UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2005/2006

November 2005

MST562 – Stochastic Processes
[Proses Stokastik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **EIGHT** pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN** muka *surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions : Answer **all FOUR [4]** questions.

*[**Arahan :** Jawab **semua EMPAT [4]** soalan].*

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1. (a) A space shuttle has two rockets. Each rocket consists of four compartments. The joint between adjoining compartments is sealed by a pair of rings. The joint is properly sealed if at least one of the two rings works. Let A_i be the event that the i^{th} joint is properly sealed, $i = 1, \cdots, 6$. The whole system fails if at least one of the joint is not properly sealed.

If the 12 rings fail independently of each other, each with failure probability 0.2, what is the probability of system failure?

[25 marks]

(b) Let $X_1, X_2, \cdots$ be independent, each with an exponential distribution, $\exp(\lambda)$. Let N be a geometric distribution, $G(p)$ and independent of the X_i's. Suppose

$$Y = X_1 + \cdots + X_N$$

(i) Derive and name the distribution of Y.
(ii) Derive the Laplace transform of the distribution obtained in (i).
(iii) Find the mean and variance of Y using (ii).

[45 marks]

(c) Briefly explain the concepts of :

(i) periodicity,
(ii) irreducible,
(iii) positive recurrent.

[30 marks]

1. (a) Sebuah kapal angkasa terdiri dari dua buah roket. Setiap roket terdiri dari empat bahagian. Penyambung antara bahagian bersebelahan dimeterai oleh sepasang gelang. Penyambung ini dimeterai dengan baik jika sekurang-kurangnya satu daripada dua gelang ini berfungsi. Katakan A_i adalah peristiwa bahawa penyambung ke-i dimeterai dengan baik, $i = 1, \cdots, 6$. Sistem kapal angkasa ini gagal jika sekurang-kurangnya satu daripada penyambung tidak dimeterai dengan baik.

Jika kerosakan 12 gelang adalah tak bersandar antara satu sama lain, setiap satu dengan kebarangkalian kerosakan 0.2, apakah kebarangkalian kerosakan sistem?

[25 markah]

(b) Katakan $X_1, X_2, \cdots$ adalah tak bersandar, setiap satu dengan suatu taburan eksponen, $\exp(\lambda)$. Biar N sebagai suatu taburan geometri, $G(p)$ dan tak bersandar dengan X_i. Andaikan

$$Y = X_1 + \cdots + X_N$$

(i) Terbitkan dan namakan taburan bagi Y.
(ii) Terbitkan jelmaan Laplace bagi taburan yang diperolehi di (i).
(iii) Dapatkan min dan varians bagi Y menggunakan (ii).

[45 markah]

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(c) Secara ringkas, terangkan konsep-konsep berikut :

(i) perkalaan,
(ii) tak terturunkan,
(iii) jadi semula positif.

[30 markah]

2. (a) Suppose that three girls *A, B* and *C* are throwing a ball from one to another. Whenever *A* has the ball, she throws it to *B* with probability of 0.3 and to *C* with a probability of 0.7. Whenever *B* has the ball, she throws it to A with a probability of 0.8 and to *C* with a probability of 0.2. Whenever *C* has the ball, she is equally likely to throw it to either *A* or *B*.

(i) Consider this process to be a Markov chain and construct the transition matrix.
(ii) If each of the three girls is equally likely to have the ball at a certain time n, which girl is most likely to have the ball at time $n+2$?

[25 marks]

(b) Suppose X_n is a Markov chain with state space $S=\{0,1,2\}$ and transition probability matrix

$$\begin{vmatrix} 0.0 & 0.6 & 0.4 \\ 0.8 & 0.0 & 0.2 \\ 0.3 & 0.7 & 0.0 \end{vmatrix}$$

(i) Draw the transition diagram for this process.
(ii) Assume that the initial distribution is $p_0 = 0.2$, $p_1 = 0.3$ and $p_2 = 0.5$. Compute $P(X_0 = 2, X_1 = 1, X_2 = 0)$ and $P(X_2 = 2 | X_0 = 1)$.
(iii) Calculate $E(X_i)$.

[40 marks]

(c) (i) Classify the communicating classes and determine the periodicity of each class for Markov chain represented by the following transition matrix :

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

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(ii) The transition matrix of a Markov chain is

$$P = \begin{vmatrix} q & p & 0 & 0 & 0 & . & . & . \\ q & 0 & p & 0 & 0 & . & . & . \\ q & 0 & 0 & p & 0 & . & . & . \\ q & 0 & 0 & 0 & p & . & . & . \\ . & . & . & . & . & . & & \\ . & . & . & . & . & & . & \\ . & . & . & . & . & & & . \end{vmatrix}$$

Is state 0 recurrent or transient? Verify using f_{ii}, the probability that the process will ever reenter state i.

[35 marks]

2. *(a) Katakan tiga kanak- kanak perempuan A, B dan C sedang membaling bola kepada satu sama lain. Bila mana A memperolehi bola, dia membalingkannya kepada B dengan kebarangkalian 0.3 dan kepada C dengan kebarangkalian 0.7. Bila mana B memperolehi bola, dia membalingkannya kepada A dengan kebarangkalian 0.8 dan kepada C dengan kebarangkalian 0.2. Bila mana C memperolehi bola, kebarangkalian dia membaling kepada A atau B adalah sama rata.*

(i) Pertimbang proses ini sebagai suatu rantai Markov dan bina matriks peralihan.

(ii) Jika peluang bagi setiap kanak-kanak perempuan ini memperolehi bola adalah sama pada suatu masa tertentu n, kanak-kanak manakah yang paling berkemungkinan akan memperolehi bola pada masa $n+2$?

[25 markah]

(b) Katakan X_n adalah suatu rantai Markov dengan ruang keadaan $S = \{0,1,2\}$ dan matriks kebarangkalian peralihan

$$\begin{vmatrix} 0.0 & 0.6 & 0.4 \\ 0.8 & 0.0 & 0.2 \\ 0.3 & 0.7 & 0.0 \end{vmatrix}$$

(i) Lukis gambarajah peralihan bagi proses ini.

(ii) Andaikan bahawa taburan permulaan adalah $p_0 = 0.2$, $p_1 = 0.3$ dan $p_2 = 0.5$. Hitung $P(X_0 = 2, X_1 = 1, X_2 = 0)$ dan $P(X_2 = 1 | X_0 = 1)$.

(iii) Hitungkan $E(X_i)$.

[40 markah]

(c) (i) *Tentukan kelas-kelas komunikasi dan kala setiap kelas bagi rantai Markov yang diwakili oleh matriks peralihan berikut :*

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

(ii) *Matriks peralihan bagi suatu rantai Markov ialah*

$$P = \begin{vmatrix} q & p & 0 & 0 & 0 & . & . & . \\ q & 0 & p & 0 & 0 & . & . & . \\ q & 0 & 0 & p & 0 & . & . & . \\ q & 0 & 0 & 0 & p & . & . & . \\ . & . & . & . & . & . & & \\ . & . & . & . & . & & . & \\ . & . & . & . & . & & & . \end{vmatrix}$$

Adakah keadaan 0 jadi semula atau fana? Sahkan dengan menggunakan f_{ii}, kebarangkalian bahawa proses akan masuk semula keadaan i.

[35 markah]

3. (a) A man moves according to a simple random walk with

$$P(Z=1)=p \quad ; \quad P(Z=-1)=q=1-p$$

Estimate the probability that the man is less than 10 units from the position he started after 30 steps if $p=0.6$.

[25 marks]

(b) Write short notes on branching processes and include the notions of criticality.

[30 marks]

(c) A population of cows behaves as branching process in which each mature (female) cow gives birth to a random number W (female) cows (which actually survive to maturity) with probabilities

$$P(W=w)=\begin{cases} \frac{1}{8} & \text{if} \quad w=0 \\ \frac{1}{2} & \text{if} \quad w=1 \\ \frac{3}{8} & \text{if} \quad w=2 \end{cases}$$

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(i) Calculate the mean and variance of the size of the population at generation n.
(ii) Compute the probability of extinction of the population.

[45 marks]

3. *(a) Seorang lelaki bergerak mengikut suatu perjalanan rawak mudah dengan*

$$P(Z=1)=p \quad ; \quad P(Z=-1)=q=1-p$$

Anggarkan kebangkalian bahawa lelaki tersebut adalah kurang daripada 10 unit dari tempat permulaannya selepas 30 langkah jika p=0.6.

[25 markah]

(b) Tuliskan nota ringkas tentang proses bercabang dengan sertakan maksud keterkritikalan.

[30 marks]

(c) Suatu populasi lembu adalah mengikut proses bercabang yang mana setiap lembu (betina) yang cukup umur melahirkan suatu nombor rawak lembu (betina) W (yang mana hidup hingga cukup umur) dengan kebarangkalian

$$P(W=w)=\begin{cases}\frac{1}{8} & \text{jika} \quad w=0\\ \frac{1}{2} & \text{jika} \quad w=1\\ \frac{3}{8} & \text{jika} \quad w=2\end{cases}$$

(i) Hitung min dan varians bagi saiz populasi pada generasi ke-n.
(ii) Kira kerbarangkalian kepupusan populasi ini.

[45 markah]

4. (a) An insurance company pays out claims at times of a Poisson process with rate 4 per week.

(i) What is the expected time until the tenth claim arrives?
(ii) What is the probability that no claim arrives for 1 month?
(iii) What is the probability that the elapsed time between the 9^{th} and the 10^{th} claim exceeds 3 weeks?

[30 marks]

(b) (i) Explain what M/M/s queue is and how it can be modeled as a birth and death process.
(ii) Briefly explain the concepts of closed queuing network and open queuing network.

[30 marks]

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(c) Let $Z = \{Z(t), t > 0\}$ be a standard Brownian motion process. Define $Y(0) = 0$ and let

$$Y(t) = t\,Z\left(\frac{1}{t}\right)$$

(i) Show that $Y(t) = \{Y(t), t \geq 0\}$ is also a standard Brownian motion process.

(ii) For, $0 \leq s < t$, find $\mathrm{Cov}\left[Y(s), Y(t)\right]$.

[40 marks]

4. *(a) Sebuah syarikat insuran membayar tuntutan pada suatu masa mengikut suatu proses Poisson dengan kadar 4 setiap minggu.*

(i) Apakah masa jangkaan sehingga tuntutan kesepuluh tiba?

(ii) Apakah kebarangkalian bahawa tiada tuntutan tiba bagi 1 bulan?

(iii) Apakah kebarangkalian bahawa masa elaps antara tuntutan ke-9 dan ke-10 melebihi 3 minggu?

[30 markah]

(b) (i) Terangkan maksud giliran M/M/s dan bagaimana ia boleh dimodelkan sebagai suatu proses kelahiran dan kematian.

(ii) Terangkan secara ringkas konsep rangkaian giliran tertutup dan rangkaian giliran terbuka.

[30 markah]

(c) Katakan $Z = \{Z(t), t > 0\}$ *adalah suatu proses gerakan Brown piawai. Ditakrifkan* $Y(0) = 0$ *dan katakan*

$$Y(t) = t\,Z\left(\frac{1}{t}\right)$$

(i) Tunjukkan bahawa $Y = \{Y(t), t \geq 0\}$ *juga adalah suatu proses gerakan Brown piawai.*

(ii) Bagi $0 \leq s < t$, *cari* $Cov\left[Y(s), Y(t)\right]$.

[40 markah]

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APPENDIX

1. If X is distributed as Poisson with parameter $\lambda > 0$, then
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0,1,2,\cdots$$

2. If X distributed as geometric with parameter p, $0 < p < 1$, then
$$P(X = x) = p(1-p)^{x-1} \quad ; \quad x = 1,2,\cdots$$

3. If X distributed as Binomial with parameter p, $0 < p < 1$, then
$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0,1,2,\cdots,n$$

4. If X distributed as exponential with parameter $\lambda > 0$, then
$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

5. If X is distributed as gama with parameter $\alpha > 0$ and $\beta > 0$ then
$$f(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1} e^{-\beta x} \quad ; \quad x > 0$$

6. If X is distributed as normal with parameter μ and $\alpha^2 > 0$ then
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad ; \quad -\infty < x < \infty$$

7. Formula of geometric series
$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad ; \quad |r| < 1$$

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