us to leave the idea of global transformations and how the new non-Abelian solutions arise. Finally we close this chapter with a discussion on spontaneous symmetry breaking which completes our final form of Lagrangian in YMH model.

A very brief introduction about some of the most important monopole solutions is included in Chapter 3. Here, at the beginning, the Dirac monopole and its electric charge quantization condition is discussed. After a brief discussion about the conditions of finite energy solutions, the 't Hooft-Polyakov solution is introduced. In the next part, exact solutions in BPS limit is studied and finally a discussion about the Julia-Zee dyon is included.

In Chapter 4, we will introduce the electrically charged multimonopole/vortex-ring solutions with axial symmetry in SU(2) Yang-Mills-Higgs theory. A historical review of these solutions is included in the beginning and the mathematical frame work of these solutions are discussed in the next section.

[1] The numerical method of the Oldenburg group is slightly different than what the USM group is using for its calculations.

In the beginning of Chapter 5, we will try to define the new concepts of bifurcation, transition and joining points in the context of these axially symmetric multimonopole solutions. Also we will describe the numerical framework in which our numerical solutions are established. In this part of the 4th chapter we will see that how the numerical errors and increasing number of branches makes us to shift from an old manual numerical method to a new method in which the data analysis is totally performed with an automatic method. In the next section, we will see how the simplest electrically charged axially symmetric compound solution including one monopole and one antimonopole is constructed based on the theoretical framework given in Chapter 4. In contrast to the previous studies which found the vortex-ring solutions only for ϕ-winding numbers of $n > 2$ the presence of new purely vortex-ring solutions are introduced for the case of $n = 2$ in this chapter. There will be a detailed investigation on physical and geometrical properties of these monopole-antimonopole pair (MAP) and vortex-ring solutions when the Higgs self-coupling constant increases from zero to the maximum value of $\lambda = 144$. In order to obtain a clearer understanding of the Abelian characteristics of the newly found vortex-ring solutions, we will use the Cho Abelian decomposition analysis. Finally a unique geometrical behaviour of branches with totally MAP configuration is reported.

In Chapter 6, electrically charged three-pole monopole antimonopole chain solutions are investigated with the same method of Chapter 5. The presence of two bifurcation and therefore five distinct solutions is the novel result of the study of three-pole solutions. Also we will compare vortex-ring configurations of three-pole solutions and two-pole solutions in this chapter. There is also a detailed discussion about the variations of the electric and magnetic charge of the pole which is located at the origin when the Higgs self-coupling constant changes. For the first time in this chapter we will see how a transition changes a monopole into an antimonopole and vice versa. Also the presence of transition in more than one branch for any given case is reported for the first time in this chapter.

The results of the study of four-pole solutions are included in Chapter 7. The more complex branching pattern for these solutions are discussed. Multi branch structures of the solutions including several bifurcations and transitions are studied in detail. Some interesting analogies between the two-pole solutions and four-pole solutions are obtained in this chapter. Also some of the results of this chapter improve the results which are previously published by other research groups. Finally, Chapter 8 includes a summary of our most important results, and some prefigurations based on the patterns of available results which will be the last part of this thesis. Also a short discussion on future possible extensions is included at the end.

1.3 Natural Units and Dimensionless Calculations

Before any mathematical discussion, we need to decide which system of units is most suitable for our calculation. In high energy physics and particle physics, the system usually is not SI or cgs. There is a practice in these areas to use a system which is called natural units. Indeed this is not really a unique system of units. In fact there are different kinds of natural unit systems which differ in some details. However their characteristic is that within these systems the unit values of any physical observable are directly taken from the Nature. For example in some system the mass of the electron would be chosen as the unit of the mass. In that system the value of the mass of proton will be almost 1838. In another system the mass of proton would be the unit of the mass (atomic mass unit). In that case the value of the mass of electron will be almost 0.0005440.

Another practice which is common in natural systems of units is that usually physicists prefer to normalize some of the major constants for simplicity. This normalization usually causes some of the quantities or constants which have different values in SI, to acquire value of one. This normalization makes the calculations easier. However, it would cause some confusion at the first glance. There are restrictions in such a normalization. For example one cannot simultaneously give the unit value for the diameter of the earth and the length which light travels in a day. One of the most common normalization contracts in high energy physics is $\hbar = c = 1$. With such a contract we can write $\hbar c = 1$ while we know that in another system for example we can write $\hbar c = 0.1975 GeV fm$. One of the most important natural systems is the Planck system of units, in which the normalization contract is $\hbar = c = G = k_B = k_e = 1$ where G is gravitational constant, k_B is the Boltzmann constant and k_e is the Coulomb constant. In such a system the units of mass and energy is the same and time and length have the same unit which is proportional to the inverse of the unit of mass.

We will not use the Planck system in our calculations. Indeed there are two major methods of relating mass to charge. For one of those methods we write $U = \frac{Q}{r}$ and for the other one we have $U = \frac{Q}{4\pi r}$ in which U stands for energy. Our calculations will be based on the second method. However, we still use the normalization contract of $\hbar = c = 1$. We will add a few more constants to this contract in the following chapters.

Finally we have to mention that the values of physical observables which we report in this thesis are given in terms of these natural units but they are dimensionless because at the final step of calculations the value is devided by the natural unit. For example for the case of total mass (energy) the minimum value of mass (which is $4\pi\xi$) is supposed to be the unit of mass. However, by definition, the calculated value is devided by this value and therefore the result is dimensionless.

CHAPTER 2

THEORETICAL FRAMEWORKE

2.1 The Duality of Electricity and Magnetism

Electromagnetic field, as we know today, is a field which explains a massless particle with spin 1; the photon. The related field equations are Maxwell's equations which are formulated long ago. Here we want to have a more detailed study on the electromagnetic fields and Maxwell's equations. These equations in their classical form are given by

$$
\begin{aligned}
\vec{\nabla}.\,\vec{E} &= \rho, \\
\vec{\nabla}\times\,\vec{E}+\frac{\partial\vec{B}}{\partial t} &= 0, \\
\vec{\nabla}.\,\vec{B} &= 0, \\
\vec{\nabla}\times\,\vec{B}-\frac{\partial\vec{E}}{\partial t} &= \vec{j},
\end{aligned}
\tag{2.1}
$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic induction, ρ is electric charge density and $\vec{j}$ is electric current. The Maxwell equations of (2.1) are respectively the differential formulations for Gauss's law, Faraday's law, the absence of magnetic charge and Ampere's law. If we show the electromagnetic four-vector-potential with $A_\mu = (\bar{\phi},\vec{A})$, then we can see that electric and magnetic fields, in order to satisfy Maxwell's homogeneous equations of Faraday's law and the absence of magnetic charge, must be of the form of

$$
\begin{aligned}
E_i &= \partial_i A_0 - \partial_0 A_i, \\
B_i &= -\frac{1}{2}\varepsilon_{ijk}(\partial_j A_k).
\end{aligned}
\tag{2.2}
$$

It is tempting now to introduce the traceless electromagnetic field tensor as

$$\begin{aligned} F_{\mu\nu} &= -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\ F_{\mu\nu} &= \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \end{aligned} \tag{2.3}$$

for which the sign convention for the metric is $g_{\mu\nu} = (+---)$. Indeed, this is four dimensional curl of four-vector-potential A_μ. Now, using this new definition, Eq. (2.2) reduces to

$$E_i = F_{i0}, \quad B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}. \tag{2.4}$$

For Maxwell's inhomogeneous equations of Gauss's law and Ampere's law, by using this new notation we can write

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu, \\ \text{where,} \quad j^\nu &= (\rho,\ \vec{j}). \end{aligned} \tag{2.5}$$

From Eqs. (2.3) and (2.5), we can easily see that the case of $\nu = 0$ gives the differential form of the Gauss's law while each one of the other three vales for ν gives a component of differential form of the Ampere's law. Also, using the Eq. (2.3) we can write:

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0, \tag{2.6}$$

which is in fact a representation for Maxwell's homogeneous equations. Defining the dual electromagnetic field tensor as

$$^{*}F^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{2.7}$$

helps the Eq. (2.6) to reduces to

$$\partial_{\alpha}(^{*}F^{\alpha\beta}) = 0, \tag{2.8}$$

For example for the value of $\beta = 0$ we have:

$$\begin{aligned}
&\partial_{\alpha}(^{*}F^{\alpha 0}) = 0 \Rightarrow \frac{1}{2}\partial_{\alpha}\delta^{\beta 0}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu} = 0 \Rightarrow \\
&\frac{1}{2}\partial_{\alpha}(F_{\mu\nu} - F_{\nu\mu}) + \frac{1}{2}\partial_{\mu}(F_{\nu\alpha} - F_{\alpha\nu}) + \frac{1}{2}\partial_{\nu}(F_{\alpha\mu} - F_{\mu\alpha}) = 0 \Rightarrow \\
&\frac{1}{2}\partial_{\alpha}(2F_{\mu\nu}) + \frac{1}{2}\partial_{\mu}(2F_{\nu\alpha}) + \frac{1}{2}\partial_{\nu}(2F_{\alpha\mu}) = 0,
\end{aligned} \tag{2.9}$$

which is equivalent to the Eq. (2.6). Now for the case of $\alpha = 1, \mu = 2, \nu = 3$, Eq. (2.6) gives

$$\begin{aligned}
&\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0, \Rightarrow \\
&\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \partial_i B_i = 0,
\end{aligned} \tag{2.10}$$

which is the third Maxwell's equation in (2.1). Any other choice for β in Eq. (2.8) will give a component of the second equation of (2.1). Therefore Eq. (2.5) together with Eq. (2.8) provides another representation for Maxwell's equations of (2.1).

Now, in the absence of matter where electric current four-vector vanishes, we can keep Maxwell's equations invariant under the duality transformation which is given by

$$F_{\mu\nu} \rightarrow ^{*} F_{\mu\nu}, \qquad ^{*}F_{\mu\nu} \rightarrow -F_{\mu\nu}. \tag{2.11}$$

However, in general case of the presence of matter, obviously there is not such a symmetry between Eqs. (2.5) and (2.8). In order to make such a symmetric theory, we need to define a new magnetic current κ^{μ} in order to be able to write

$$\partial_{\nu}F^{\mu\nu} = j^{\mu}, \qquad \partial_{\nu}\,{}^{*}F^{\mu\nu} = \kappa^{\mu}. \tag{2.12}$$

In such a theory, the extended form of duality transformation of (2.11) will be expressed as

$$j^{\mu} \rightarrow \kappa^{\mu}, \qquad \kappa^{\mu} \rightarrow -j^{\mu}. \tag{2.13}$$

In such a symmetric classical theory the dynamics of system is given by

$$m\frac{d^2x^{\mu}}{dx_0^2} = (\mathfrak{q}F^{\mu\nu} + \mathfrak{m}\,{}^{*}F^{\mu\nu})\frac{dx_{\nu}}{dx_0}, \tag{2.14}$$

in which $\mathfrak{q}$ is the electric charge and $\mathfrak{m}$ is the magnetic charge of a particle with mass m. This is the generalization of Lorentz force law to the case of this symmetric theory.

The perspective of such a theory with pure positive or negative magnetic charges and therefore the possibility of existence of magnetic monopoles as magnetically charged objects, has been really pleasant for theoretical physicists and this caused widespread historical efforts to construct such a theory. In Chapter 3 we will briefly review some of the most important historical efforts to find theoretical solutions for magnetic monopoles.

Let's now study the behaviour of electromagnetic field (or Maxwell's equations) under the gauge transformation of

$$A_{\mu} \rightarrow A_{\mu} + \frac{1}{\tilde{g}}\partial_{\mu}\tilde{\Gamma}(x^{\mu}), \tag{2.15}$$

where $\frac{1}{\tilde{g}}$ is a coupling constant. Noting that $\partial_{\mu}\tilde{\Gamma}$ is the gradient of the scalar field $\tilde{\Gamma}(x^{\mu})$, and knowing that the curl of the gradient of any scalar field is zero, the field tensor $F_{\mu\nu}$, as the four dimensional curl of the field A_μ, is invariant under the transformation of (2.15). This is a very important result which we will use it in the next chapter to derive the Maxwell's equations using a more fundamental approach.

2.2 A Few Topics about Quantum Field Theory

The expression of particles as excitations of the fields has been a successful approach for particle physics during the previous decades. In such a context which is generally referred as quantum field theory (QFT), it is a reasonable expectation to find a theory in which magnetically charged fundamental particles arise as fundamental solutions.

In fact, the recent developments in magnetic monopole studies are deeply correlated with the theoretical framework of classical and quantum field theories. Monopole studies have caused important developments in those theoretical fields. For example they have caused us to learn more about the interpretation of non-Abelian gauge theories which are foundations for some of the most important achievements of recent decades, namely: electroweak theory and quantum chromodynamics (Goddard and Olive, 1978).

This profound correlation of monopole studies and quantum field theory, makes it necessary to have a fast and brief review of some of the general concepts of QFT in this section. These concepts are widely used in the next chapters of this thesis.

2.2.1 The Beginning

The first person, who realized the electromagnetic phenomena would be most simply explained in terms of electromagnetic fields, was Michael Faraday (1832). However, the one who formalized this idea and created the mathematical concept of electromagnetic field concept, was James Clerk Maxwell (1881). The previous version of physics was based on the Newtonian approach to fundamental laws of physics in which the forces among the particles explain the behaviour of the physical system.

The major importance and advantage of the field concept was not emerged until the time of appearance of special relativity in the beginning of 20th century and the fields were only used in order to illustrate the behaviour of natural forces on the material particles. In the classical picture of Newtonian physics, any particle in any position in space instantly interacts with all other particles at any arbitrary point of the space regardless of the distance between these particles. Special relativity however, indicated that no physical interaction can occur instantly and any interaction needs a minimum time of $t{=}d/c$ in which d is the distance of interactive particles and c is the speed of light in the vacuum. This implies that any interaction between two particles at a given time is deduced from their previous position at an earlier time.

This new picture of physics was only possible in case of the presence of the fields for interactions which fill the space with a limited speed. This concept of locality (or better saying the causality) was not previously formulated but somehow was accepted by experimental physicists who believed that if they isolate their experiments, any previously done experiment will be reproducible (Wilczek, 1999).

In 1920s and 1930s the new quantum mechanics theory caused a new question to rise. Are the historic ideas of fields and locality valid for extrapolated cases of subatomic world of quantum states or not? The answer to this question was not an easy one and trying to respond, caused a completely new concept to be developed; the Quantum Field Theory. Today we know that in order to be truly relativistic we need to use local symmetries in our physical theories (Wilczek, 1999).

In the relativistic speed domains the classical approximations are not valid any more. The study of fundamental particles in this energy scale is known as high energy physics. Also in particle physics, sometimes the ranges of the interactions are of the order of nuclear sizes i. e. 10^{-16} m. If we consider this length as the de Broglie wavelength of a particle, its energy will be of the order of 10^{10} GeV (Perkins, 2003). That is why we refer to this area of physical researches as *High Energy Physics*.

The successful background of relativity as a theory of high energy systems, and the quantum mechanics as the description of small scale phenomena, caused a widespread effort between physicists in order to produce a mathematical framework which somehow includes both of these illustrious theories. Such a theory must probably include a wave function for each single particle and thus, there should be a wave equation like the Schrödinger's wave equation.

2.2.2 Klein-Gordon Equation

The first step was obviously the Klein-Gordon equation. Special relativity proposes that for the energy-momentum 4-vector (p^{μ}) of any particle of mass m, we have:

$$p^{\mu}p_{\mu} = \frac{E^2}{c^2} - \vec{p}.\vec{p} = m^2c^2, \tag{2.16}$$

where E stands for total energy of the particle. Substitution of the energy and momentum operators from standard quantum mechanics enables us to generate an operational equation equivalent to Schrödinger's wave equation as below

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{m^2c^2}{\hbar}\phi = 0, \tag{2.17}$$

in which ϕ is supposed to be the wave function. In the natural units of mass, where $\hbar = c = 1$ and using the sign convention of the metric of $g_{\mu\nu} = (-+++)$, the equation reduces to

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = (-\partial^\mu \partial_\mu + m^2)\phi &= 0 \\ \text{or,} \quad (\Box + m^2)\phi &= 0, \end{aligned} \tag{2.18}$$

where the second form is written using D'alembertian operator. The most obvious and important difference of the Klein-Gordon equation with the Schrödinger's equation is that Klein-Gordon equation is second order differential equation with respect to time, while the Schrödinger's equation is a first order one. This causes a drastic difference in the physical interpretation of the wave function ϕ in this equation and the standard wave function in quantum mechanics. In the standard quantum mechanics, the probability density is defined as

$$\mathscr{P} = \phi^* \phi. \tag{2.19}$$

The probability is a locally conserved quantity and therefore must satisfy the continuity equation,

$$\frac{\partial \mathscr{P}}{\partial t} + \vec{\nabla}.\vec{j} = 0, \tag{2.20}$$

where $\vec{j}$ is the probability current. In relativistic case, $\mathscr{P}$, must be the time component of the 4-vector $\mathscr{J}^\mu$, which satisfies the continuity equation as below

$$\partial_\mu \mathscr{J}^\mu = 0. \tag{2.21}$$

Therefore the quantity, $\mathscr{P}$, or as is called in quantum mechanics, the probability density must be written in the form of

$$\mathscr{P} = \frac{i\hbar}{2m}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right). \tag{2.22}$$

We expect the probability density function to be positive definite. However, as can be seen from Eq. (2.22), $\mathscr{P}$ can acquire negative values as well as positive ones. The characteristic property of being second order for Klein-Gordon equation, makes it possible for its wave functions and their time derivatives to choose any arbitrary value at a given space-time. According to Eq. (2.22), this would cause negative values for $\mathscr{P}$. So, the interpretation of ϕ as the wave function of a single particle, cannot be a valid physical proposition.

The other fundamental difficulty which is raised with Klein-Gordon equation is the problem of negative energies. The relativistic origins of Klein-Gordon equation makes it possible for the wave function to acquire negative energy values. As can be seen from Eq. (2.16), this equation does not forbid negative values for energy.

2.2.3 Dirac Equation

The pursue for an equation of the first order which satisfies the relativity as well, led Dirac to his famous equation (Dirac, 1928). The origins of this new equation are quite different from the origins of Klein-Gordon equation. The contribution of the special relativity is tried to be imposed by its corresponding transformation, the Lorentz transformation. Indeed, the method used by Dirac in order to derive his equation, is much more important for us in comparison with the Dirac equation as the result. We will review his method briefly here (Ryder, 1996).

We know that any rotation (as a transformation) in three dimensions is correlated to a transformation in a two dimensional unitary space. This correspondence between SU(2) and SO(3), can be illustrated as

$$U = e^{(i\vec{\sigma}\cdot\vec{\theta}/2)} \leftrightarrow \quad R = e^{(i\vec{J}\cdot\vec{\theta})}, \tag{2.23}$$

where U is the operator of transformation in two dimensional space, R is the rotation in three dimensions, $\vec{J}$ is the angular momentum vector, $\vec{\sigma}$ is the vector of Pauli matrices and $\vec{\theta}$ is the angle of rotation. Both groups have the same Lie algebra and there is a two to one mapping between the elements of two groups. Now we consider a Lorentz transformation consisting of a Lorentz boost which is given by

$$
\begin{aligned}
x'^{\mu} &= \Lambda^{\mu}_{\nu}\; x^{\nu}, \\
\gamma &= (1-v^2/c^2)^{-1/2}, \\
\beta &= v/c, \\
\gamma &= \cosh\phi, \quad \gamma\beta = \sinh\phi, \quad \beta = \tanh\phi,
\end{aligned} \tag{2.24}
$$

where v is the relative velocity of two inertial frames, in direction of the Lorentz boost and ϕ is an angle in Minkowski space. Note that the condition of $\gamma^2 - \beta^2\gamma^2 = 1$, is satisfied by all of the definitions of γ and β in Eq.(2.24). If we call the generators of this transformation in $x,\ y$ and z directions, $K_x,\ K_y$ and K_z, we will see that these generators do not form a closed algebra under commutation. This means that the pure Lorentz boost is not a group. Indeed the commutator of the boost matrices in two different directions, gives the component of the angular momentum in the third direction. So, the commutation relations are

$$
\begin{aligned}
[K_i, K_j] &= -i\, \varepsilon_{ijk}\, J_k, \\
[J_i, K_j] &= i\, \varepsilon_{ijk}\, K_k, \\
[J_i, J_j] &= i\, \varepsilon_{ijk}\, J_k.
\end{aligned} \tag{2.25}
$$

This means that 'J's and 'k's form a closed algebra under commutation and therefore, Lorentz boosts beside rotations in three directions, form a group. This group with the Lie algebra of Eq.(2.25) is known as Poincaré group. Like the relationship between SO(3) and SU(2), there is a similar relationship between Poincaré group and SL(2,C). The generators of SL(2,C) form a closed commutation algebra as below (Mulders, 2012)

$$
\begin{aligned}
[\Sigma_i/2, \Sigma_j/2] &= -\,\varepsilon_{ijk}\,\sigma_k/2, \\
[\sigma_i/2, \Sigma_j/2] &= \varepsilon_{ijk}\,\Sigma_k/2, \\
[\sigma_i/2, \sigma_j/2] &= \varepsilon_{ijk}\,\sigma_k/2,
\end{aligned} \tag{2.26}
$$

in which $\Sigma_i = -i\sigma_i$ where $\sigma_i,\ \sigma_j$ and σ_k are Pauli matrices. Noting that, Pauli matrices are the generators of SU(2), SU(2) is a subgroup of SL(2,C). SL(2,C) (or roughly saying, the Lorentz group) is originally SU(2)$\times$ SU(2) with two inequivalent representations of

$$
\begin{aligned}
\phi_R &\rightarrow e^{(i\vec{\sigma}/2\cdot(\theta\hat{n}-i\vec{\phi}))}\phi_R, \\
\phi_L &\rightarrow e^{(i\vec{\sigma}/2\cdot(\theta\hat{n}+i\vec{\phi}))}\phi_L.
\end{aligned} \tag{2.27}
$$

where ϕ_R and ϕ_L are two different kinds of two-component spinors and **n** is the unit vector in the direction of the transformation boost. Each one of these two spinors transform to the other one under the parity transformation so, we call ϕ_R as the right handed spinor and ϕ_L as the left handed one. In order to form a spinor including both parity states, we define the four-spinor, ψ,

$$
\psi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}. \tag{2.28}
$$

Now, suppose that we study a case with a pure boost without any rotation. Then the transition reduces to

$$
\phi_R \rightarrow e^{\vec{\sigma}\cdot\vec{\phi}/2}\phi_R = [cosh(\phi/2) + \vec{\sigma}\cdot\hat{n}\ sinh(\phi/2)]\phi_R, \tag{2.29}
$$

where $(\vec{\sigma\cdot} n)^2 = 1$, is used. If we show the spinor of a particle at rest with $\binom{\phi_R(0)}{\phi_L(0)}$ and the spinor of a particle which has acquired the momentum $p\mathbf{n}$ with $\binom{\phi_R(\mathbf{p})}{\phi_L(\mathbf{p})}$, then using Eq.(2.24) we can write (Ryder, 1996)

$$\begin{aligned}\phi_R(\vec{p}) &= \frac{E+m+\vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}}\,\phi_R(0),\\ \phi_L(\vec{p}) &= \frac{E+m-\vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}}\,\phi_L(0).\end{aligned} \tag{2.30}$$

Since, it is not possible to distinguish between left handed and right handed spin for a particle at rest (refer to the Stern-Gerlach experiment for example), then $\phi_R(0) = \phi_L(0)$ and we can write

$$\begin{aligned}\phi_R(\vec{p}) &= \frac{E+\vec{\sigma}\cdot\vec{p}}{m}\,\phi_L(\vec{p}),\\ \phi_L(\vec{p}) &= \frac{E-\vec{\sigma}\cdot\vec{p}}{m}\,\phi_R(\vec{p}).\end{aligned} \tag{2.31}$$

where $(\vec{\sigma}.p)^2 = \mathbf{p}^2$, is used. In the matrix form we have

$$\begin{pmatrix} -m & p_0+\vec{\sigma}\cdot\vec{p} \\ p_0-\vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix}\phi_R(\vec{p}) \\ \phi_L(\vec{p})\end{pmatrix} = 0. \tag{2.32}$$

This is the matrix form of the Dirac equation for a particle with mass m and spin $\hbar/2$. For the massless particles two separate equations will be obtained from Eq.(2.32). Using the 4×4 Dirac matrices (in chiral representation) which are defined as

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},\ \ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.33}$$

we can simplify Dirac equation for the four-spinor $\psi(p)$ as

$$(\gamma^\mu p_\mu - m)\psi(p) = 0. \tag{2.34}$$

It is easy to check that Dirac matrices of Eq.(2.33), satisfy the condition of $\{\gamma^\mu,\ \gamma^\nu\}$=2 $g^{\mu\nu}$

which is imposed by Eq.(2.16). The differential form of the Dirac equation then becomes

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(p) = 0, \tag{2.35}$$

which is a first order equation. Using this form of the Dirac equation we can show that the continuity equation of the form of (2.21) is valid for Dirac equation and the $\mathscr{P}$ is given by

$$\mathscr{P} = \mathscr{J}^{0} = \psi^{\dagger}\psi. \tag{2.36}$$

Obviously, this is positive definite and we can consider $\mathscr{P}$ as the probability density. To find the energy eigenstates of Dirac equation, we have to move into standard representation of Dirac matrices (instead of chiral representation) which is given by

$$\gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}, \ \gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.37}$$

Using Eqs. (2.34) and (2.37), for a particle at rest, it's now straightforward to see that the energy eigenvalues are $+m$ (twice) and $-m$ (twice). So, the negative energy problem still remains. Dirac's idea of considering the vacuum as a sea of particles with spin $\hbar/2$ and negative energy however, solves this problem. Based on this idea an occasionally not occupied state in the sea of particles, can be filled by an electron (or any particle with spin $\hbar/2$) and during this process, a quantum of energy must be emitted. Since the vacuum is electrically neutral, the hole which is later occupied with the electron, must have originally positive electric charge.

This is the prediction of antiparticles for all kinds of particles with spin $\hbar/2$. Experimental observation of all these predicted particles, brought a great reputation for Dirac equation. Indeed the Dirac spinors, are not single particle wave functions and a two-particle (particle and antiparticle) state is indicated by them. Another significant finding of field theory as mentioned above, was the new interpretation of the vacuum while before Dirac's finding, it was generally believed that vacuum is nothing.

For particles with spin 1, two different sets of equations must be used. We know that photons as spin 1 particles, obey the Maxwell electromagnetic equations. Indeed Maxwell equations are suitable only for photons as massless particles with spin 1. For those massive particles which have spin 1, the corresponding field equations are Proca equations. With the same method of the study of the behaviour of spinors under the Lorentz transformation (which we used to derive Dirac equation), it is possible to derive both of above mentioned equations for spin 1 particles (Ryder, 1996). All of these equations are relations for spinors in each case. Proca equation is not gauge invariant and it's out of our survey. However, we will briefly discuss about the Maxwell equations in the next chapter and we will see how we can derive them from fundamental concept of the symmetry.

The next important step in quantum field theory, was canonical quantisation. Theoretically, Klein-Gordon equation cannot be meaningless because it is constructed based on accepted physical concepts of special relativity and quantum mechanics. Then, why the probability density becomes negative? Physicists, after Dirac's important findings, realized that the wave function in new relativistic quantum mechanics cannot be interpreted as a single particle function. Therefore a new interpretation of the wave functions aroused. This new concept indicates that any wave function given by Klein-Gordon (or Dirac) equation explains a group of particles instead of a single particle.

2.2.4 Canonical Quantisation

Indeed, any wave function in quantum field theory, helps us to calculate the number of particles in each quantum state at any given time. In this new perspective, any wave function, including

Dirac and Klein-Gordon functions, can be expanded in terms of the solutions of wave equations. The coefficients of this expansion are the annihilation and the creation operators which destroy or produce the particles with given energy (Srednicki, 2007).

Canonical quantisation and its consequences, shed light on the processes of radiation and absorption and in wider perspective on all physical interactions. Quantum electrodynamics (QED) as the first step, obtained great success in accurate calculation of natural quantities. For example, the magnetic dipole moment of muon predicted by QED matches very well with recent experimental results (Roberts, 1992). This theory, beside its triumphs in explanation of emission and absorption of the photon as a massless particle, prepared the basis for the next level of theories in which the emission and absorption of massive particles were investigated. Fermi's theory of beta decay was the first try in this context.

Explanation of the force between two objects with the idea of particle exchange between them was the direct consequence of the new interpretation of the annihilation and creation of particles. This relation between fields and particles proposes that for example the origin of electromagnetic field which is supposed to be the responsible for electromagnetic interaction between two charged objects, is in fact the exchange of virtual photons between them. Yukawa who was thinking that the pions are responsible for strong nuclear forces, used this idea to calculate the mass of pions considering the short range of these interactions inside the nuclei.

The success of electroweak theory which predicted the existence of intermediate vector bosons, as the particles which are responsible for weak forces, was the most powerful evidence for the validity of this theoretical postulate.

Another general finding of quantum field theory has been the Charge-Parity-Time Reversal CPT theorem. The violation of some of the conservation laws of physics (which previously used to be assumed as fundamental conservation laws) caused mixed conservation laws to arise. As we will see in the next chapter briefly, any symmetry in the nature is responsible for a conservation law. The *CPT* theorem indicates that the product of charge conjugation, parity and time reversal is the essential symmetry of the world. Within this context, a new (and more exact) definition is introduced for antiparticles. An antiparticle is *CPT* conjugate of its corresponding particle (Wilczek, 1999).

2.3 Principle of Stationary Action

Principle of stationary action which is the modified version of variational principle, indicates that for any variation of a physical system between an initial point and a final point in space-time, the system takes the path through which the variation of action (to first order) is zero. The action of a particle in classical mechanics is given by

$$\mathscr{S} = \int_{t_1}^{t_2} L(q, \dot{q}) dt, \tag{2.38}$$

where L is the Lagrangian of the system and $\dot{q}$ is the time derivative of the generalized coordinate q. The principle of stationary action provides the basis for a very powerful framework for classical dynamics in which using the generalized coordinates, enables us to work in any arbitrary coordinate system. Also this method makes it easier to impose a constraint to the dynamical systems and study the consequences.

This framework which has an *energy approach* to the physics (instead of Newtonian *force approach*) is known as Lagrangian formulation (or equivalently Hamiltonian formulation). The principle of stationary action can be directly used to derive the Euler-Lagrange equation which is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{2.39}$$

Euler-Lagrange equation is indeed equation of motion of individual particle characterised with $L(q,\dot{q})$. The Lagrangian method because of its *energy approach* (in comparison with the *force approach*) is much more flexible and therefore is widely used in quantum mechanics and relativity.

In order to use this flexible method for the fields, it's enough to substitute the generalized coordinate $q(t)$ with the wave function $\phi(x^\mu)$. Therefore, the form of the action and the Euler-Lagrange equation changes to

$$\begin{aligned} \mathscr{S} &= \int \mathscr{L}(\phi, \partial_\mu \phi) d^4x, \\ \frac{\partial \mathscr{L}}{\partial \phi} &= \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right], \end{aligned} \tag{2.40}$$

in which $\mathscr{L}$ is the Lagrangian density function but for simplicity we will call it as Lagrangian. This means that by using the Eq. (2.40) and a proper Lagrangian for any physical system, we can derive the equivalent wave equation. For example for the Lagrangian of

$$\mathscr{L} = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \; \partial_\beta \phi - \frac{m^2}{2} \phi^2, \tag{2.41}$$

the Euler-Lagrange equation reduces to

$$(\partial_\mu \partial^\mu + m^2)\phi = 0. \tag{2.42}$$

which is the Klein-Gordon equation.[1]

One of the reasons which makes us highly interested in this formulation is that using this method we can investigate the symmetries of the physical systems and as we mentioned briefly in previous section, symmetries play an important role in different versions of field theories.

[1]Note that the sign convention for metric which is used here is $g^{\alpha\beta} = (+---)$. That is why the forms of Eqs. (2.18) and (2.42) are slightly different.