

**EXPLORING A-LEVEL STUDENTS'
MATHEMATICAL FLEXIBILITY AND
ADAPTIVITY THROUGH CONTINUAL
EXPOSURE TO MULTIPLE-SOLUTION TASKS
AND STRATEGIES**

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UNIVERSITI SAINS MALAYSIA

2016

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by

LOW CHEE SOON

**Thesis submitted in fulfillment of the requirements
for the degree of
Doctor of Philosophy**

January 2016

ACKNOWLEDGMENT

It was probably a predestined click on the USM web page to find a fatherly man who accepted me, without much hesitation, as his first PhD student some four years ago. I am deeply appreciative to my main supervisor, Dr. Chew Cheng Meng, and my co-supervisor, Professor Dr. Lim Chap Sam, for their invaluable thrust which infused my academic journey with their encouragement, trust, and confidence. Their remarks and sharing were always so concise but highly relevant, in-depth, and thought-provoking. The learning process, with seemingly incessant trials on the rise and fall, has been a great learning experience and a booster to my determination to complete the study.

I am equally appreciative to Associate Professor Dr. Ong Saw Lan for her generous sharing at the steepest point of my PhD journey. Her sharing has injected much courage for me to continue the journey not to the end but a new beginning.

I owe huge debt to the Ministry of Education for sponsoring my studies via MyBrain15, which has to a great extent been financially supportive.

I am also greatly appreciative to my immediate superiors, Mr. Yeoh Ewe Aun and Ms Lee Bui Bui, for their invaluable supports and trusts. Without the adjustment of my workload and the permission to carry out the research during work hours, I would not be able to complete the study smoothly.

Certainly, my deep appreciation is devoted to my lovely Nei Tze and two pet daughters, Cordy and Lily. Their company, encouragement, and laughter have been such enlightening and energizing. They have colored my world with a meaningful journey.

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**MENEROKA TAHAP FLEKSIBILITI DAN ADAPTIVITI MATEMATIK
PELAJAR-PELAJAR PARAS-A MELALUI PENDEDAHAN BERTERUSAN
KEPADA MASALAH DAN STRATEGI PELBAGAI PENYELESAIAN**

ABSTRAK

Kajian ini menggunakan reka bentuk kajian gabungan untuk meneroka strategi penyelesaian pelajar Paras-A bagi menyelesaikan masalah pelbagai penyelesaian, tahap fleksibiliti dan adaptiviti matematik, punca-punca pencapaian dan pemilihan strategi penyelesaian mereka. Sampel kajian melibatkan 24 hingga 32 pelajar Paras-A pada peringkat kajian yang berlainan. Kajian ini melalui tiga fasa kual, QUAN dan QUAL yang bertindihan di mana strategi penyelesaian peserta bagi masalah pelbagai penyelesaian, secara berulang-ulang, dikaji untuk ditaksiran dalam tahap fleksibiliti dan adaptiviti bagi analisis statistik dan perbandingan, dan dikajikan selanjutnya dengan temu duga. Para peserta didedahkan masalah pelbagai penyelesaian di sepanjang masa kajian ini. Hasil kajian awal menunjukkan tahap fleksibiliti dan adaptiviti yang rendah dalam penyelesaian peserta. Adalah didapati bahawa peserta jarang mencuba strategi pelbagai penyelesaian semasa sekolah menengah disebabkan mentaliti ujian dan soalan peperiksaan yang cuma memerlukan strategi penyelesaian yang tunggal. Walau bagaimanapun, tahap fleksibiliti dan adaptiviti didapati meningkat di sepanjang kajian, walaupun tahap peningkatan didapati berbeza bagi masalah-masalah matematik yang berlainan. Keputusan ini mencadangkan bahawa pendedahan yang berterusan terhadap pelbagai strategi dan penyelesaian masalah matematik meningkatkan tahap fleksibiliti dan adaptiviti. Secara umumnya, elemen-elemen matematik yang mempengaruhi tahap fleksibiliti dan adaptiviti peserta-peserta secara lebih ketara merangkumi persepsi

yang fleksibel, pengetahuan konsep, pembelajaran masalah dengan pelbagai penyelesaian, kecenderungan untuk meneroka, dan kebolehan menggambarkan masalah matematik. Khususnya, strategi-strategi penyelesaian yang memerlukan pengaturan semula masalah matematik dan penukaran representasi (contohnya, representasi simbolik kepada geometri) didapati lebih memerlukan kebolehan pemikiran. Terutamanya, keupayaan untuk mengalih persepsi di antara elemen-elemen masalah matematik ‘gestalt’ berkemungkinan merupakan faktor kritikal dalam mendirikan penyambungan dengan pengetahuan konsep yang sedia ada. Tanpa fleksibiliti persepsi sebegini, pengetahuan konsep tidak dapat diaktifkan. Akan tetapi, peserta-peserta didapati menunjukkan adaptiviti persendirian. Walaupun mereka mencapai tahap fleksibiliti dan adaptiviti yang tinggi, namun penggunaan strategi-strategi penyelesaian didapati bergantung kepada konteks. Terutamanya, peserta-peserta didapati lebih cenderung kepada penggunaan strategi-strategi penyelesaian yang sering digunakan dan lebih yakin, khasnya semasa pemeriksaan walaupun mereka memiliki pengetahuan strategi yang lebih efisien. Pemerhatian ini menunjukkan kepentingan dalam memupuk dan menggalakkan fleksibiliti dan adaptiviti pada peringkat awal sebelum berlakunya pelekatan kukuh kepada strategi penyelesaian yang tunggal.

**EXPLORING A-LEVEL STUDENTS' MATHEMATICAL FLEXIBILITY
AND ADAPTIVITY THROUGH CONTINUAL EXPOSURE TO
MULTIPLE-SOLUTION TASKS AND STRATEGIES**

ABSTRACT

This study employed a mixed methods research design to explore A-Level students' solution strategies for multiple-solution tasks, their levels of flexibility and adaptivity, and the reasons for their performance and solution choices. The sample included 24 to 32 A-Level students who participated at different stages of the study. The study underwent three overlapping qual, QUAN, and QUAL phases, whereby the participants' solution strategies for multiple-solution tasks were iteratively analyzed, quantified into levels of flexibility and adaptivity for statistical analysis and comparison, and further analyzed with ensuing interviews. The participants were continually exposed to multiple-solution tasks and strategies throughout the study. Initial findings generally revealed low levels of flexibility and adaptivity in the participants' solutions. It was found that they rarely attempted multiple-solution strategies in their secondary studies because of for-the-test mentality and examination requirements for only a solution to a task. Throughout the study, however, their levels of flexibility and adaptivity increased considerably, though to varying extents across tasks, suggesting positive impact of regular exposure to multiple-solution tasks and strategies. The mathematical elements which appeared to be more saliently influential on the participants' flexibility and adaptivity include perceptual flexibility, conceptual knowledge, learning of multiple-solution tasks and strategies, exploratory disposition, and problem representational skills. In particular, solution strategies which required task reconfiguration and representational

switching (i.e. symbolic to geometric representations) were found to be more cognitively demanding. Of particular interest was the finding that the ability to flexibly switch perceptually among possible gestalts of task elements could be critical in establishing connections with existing conceptual knowledge, which may not necessarily be activated without such perceptual flexibility. However, the participants were found to carry with them self-perceived adaptivity whereby a solution strategy may not be favored owing to its efficiency per se. While the participants had attained higher levels of flexibility and adaptivity, their use of solution strategies appeared to be contextually dependent. In particular, the participants preferred familiarity and certitude to efficiency of solution strategies during examinations despite the knowledge of more efficient solution strategies. The observation implies the significance of nurturing flexibility and adaptivity from young before a strong rigid attachment to—the only—learned strategy is established.

CHAPTER 1

INTRODUCTION

1.1 Introduction

The inclination toward strategic flexibility is a part of human nature. Humans naturally employ various strategies in solving problems flexibly by adapting to situational factors (Lemaire, Arnaud, & Lecacheur, 2004; Siegler, 1999; Verschaffel, Torbeyns, Smedt, Luwel, & Dooren, 2007). This nature makes flexibility and adaptivity important in problem solving. Without being flexible, a person will hardly be creative, nor will he be adaptive, owing to flexibility being a key component of both creativity (Guilford, 1959; Torrance, 1969) and adaptivity (Baroody & Dowker, 2003). Both constructs of creativity and adaptivity embody the ability of transfer to apply learned (mathematical) concepts *flexibly* in dealing with new, unfamiliar situations and domains. Such emphasis on flexibility in the learning of mathematical problem solving as a stepping stone to creativity and adaptivity should therefore receive due attention.

The aspiration of Malaysia to attain a full-fledged industrialized and developed status has been leading the nation to steadily progress toward Vision 2020. One critical thrust driving toward this direction is no other than the role played by an educational system which embodies sound educational principles, policies, and plans. In the blueprint of Education Development, the Ministry of Education in 2007 explicitly proclaimed the importance of, among others, critical and creative thinking as well as the mastery of problem solving skills to be able to face with the challenges and demands of the ever-changing global environment (Ministry of Education, 2007). Further, effort has been intensified since the inception of Curriculum Development

Centre toward shaping an integrated curriculum aimed at the provision of holistic, quality education (Rajendran, 2011; Zamrus & Mokelas, 2011). In particular, the secondary Mathematics curriculum has been designed and undergoing progressive improvement to facilitate the acquisition of not only mathematical knowledge, but higher-order mathematical thinking and problem solving skills so as to prepare students with sound decision making capability applicable in their daily lives (Ministry of Education, 2003, 2004a, 2004b, 2005, 2006). As explicitly stated in *Integrated Curriculum for Secondary Schools* (Curriculum Development Centre, 1989, p. 2), one objective of secondary school education in Malaysia is to “develop and enhance their (student’s) intellectual capacity with respect to rational, critical and creative thinking”, which is an integral element in sound problem solving. This educational objective has underlined the school opportunity of learning flexibility and adaptivity, which essentially form the backbone of creativity and sound problem-solving skills.

Rousseau and Tate (2008) commented that the lack of access to quality mathematics education is likely to limit human potential and individual economic opportunity. This notion is particularly true in view of the ever-increasing demands of rapidly-evolving work environments which require knowledgeable, creative, flexible and adaptive workforce. Replication in use of knowledge is no longer adequate to support problem solving. Conversely, individuals are required to be able to apply knowledge *flexibly* in new domains and different situations (Grabinger & Dunlap, 1995) and it is important to ensure students acquire critical and creative thinking in their learning experience as espoused in the Malaysian Mathematics and Science curriculums (Hong, Ting, & Hasbee Hj Usop, 2009).

Any educational vision and aims, as mentioned above, will remain as grandiose, conceptual ideals without careful, systematic plan and mechanism to transform them into pragmatic classroom experience. A consonant educational view was expressed by the National Council of Teachers of Mathematics (NCTM, 1991) with further elaboration that students should be engaged in mathematical discourse about problem-solving, which includes discussing *different solutions and solution strategies* for a given problem, as well as how solutions can be extended and generalized. Similarly, the National Research Council (1989) asserted that mathematical learning should entail motivation for moving beyond just mathematical rules to also focus on seeking solutions (i.e. not just *a* solution by memorizing procedures), exploring patterns (i.e. not just memorizing formulas) and formulating conjectures (i.e. not just doing exercises).

Obviously, educational reforms in mathematics learning has generally converged to an endeavor to produce students who are *real* problem solvers with rich experience in exploring mathematics dynamically rather than rigidly with some absolute, closed body of laws to be memorized. As stated by Schoenfeld (1992, p. 335), “learning mathematics is empowering. Mathematically powerful students are quantitatively literate They are *flexible* thinkers with a broad repertoire of techniques and perspectives for dealing with novel problems and situations.” This study is an initiative orientated towards such a goal: To cultivate in students attributes of flexibility and adaptivity armed with diverse mathematical thinking and strategies readily accessible in the process of mathematical problem solving. Such emphasis on conceptual diversity is important as it would facilitate the understanding of mathematical nuances and their applications (Zbiek & Shimizu, 2005).

This study was based on a mixed methods design aimed at exploring A-Level students' flexibility and adaptivity specifically in the use of solution strategies for solving multiple-solution (mathematical) tasks at secondary level. In the study, flexibility was referred to as the ability to produce conceptually-varied solutions, while adaptivity the ability to employ relatively more efficient strategies. Throughout the study, the participants were continually exposed to alternative solutions to A-Level mathematical tasks which were amenable to multiple strategies founded on varied concepts. The benefits of multiple-solution approach to learning mathematics have well been established (Alibali, 1999; Elia, Heuvel-Panhuizen, & Kolovou, 2009; Greer, 2009; Newton, Star, & Lynch, 2010; Rittle-Johnson, Star, & Durkin, 2009; Star & Newton, 2009; Verschaffel et al., 2007). The learning of multiple-solution tasks and strategies draw on alternative concepts, whereby students would gain greater opportunity for conceptual development via the applications of diverse mathematical concepts in problem solving. The researcher hypothesized that such multiple-solution approach to learning mathematics would provide students the opportunity to conceptually compare and contrast among various solution strategies, thus facilitate the development of adaptivity undergirded by flexibility. This study was thus in line with the ever-increasing focus on conceptual understanding, not only basic computational skills, in mathematics education in Malaysia over the past decades (Noor Azlan Ahmad Zanzali, 2011).

1.2 Flexibility and Adaptivity in Mathematical Problem Solving

A problem exists when it constitutes a situation in which one needs to find a means to reach some goal (Chi & Glaser, 1985). Reiterated in another way, a problem is a confrontation, the solution of which is not immediately obvious

(Garnham & Oakhill, 1994). Problem solving is then a process whereby an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation (Krulik & Rudnick, 1989). A parallel description has been stated by Polya (1980):

To solve a problem is to find a way where no way is known off-hand, to find a way out of a difficulty, to find a way around an obstacle, to attain a desired end, that is not immediately attainable, by appropriate means. (p.1)

Based on the above definition of problem solving, mathematical problem solving can be construed as a process in which some mathematical problem is to be solved, without an obvious initial clue of solution, by means of some appropriate mathematical concepts. Schoenfeld (1985), however, cautioned that a task, which constitutes a *problem*, must impose an intellectual impasse, which occurs in relation to the problem solver. As such, a mathematical task could be a *problem* to the weaker students, but an *exercise* to more capable students, who immediately know about the solution learned from their past experience. A similar distinction between real *problems* and mere *exercises* has also been highlighted by Kantowski (1980) earlier.

Owing to the presence of intellectual impasse and the need for a wide range of such problem-solving skills and attributes as heuristic skills, conceptual understanding, and procedural fluency, mathematical problem solving is generally viewed as a complex process. It is thus a great challenge from the educational perspective to impart problem solving skills to students. In particular, the relation between conceptual and procedural knowledge is really an intricate issue (Star, 2000). However, the nature and characterization of mathematical problem solving seem to have undergone a long evolutionary journey, seeing a battle between procedural skills and conceptual understandings, with only the latter perceived as being

associated with more meaningful mathematical thinking and reasoning (Schoenfeld, 1992). In the early years of mathematics education, problems were basically viewed as routine exercises consisting of tasks to be done by students based on specific mathematical techniques demonstrated by the teacher (Schoenfeld, 1992). The emphasis was mainly on learning and applying computational algorithms (Musser & Shaughnessy, 1980). The ability to apply the taught procedures to solve problems in relation to particular task features essentially formed the yardstick for the acquisition of mathematical skills. The key means to success was by drill and practice. Based on the historical review by Stanic and Kilpatrick (1989), however, problem solving has played various roles, from supporting contextual problem solving germane to real-world experience to serving as a skill in its very own right, and as an artistic vehicle for tackling problems of considerable complexity. That is this last role of mathematical problem solving with underlying reasoning and conceptual understanding, which was critically called for by Schoenfeld (1992), who further asserted the need and importance of thinking about how students build their own understandings of mathematical topics (from a constructivist viewpoint), rather than just simply focusing on the way to present material clearly (Schoenfeld, 2001). The essence of conceptual understanding and the need for incorporating flexibility in teaching and learning with extensive problem-solving techniques were assertively highlighted:

Instruction should be aimed at conceptual understanding rather than at mere mechanical skills, and developing in students the ability to apply the subject matter they have studied with flexibility and resourcefulness.

Mathematics instruction should provide students the opportunity to explore a broad range of problems and exploratory situations. It should provide students with a broad range of approaches and techniques (ranging from the straightforward application of the appropriate algorithmic methods to the use of approximation methods, various modeling techniques, and the use of heuristic problem solving strategies) for dealing with such problems . . .

(Schoenfeld, 1992, p. 32)

The use of multiple solution strategies as an instrumental pathway for students to experience the flexible application of various strategies based upon sound conceptual understanding seems to fit in nicely with Schoenfeld's arguments. After all, students should not be misled into thinking that a problem can only be solved in a single way, as conveyed by Posamentier and Krulik (1998) that teachers should be aware of and students made known to the many problem-solving strategies (i.e. flexibility) that can be used to provide efficient and elegant solutions to many problems (i.e. adaptivity).

In the development of educational and psychological research pertinent to problem solving, strategy flexibility appears to be a central focus and a key implication of problem solving for teaching mathematics. For instance, upon a lengthy review on various interpretations of problem solving, Branca (1980) concluded with two objectives, among others, of teaching mathematics with problem solving based on a School Mathematics Study Group (MSG) project, namely (a) To provide the student with a variety of strategies for problem solving, and (b) To develop some flexibility in the student's approach to problem solving. Such emphasis on teaching flexibility in problem solving, however, either may not have led to satisfactory success in classroom teaching and learning, or may have remained critical over the past. Literature shows evidence that strategy flexibility (such as use of multiple solutions) has continued to receive due recognition in view of its educational value and potential contribution to problem solving, even over some 30 years since Branca's (1980) article (e.g. Afamasaga-Fuata'i, 2009; Elia et al., 2009; Heinze, Star, & Verschaffel, 2009; Newton et al., 2010; Leikin, 2007, 2011; Star & Newton, 2009; Star & Seifert, 2006; Tsamir, Tirosh, Tabach, & Levenson, 2010;

Verschaffel et al., 2007). Unfortunately, some educators do not seem to value the educational potential of strategy flexibility in teaching mathematics, resulting in a gap between research ideals and classroom practice (Bingolbali, 2011). There is certainly still much effort required to fill the gap between research implications and classroom practice.

The term flexibility, which has been mostly viewed as a key dimension embedded in the broader sense of creativity, primarily refers to switching smoothly between different strategies (Guilford, 1959; Stein, 1974; Torrance, 1969). To operationally define flexibility and distinguish it from adaptivity upon an extensive literature review, Verschaffel, Luwel, Torbeyns and Dooren (2009a) referred *flexibility* to use of multiple strategies, and *adaptivity* to selection of most appropriate strategy. The term ‘appropriate’, however, has not been specifically defined, but is broadly referred to as (a solution choice) dependent on the task in hand, for that particular problem solver and in a particular context (Verschaffel et al., 2009a, Verschaffel, Luwel, Torbeyns, & Dooren, 2009b).

Despite the difficulty to attain unanimous conceptualization and definitions of creativity, adaptivity and flexibility among the researchers owing to the inextricably intertwining attributes of the three constructs, Selter (2009) attempted to extend and refine the concepts of adaptive expertise by Verschaffel et al. (2009a) and Hatano (in Baroody & Dowker, 2003) to lay out more refined, distinctively intricate attributes for the three constructs, namely *creativity* for the ability to invent new or modify known strategies, *flexibility* for the ability to switch between different strategies, and

adaptivity for the ability to use appropriate strategies the individual has creatively developed or flexibly selected.

In an effort to promote flexibility particularly in equation solving, Star and Rittle-Johnson (2007) as well as Star and Seifert (2006) defined flexibility in problem solving as knowledge of (a) multiple strategies and (b) the relative efficiency of these strategies. Apparently, these researchers have perceived efficiency as an attribute within flexibility and a much desired quality in problem solving. However, it seems to others that inculcating the predisposition toward *possible strategies* with profound reasoning rather than just emphasizing *correct strategies* in the learning culture should be of primary concern (such as Baron, 1988; Garnham & Oakhill, 1994; Lithner, 2003, 2008; Stacey & Vincent, 2009; Stein, Grover, & Henningsen, 1996). Such concern coincides with the modern view of arithmetical expertise by Cowan (2003): “By encouraging diverse strategic solutions and requiring students to explain these to others, children will realize that there can be more than one way to work things out and that mathematics is about methods as much as it is about right answers.” (p. 44)

In light of the various views on flexibility, it is necessary to determine a working definition of mathematical flexibility for use in this study. The researcher of this study is in absolute agreement that a student’s mathematical solutions must be associated with sound reasoning and conceptual understanding. And it is a reasonable assumption that conceptual understanding is an implicit attribute in the ability to produce *multiple* solutions and to provide rationales for solution choices.

As such, a student is deemed to be mathematically flexible and adaptive if he exhibits:

- (a) the knowledge of multiple solution strategies,
- (b) the ability to use multiple solution strategies accurately,
- (c) the ability to employ relatively more efficient solution strategies;
- (d) the ability to provide (conceptual or psychological) explanations for his solution choice.

Unlike Star and Rittle-Johnson (2007) and Star and Seifert (2006) who have considered knowledge of relative efficiency of various solution strategies as an intrinsic characteristic of flexibility, the conceptual efficiency of a solution choice is considered as an adaptive quality in this study. In other words, the operational definition of adaptivity by Verschaffel et al. (2009a) has been adopted, conditionally though. That is, the researcher has taken a dual perspective of adaptivity. In light of the nature and requirements of Cambridge International Examinations (CIE), which highly emphasize both solution accuracy and efficiency, an adaptive choice of solution is referred to as a relatively more efficient solution (i.e. requiring relatively fewer concepts, properties and operations) compared with other possible solutions. On the other hand, an adaptive choice of solution is also taken as the most appropriate choice for a particular task, by a particular problem solver, in a particular context. This latter perspective recognized subjective views on solution qualities. Other than strategy efficiency, the student participants in this study may perceive other solution attributes as being adaptive (e.g. familiarity over efficiency during examinations), which formed part of the investigation in this study. In other words,

this study also aimed to investigate the students' perspectives of adaptivity or self-perceived adaptivity.

1.3 Background Information of the Study

This study involved a private college in Penang which offers a wide range of programs, including the Cambridge GCE A-Level program. The college offers three main intakes in a year, namely January, April, and July intakes. Students who join the A-Level program are mainly secondary school-leavers who have just completed their Sijil Pelajaran Malaysia (SPM) examinations (i.e. equivalent to GCE O-Level). They are required to take a minimum of three subjects for entrance into their targeted undergraduate courses as entry requirements set forth by most universities. In recent years, however, increasingly stiff competition has driven more students to take up four subjects in order to gain a competitive edge for entrance into high-rank universities. Mathematics has been one of the most popular subjects, especially among the students from science background. This study involved the post-SPM students from twenty public secondary schools who were enrolled in the Cambridge A-Level Science program at the college in April 2013. They carried with them varied mathematical foundations, contributing to a composition of well-mixed academic strengths, with a higher proportion of high achievers. It was believed that any common mathematical behaviors among the students would reflect to certain extent a prevalent practice among secondary schools.

Some of the A-Level students applied for scholarships offered by the Department of Public Service [Jabatan Perkhidmatan Awam (JPA)] for subsequent pursuance to their first degree courses. The successful applicants of JPA scholarship

were likely to quit the A-Level program sometime April to June to be enrolled in a foundation course (i.e. equivalent to the A-Level or STPM program) at a college or university designated by JPA.

The researcher in his teaching experience has consistently found that post-SPM students who join A-Level program generally demonstrate mathematical solutions characterized by low flexibility and rigid procedures—mostly without sufficient reasoning. The students, including even the academically-high achievers (based on the SPM results), mostly do not have difficulty in producing accurate answers in dealing with typical mathematical tasks. They, however, show a high propensity for monolithic, predetermined solutions to similar problems—a typical trend of mindless reproduction of standard methods, apparently without conscious, deliberate analysis prior to solving mathematical tasks. Such problem-solving behavior is not surprisingly unexpected though. In educational institutions, mathematical performance is rated heavily on accuracy over other qualities (such as efficiency, elegance, clarity). The need for exploring multiple solutions to a task generally is not perceived as necessary. The learning of flexibility in mathematical problem solving with particular respect to use of multiple solutions hence has not been really valued (Bingolbali, 2011). One could reasonably presume that it is by no means a common classroom practice whereby learners are required to produce multiple solutions to the same mathematical tasks in an effort to cultivate flexibility and adaptivity, and to encourage the application of various mathematical concepts which could be equally robust and capable of producing the required answers (Stigler & Hiebert, 1999). As a result, students are deprived of sufficient opportunity to attempt multiple solutions and to make connections among mathematical concepts

and properties. Equally lost is the opportunity to learn by comparing and contrasting essential problem characteristics of various solutions to the same problems which has empirically proven to be critical for conceptual development (Levav-Waynberg & Leikin, 2012; Rittle-Johnson & Star, 2007, 2009). Consequently, students usually do not have the conception of alternative solutions and their potential benefits (i.e. being a vehicle for verification), not to mention the propensity for attempting alternative means when encountering a mathematical impasse and when there is a need for an adaptive solution. That is all this experience with the concern for a positive change in the students' mathematical competency which has mustered the motivation for initiating this study. The ultimate purpose of the study is to optimize the students' performance in their Cambridge examinations.

Simply put, this study explored the A-Level participants' flexibility and adaptivity through the lens of their ability to produce conceptually-varied solutions and employ relatively more efficient strategies. Of particular interest were also to identify the reasons for their performance and the factors determining their solution choices. Throughout the study, the participants were continually exposed to multiple-solution tasks and strategies.

This study adopted a mixed methods research design which drew on the strengths of both quantitative and qualitative studies (Creswell, 2014; Gay, Mills, & Airasian, 2009). The quantitative and qualitative phases were run in parallel with both the quantitative and qualitative data weighted equally. The quantitative and qualitative data were collected iteratively, as schematically illustrated in Figure 1.1.

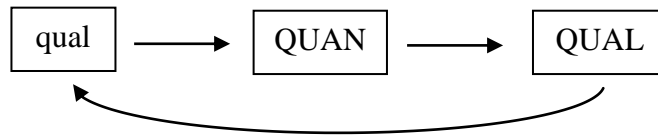


Figure 1.1 An iterative mixed methods research design

A more elaborative representation of the research design which includes timelines of core activities will be presented in Chapter 3.

In the initial qualitative (qual) phase, the participants' solutions to multiple-solution tasks were progressively collected and thoroughly analyzed to identify the kinds of mathematical concepts employed in the participants' solution strategies. The solution strategies were subsequently scored and quantified into various levels of flexibility and adaptivity in the quantitative (QUAN) phase. Such quantification of qualitative data facilitated evaluation and enabled statistical comparison of the participants' performance throughout the study. However, quantifying the participants' levels of flexibility and adaptivity per se would not reveal much of the underlying reasons for their performance and solution choices. Therefore, the qualitative (QUAL) phase, with sampling guided by the quantitative analysis, was intended for an in-depth exploration of those underlying reasons with a few cases. In the qualitative (QUAL) phase, the participants' use of solution strategies and verbal transcripts from interviews formed the main source of data for analyses (Gay, Mills, & Airasian, 2009; Judith, 2008). Interviews conducted were mainly to explore the underlying reasons for the observed (extreme) phenomena (i.e. low levels of flexibility and failure to relate to *learned* mathematical concepts), and the factors underlying the participants' (adaptive) solution choice (i.e. participants' concern

about accuracy, efficiency, familiarity, certitude, etc.). Specific details will be addressed in Chapter 3.

1.4 Problem Statements

Numerous studies have revealed low mathematical achievements and inadequate problem-solving skills among students. Unsatisfactory academic achievements in mathematics have been consistently reported, be it in national examinations or international comparative studies (Hong et al., 2009; Hwa, 2010; Mullis, Martin, Gonzalez, & Chrostowski, 2004). For instance, Trends in International Mathematics and Science Study (TIMSS) recorded a steady decline in Malaysian eighth-graders' mathematics results in 1999, 2003, 2007 with average scores of 519, 508 and 474, respectively (Hwa, 2010). Despite the percentile improvement of Malaysian students from 1999 to 2003 (M. Najib, Rohani, & Ebrahim, 2011), the improvement, however, was attributed to an overall international decline in mathematics achievements according to Mullis et al. (2004). Moreover, the further decline of Malaysian students in 2011 TIMSS results with scores of 440 and 426 in Mathematics and Science, respectively, was remarked by the chairman of Parent Action Group for Education (PAGE), Datin Noor Azimah Abdullah Rahim (2012, December 23), as much below the acceptable average score of 500. Despite the key objective of international testing, i.e. TIMSS and Programme for International Student Assessment (PISA), to emphasize educational effort aimed at promoting pragmatic skills and knowledge to foster the ability to solve problems (Rojano, 2008), there appears to be a confounding gap between students' achievements in mathematics and the intended outcomes (English, 2008). It is frustrating to notice that students who are known to have knowledge sufficient

to solve a problem are unable to employ or adapt this knowledge to solve unfamiliar problems, even with the same concepts required (Even & Tirosh, 2008). Such phenomenon is rather common and could be attributed to cognitive inflexibility, which often leads to failure in transfer. Elen, Stahl, Bromme and Clarebout (2011) described cognitive flexibility as “the disposition to consider diverse context-specific information elements while deciding on how to solve a problem or to execute a (learning) task in a variety of domains and to adapt one’s problem solving or task execution in case the context changes or new information becomes present” (p. 2). The remark obviously speaks highly on the importance of flexibility and adaptivity.

Students generally demonstrate high in procedural but low in conceptual understanding (Beh, Tong, & Che Noorlia, 2006). Such scenario points to a bleak academic outlook in view of the fact that students with low conceptual understanding are likely to have difficulty dealing with problems of greater complexity which demand higher-order thinking and sound conceptual understanding. Noor Azlan Ahmad Zanzali and Lui (2012) evaluated the levels of mathematical problem-solving abilities among 242 Form Four science and non-science students from four schools in an urban district. It was found that the students had limited exposure to problem solving instruction. Despite having fairly good command of basic knowledge and skills, the students generally demonstrated inadequate command of problem solving skills and they were unable to provide reasons and explanations for certain problem-solving procedures with correct and suitable mathematical symbols and vocabulary. Such scenario implied a generally

lacking conditional knowledge among students despite the availability of declarative (i.e. factual) and procedural (i.e. process) knowledge (Schunk, 2009).

The researcher's teaching experience consistently reveals prevalent failures of learners in bridging their learned mathematical knowledge with its application in problem solving—a phenomenon described by Even and Tirosh (2008) as “knowing-about and knowing-to: knowing facts versus knowing to act” (p. 207). Students are commonly found to lack the capacity to integrate learned mathematical concepts for solving mathematical tasks flexibly and adaptively. Students hardly show sufficient experience in the learning of flexibility via the exploration of alternative solutions which is instrumental in promoting divergent thinking (Hopkins, 2010). This observed limitation was supported by the study by Elia et al. (2009) which revealed that intra-task flexibility (i.e. changing strategies within problems) was rare in students' solutions and that students mostly showed single strategies. Learning by comparing, contrasting, and discussing alternative solutions has been shown to be effective for conceptual development (Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson et al., 2009). Unfortunately, such precious learning opportunity does not seem to have been highly valued (Bingolbali, 2011; Stigler & Hiebert, 1999). Majority of the students who join a college for an A-Level program show excellent SPM results (i.e. with high numbers of A's). It is hence not unreasonable to presume that these SPM school leavers—especially those academically-high achievers—should have gathered a rich arsenal of mathematical concepts and strategies at their disposal. Ironically, students tend to elicit surface rather than intrinsic properties of tasks without meaningful relation to deep features and relevant concepts when solving a problem (Hopkins, 2010; Lithner, 2003, 2008;

Yudariah, 1997). Reiterating in Skemp's (1978) terms, they tend to rely more on instrumental rather than relational understanding. The solution strategies employed, though may still produce results in some situations (e.g. examinations), heavily imply conditioned capability to spot key words and mechanical response to surface features of problems rather than with the ability to identify relevant facts, analyze contextual information, establish relationships and explore possible solution choices before deciding on an optimal solution strategy. Such inflexible approach to problem solving without reasoning is believed to have been systematically *learned*, such as via intense training for efficient, automated response, and fueled by an over-emphasis on examinations or for-the-test mentality (Lim, Fatimah, & Tan, 2004). The concomitant results of for-the-test mentality could be the propensity for direct instructions and strategic reliance on single solutions and intense practice, with an aim to optimize examination results. Unfortunately, such for-the-test mentality would lead to unintended consequences, such as teaching-to-the-test syndrome and learned paralysis (Lim, 2009), and be a great hurdle to the development of advanced mathematical or higher-order thinking and to the mastery of mathematical concepts (Peterson, 1988; Tall, 1991).

The rigid, result-orientated approach to teaching and learning mathematics based upon the belief that learning can take place with a mere transfer of facts and students resorting to rote learning are prevalent. Unfortunately, such culture of teaching and learning is far from, if not completely antithetical to, the principles and standards set forth by the National Council of Teachers of Mathematics (Cangelosi, 2003; NCTM, 2000). By encouraging strategy flexibility and adaptivity, this study aimed to help students establish a predisposition to consciously explore various

mathematical concepts/properties germane to a problem and to evaluate alternative solutions before deciding on a most plausible or appropriate solution in a particular context. After all, students should not be led into thinking that there is always one possible solution to most mathematical problems and that a mathematical problem can always simply be solved very quickly (Schoenfeld, 1985; Silver, 1987). And such (inappropriate) belief systems could stand in the way of students' meta-cognitive efforts and be highly influential in mathematical problem-solving performance (Ee, Chang, & Tan, 2004; Garofalo & Lester, 1985; Goos & Galbraith, 1996; Schoenfeld, 1985, 1987a, 1988, 1992).

All the problems discussed above apparently point to the need for an educational effort to ensure the effective learning of mathematics. The researcher conjectured that continual exposure to multiple-solution tasks and strategies would help consolidate students' flexibility and adaptivity and thus enhance mathematical competency. When attempting at solving a task in multiple ways (i.e. being flexible), students would learn that it is the solution and not just the answer which is valued (i.e. a deviation from exam-driven mentality) and that there is not always only one solution to a task. On the contrary, many mathematical tasks could indeed be solved in a variety of ways (i.e. a cognitive disequilibrium to an improper mathematical world view). It is a reasonable belief that attempting at multiple solutions, students are less likely to deliver solutions from memorized facts because of the higher cognitive demand on the students to rummage in the memory any learned mathematical concepts and strategies which could be relevant and effective. Thus, such exercise would orientate learners towards the need for conceptual

understanding and reasoning in mathematical problem solving and naturally facilitate adaptive choice of solution.

1.5 Research Objectives

This research, based upon a mixed methods data analysis design, set out to explore A-Level students' flexibility and adaptivity when they first enrolled in the A-level program at a private college and throughout continual exposure to multiple-solution tasks and strategies. In particular, the study explored the participants' use of solution strategies for multiple-solution tasks, mathematical flexibility (i.e. the ability to produce conceptually-varied solutions to multiple-solution tasks), mathematical adaptivity (i.e. the ability to employ relatively more efficient strategies), as well as the reasons for their varied performance and solution choices. The researcher hypothesized that mathematical competency, thus mathematical performance, could be enhanced if students *learn* to be flexible via deliberate, conscious effort to explore alternative solutions when solving mathematical problems. Specifically, this study was aimed at investigating,

- 1) the solution strategies employed by the A-Level students in dealing with multiple-solution tasks,
- 2) the extents of the A-Level students' mathematical flexibility and adaptivity throughout the study,
- 3) if there are changes in the participants' levels of flexibility and adaptivity throughout the study,
- 4) the possible reasons (if any) underlying the A-Level students' mathematical flexibility and adaptivity,
- 5) the factors determining the A-Level students' solution choices.

1.6 Research Questions

Based on the above research objectives, the following research questions were formulated:

- 1) What are the solution strategies employed by the A-Level students in dealing with multiple-solution tasks?
- 2) To what extents are the A-Level students' mathematical flexibility and adaptivity throughout the study?
- 3) Are there changes in the participants' levels of flexibility and adaptivity throughout the study?
- 4) What are the possible reasons (if any) underlying the A-Level students' mathematical flexibility and adaptivity?
- 5) What are the factors determining the A-Level students' choices of solution?

1.7 Significance of the Study

Numerous studies have linked strategy flexibility to positive student learning and performance. Silver, Ghouseini, Gosen, Charalambous and Strawhun (2005) claimed that comparing, reflecting on, and discussing multiple solution methods would improve student learning. Similarly, Alibali (1999) and Siegler (1995) found that instructional interventions are more effective with students having knowledge of multiple strategies. Students without flexible knowledge are faced with great difficulties in dealing with both near- and far-transfer problems across a range of ages and domains (Hiebert & Carpenter, 1992), pointing to a high possibility that strategy inflexibility is closely related to low academic achievement in mathematics. For instance, in a study of pupils' constructed definitions and use of functional

concepts and their various representations in solving functional problems, Elia, Panaoura, Eracleous and Gagatsis (2007) attributed the pupils' low competence in using different representations of functions in problem solving to the lack of flexibility between different ways of approaching functions. In this current study, the researcher further argues that attempts at various strategies would require connections with and integration of varied mathematical concepts, hypothesizing that strategy variability would accelerate and consolidate the learning and mastery of mathematical concepts. In recent years, many researchers have recognized the importance of strategy flexibility and experimentally assessed the impact of exposing students to multiple solutions as well as students' flexible use of strategies and choice of solution. Strategy flexibility is deemed to have an inextricable relation to gain in procedural and conceptual knowledge and thus enhanced mathematical performance (Elia et al., 2009; Greer, 2009; Heinze et al., 2009; Newton et al., 2010; Rittle-Johnson & Star, 2007, 2009; Star & Rittle-Johnson, 2007). Most of these studies however targeted at mathematics of lower levels with a primary focus on rudimentary summation and subtraction strategies (Baroody & Dowker, 2003) and basic algebraic flexibility, such as in linear equation solving (Rittle-Johnson & Star, 2007, 2009; Star & Rittle-Johnson, 2007). This current study sees an extended effort in the same direction. Furthermore, while conceptual complexity increases in higher-level mathematics (i.e. A-Level mathematics), the learning of algebraic flexibility per se does not seem to be adequate. Higher-level mathematics is generally amenable to conceptually-varied strategies. For instance, symbolic and geometric approaches to solving some mathematical tasks could involve rather different concepts.

In short, this study did not exclusively aim at students' ability to generate unusual, novel solutions but to exhibit some mastery levels of flexibility and adaptivity in relation to fundamental mathematical concepts. It was an effort to explore the possible impacts of increased learners' experience with multiple-solution tasks and strategies in mathematics learning. While students rarely deal with strategy flexibility (Bingolbali, 2011; Elia et al., 2009; Stigler & Hiebert, 1999), attempts at multiple-solution tasks and strategies in association with fundamental concepts are believed to be challenging but motivating (Reeve, 1999)! It is hoped that this study would lead to further insight and research into means of enhancing mathematical thinking and competency in a typical classroom context.

1.8 Limitations of the Study

This study is limited in the following aspects.

External validity

In view of the small sample size and the specific sample of A-Level students, the findings are restricted to low external validity (Bogdan & Biklen, 2007; Merriam, 1998). It is possible that the results produced with participants of a different educational level would differ from those of this study, i.e. younger students might rely more on heuristics than conceptual knowledge owing to limited conceptual structures available to them. In addition, the results could be relatively specific to the tasks employed in the study. It is hence important not to generalize the results to domains beyond the kinds of tasks used in this study without further empirical evidence.

Time constraint

Attempts at conceptually-varied solutions to the same tasks during the study have been found to be extremely time-consuming. At times, the learning process had to be hastened so that the participants' learning pace could reasonably keep up with the syllabus in time for the external Cambridge examinations. This situation concurs with the experience by Ward and Herron (1980) who attributed the lower-than-expected scores for the participants in the Science Curriculum Improvement Study (SCIS) learning cycle to limited time spent on activities. It is not immediately certain if the typical amount of time allocated for A-Level studies needs to be increased should a multiple-solution approach to mathematics learning be incorporated into the curricular structure. Further research is necessary to ascertain if more time allocation would lead to better learning effects.

Inconclusive specific sources of learning effect

In spite of the findings showing some levels of improvement in the participants' mathematical flexibility and adaptivity, the study has not been adequate to point to specific sources of the learning outcomes except continual exposure to multiple-solution tasks and strategies. It is not immediately conclusive, at a more refined level, as to what attributes of multiple-solution tasks and strategies have actually contributed to the participants' improvement. Further studies are warranted to determine if such factors as reasoning, the opportunity to explore and compare various strategies, instructional explanations, or other attributes have individually or jointly contributed to the participants' improvement.