UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Year 2003/2004

October 2003

MST 562 STOCHASTIC PROCESSES

Time: - 3 hours

Please ensure that this exam paper consists of **SIX [6]** printed pages before you begin the exam.

Answer all **FIVE (5)** questions.

1. (a) State and prove the Weak Law of Large Numbers. How does the Weak law differs from the Strong Law of Large Numbers?

(b) Let X_i, i=1,2,...., 10 be independent random variables, each being uniformly distributed over (0,1). Calculate

$$P\left\{\sum_{i=1}^{10} X_i > 7\right\}.$$

(c) Suppose that X is a random variable with mean 10 and variance 15. What can we say about $P\{5 < X < 15\}$?

1. (a) Nyatakan dan buktikan Hukum Lemah Bilangan Besar. Bagaimana Hukum Lemah ini berbeza daripada Hukum Kuat Bilangan Besar ?

(b) Biarkan X_i, i=1,2,...., 10 sebagai pembolehubah rawak (p.u.r) tak bersandar, setiap dengan taburan seragam selanjar pada selang (0,1). Cari

$$P\left\{\sum_{i=1}^{10} X_i > 7\right\}.$$

(c) Jika X ialah pembolehubah rawak (p.u.r) dengan min 10 dan varians 15. Apakah

$$P\{5 < X < 15\}?$$

2. (a) Define the Markov chain and state the Markov property. Show that

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}$$

for all n,m$\geq$0, for all i,j.

(b) A Markov chain X_0, X_1,.... on states 0,1,2 has the transition probability matrix.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array}\right] \end{array}$$

...3/-

Determine the conditional probabilities $P(X_1 = 1, X_2 = 1 / X_0 = 0)$ and $P(X_2 = 1, X_3 = 1 / X_1 = 0)$.

(c) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. Let $\alpha = 0.7$ and $\beta = 0.4$, then calculate the probability that it will rain four days from today given that it is raining today. If we say that the state is 0 when it rains and 1 when it does not rain, then find the limiting probabilities π_0 and π_1.

2. (a) *Takrifkan rantai Markov dan nyatakan ciri-cirinya. Tunjukkan bahawa*

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}$$

bagi semua $n,m \geq 0$, untuk semua i,j.

(b) *Suatu rantai Markov $X_0, X_1, \ldots$ pada keadaan 0,1,2 mempunyai matriks kebarangkalian peralihan*

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

Cari kebarangkalian bersyarat $P(X_1 = 1, X_2 = 1 / X_0 = 0)$ dan $P(X_2 = 1, X_3 = 1 / X_1 = 0)$.

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(c) *Jika kebarangkalian untuk hujan pada keesokan hari bergantung pada keadaan cuaca sebelumnya melalui hanya pada sama ada terdapat hujan pada hari ini dan bukan pada keadaan cuaca pada hari yang lain. Andaikan bahawa jika terdapat hujan pada hari ini, maka wujud kebarangakalian* α *bahawa esok akan berhujan, dan jika hari ini tidak berhujan, maka wujud kebarangkalian* β *bahawa esok akan berhujan. Biarkan* $\alpha = 0.7$ *dan* $\beta = 0.4$. *Kira kebarangkalian bahawa hujan akan turun empat hari dari hari ini jika diketahui bahawa hujan turun pada hari ini. Jika kita membiarkan keadaan "0" apabila hujan turun dan "1" apabila tiada hujan, cari kebarangkalian penghad* π_0 *dan* π_1.

3. (a) Show that the symmetric random walks in one and two dimensions are both recurrent. (Use Stirling's approximation $n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi}$).

(b) If $X_n = i$, then for $i \geq 1$, $P(X_{n+1} = i+1 / X_n = i) = p_i$, $P(X_{n+1} = i-1 / X_n = i) = q_i$, $P(X_{n+1} = i / X_n = i) = r_i$. If the state space is taken as the nonnegative integers, show the transition probability matrix.

3. (a) Tunjukkan bahawa perjalanan rawak bersimetri dengan satu dan dua dimensi adalah kedua-duanya "recurrent" (gunakan anggaran Stirling $n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi}$ *).*

(b) Jika $X_n = i$, *maka untuk* $i \geq 1$, $P(X_{n+1} = i+1 / X_n = i) = p_i$, $P(X_{n+1} = i-1 / X_n = i) = q_i$, $P(X_{n+1} = i / X_n = i) = r_i$. *Jika ruang keadaan merupakan integer-integer yang bukan negatif, tunjukkan matriks kebarangkalian peralihan.*

4. (a) Define a branching process. If μ is the mean number of offspring of a single individual and σ^2 is the variance of the number of offspring produced by a single individual then for $X_0 = 1$, show that

$$E[X_n] = \mu^n \text{ and } Var(X_n) = \sigma^2 \mu^{n-1}\left(\frac{\mu^n - 1}{\mu - 1}\right).$$

(b) Let π_0 denote the probability that the population will eventually die out under the assumption that $X_0 = 1$. Show that if $P\{\text{population dies out} / X_1 = j\} = \pi_0^j$ and $P\{X_1 = j\} = P_j$ then

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j . P_j .$$

(c) Determine π_0 if

(i) $P_0 = 1/2, P_1 = 1/4, P_2 = 1/4$ and

(ii) $P_0 = 1/4, P_1 = 1/4, P_2 = 1/2$

4. *(a)* *Takrifkan proses bercabang. Jika μ ialah min bilangan anak dari individu tunggal dan σ^2 ialah varins untuk bilangan anak dari individu tunggal, maka untuk $X_0 = 1$, tunjukkan bahawa*

$$E[X_n] = \mu^n \text{ dan } Var(X_n) = \sigma^2 \mu^{n-1}\left(\frac{\mu^n - 1}{\mu - 1}\right).$$

(b) *Biarkan π_0 mewakili kebarangkalian bahawa populasi akan hapus pada akhirnya di bawah andaian $X_0 = 1$. Tunjukkan bahawa jika* $P\{\text{population dies out} / X_1 = j\} = \pi_0^j$ *dan* $P\{X_1 = j\} = P_j$ *maka*

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j . P_j .$$

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(c) *Cari π_0 jika*

(i) $P_0 = 1/2, P_1 = 1/4, P_2 = 1/4$, *dan*
(ii) $P_0 = 1/4, P_1 = 1/4, P_2 = 1/2$.

5\. (a) What are the postulates for a Poisson process? Let N(t) be the total number of occurrences of the event E in an interval (0,t). If the events occur at times $t_1, t_2, ..., t_n, ..$ then show that for a Poisson process

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

(b) Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile.

(i) What is the probability that no defects appear in the first two miles of cable?
(ii) Given that there are no defects appear in the first two miles of cable, what is the conditional probability of no defects between mile points two and three?

5. *(a)* *Apakah postalat untuk proses Poisson? Biarkan N(t) mewakili bilangan kejadian peristiwa E dalam selong (0,t). Jika peristiwa-peristiwa berlaku pada masa* $t_1, t_2, ..., t_n, ..$ *tunjukkan bahawa untuk proses Poisson*

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

(b) *Kecacatan berlaku pada kabel dibawah lautan mengikut suatu proses Poisson dengan kadar* $\lambda = 0.1$ *per batu.*

(i) Apakah kebarangkalian bahawa tiada kecacatan dalam dua batu yang pertama?
(ii) Jika diketahui bahawa tiada kecatatan yang berlaku dalam dua batu yang pertama, apakah kebarangkalian bersyarat tiada kecacatan antara titik-titik pada batu kedua dan ketiga?

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