UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2005/2006

November 2005

MST561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **NINE** pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ***SEMBILAN*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions : Answer **all FIVE [5]** questions.

[Arahan : Jawab ***semua LIMA [5]*** *soalan].*

...2/-

1. (a) A coin is tossed two times. Let *H* and *T* represent head and tail respectively.
   (i) Find the set of all possible outcomes, Ω.
   (ii) Find the σ-field, *S* of this experiment based on (i).
   (iii) State the meaning of a σ-field.
   (iv) State the meaning of a sample space.

[40 marks]

(b) Let *A* and *B* be events. Show that

$$P[(A \cap B) \mid (A \cup B)] \leq P[(A \cap B) \mid A].$$

When does equality hold?

[20 marks]

(c) Let I_A denotes the indicator function of event *A*, i.e.,

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}.$$

Let *D*, *E* and *F* represent independent events, with $P(D) = P(E) = P(F) = 0.5$. Define the random variable *Y* by $Y(\omega) = I_D(\omega) + 2I_E(\omega) - I_F(\omega)$.
   (i) What are the values taken by the random variable *Y*?
   (ii) Find the probability mass function, $p_Y(y)$ of the random variable *Y*.
   (iii) Find the cumulative distribution function $F_Y(y)$ of the random variable *Y*.

[40 marks]

1. (a) Sekeping duit syiling dilambungkan dua kali. Biarkan H dan T masing-masing mewakili kepala dan bunga.
   (i) Cari Ω, set kesemua kesudahan yang mungkin.
   (ii) Cari S, medan-σ bagi eksperimen ini berdasarkan (i).
   (iii) Nyatakan maksud suatu medan-σ.
   (iv) Nyatakan maksud suatu ruang sampel.

[40 markah]

(b) Biarkan A dan B sebagai peristiwa-peristiwa. Tunjukkan bahawa

$$P[(A \cap B) \mid (A \cup B)] \leq P[(A \cap B) \mid A]$$

Bilakah persamaan benar?

[20 markah]

(c) Biarkan I_A mewakili fungsi petunjuk bagi peristiwa A, yakni

$$I_A(\omega) = \begin{cases} 1 & \text{jika } \omega \in A \\ 0 & \text{sebaliknya} \end{cases}.$$

Biarkan D, E dan F mewakili peristiwa tak bersandar, dengan P(D) = P(E) = P(F) = 0.5. Takrifkan pembolehubah rawak Y sebagai $Y(\omega) = I_D(\omega) + 2I_E(\omega) - I_F(\omega)$.
   (i) Apakah nilai yang diambil oleh pembolehubah rawak Y?
   (ii) Cari $p_Y(y)$, fungsi jisim kebarangkalian bagi pembolehubah rawak Y.
   (iii) Cari $F_Y(y)$, fungsi taburan longgokan bagi pembolehubah rawak Y.

[40 markah]

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2. (a) The joint probability density function (pdf) of X and Y is

$$f(x,y)=\frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(x^2+y^2\right)\right],\ -\infty<x,y<\infty.$$

If $S=X+Y$ and $T=X-Y$, find the joint pdf of S and T.

[30 marks]

(b) The distribution of the random variable X is

$$P(X=m)=\frac{3p^m}{(3+p)^{m+1}},\ m=0,1,2,\ldots;0<p<1.$$

(i) Find the moment generating function of X, $M_X(t)$.

(ii) Find $E(X)$ and $\mathrm{Var}(X)$ using (i).

[30 marks]

(c) Let $X_1,X_2,\ldots,X_n$ be a random sample from a Bernoulli, b(1,θ) distribution, where $0<\theta<1$. Then $Y=\sum_{i=1}^{n}X_i$ follows a binomial, Bin(n,θ) distribution with probability mass function (pmf)

$$f(y\mid\theta)=\begin{cases}\binom{n}{y}\theta^y(1-\theta)^{n-y}, & y=0,1,2,\ldots,n\\ 0, & \text{elsewhere}\end{cases}.$$

Assume that the prior probability density function (pdf) of the random variable Θ is given by

$$h(\theta)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0<\theta<1\\ 0, & \text{elsewhere}\end{cases}$$

where α and β are known positive constants.

(i) Find the posterior pdf for Θ.

(ii) Find the Bayes' solution $w(y)$ for θ with respect to the prior pdf $h(\theta)$ using the loss function $L[\theta,w(y)]=[\theta-w(y)]^2$.

[40 marks]

2. *(a) Fungsi ketumpatan kebarangkalian (fkk) tercantum bagi X dan Y ialah*

$$f(x,y)=\frac{1}{2\pi}\text{eks}\left[-\frac{1}{2}\left(x^2+y^2\right)\right],\ -\infty<x,y<\infty.$$

Jika $S=X+Y$ dan $T=X-Y$, cari fkk tercantum bagi S dan T.

[30 markah]

(b) Taburan bagi pembolehubah rawak X ialah

$$P(X=m)=\frac{3p^m}{(3+p)^{m+1}},\ m=0,1,2,\ldots;0<p<1.$$

(i) Cari $M_X(t)$, fungsi penjana momen bagi X.

(ii) Cari E(X) dan Var(X) dengan menggunakan (i).

[30 markah]

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(c) *Biarkan* $X_1, X_2, \ldots, X_n$ *mewakili sampel rawak daripada taburan Bernoulli,* $b(1,\theta)$, *yang mana* $0 < \theta < 1$. *Maka,* $Y = \sum_{i=1}^{n} X_i$ *mempunyai taburan binomial,* $\text{Bin}(n,\theta)$ *dengan fungsi jisim kebarangkalian (fjk)*

$$f(y \mid \theta) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, & y = 0,1,2,\ldots,n \\ 0, & \textit{di tempat lain} \end{cases}.$$

Andaikan bahawa fungsi ketumpatan kebarangkalian (fkk) priori bagi pembolehubah rawak Θ *diberi oleh*

$$h(\theta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, & \textit{di tempat lain} \end{cases}$$

yang mana α *dan* β *merupakan pemalar positif dengan nilai yang diketahui.*

(i) *Cari fkk posterior bagi* Θ.

(ii) *Cari w(y), penyelesaian Bayes bagi* θ *terhadap fkk priori* $h(\theta)$ *dengan menggunakan fungsi kerugian* $L[\theta, w(y)] = [\theta - w(y)]^2$.

[40 markah]

3. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a $N(\mu,1)$ distribution. Define the sample average as $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the distribution of the following statistics:

(i) $$\frac{m\sum_{i=1}^{n}(X_i-\mu)^2}{n\sum_{i=1}^{m}(X_i-\mu)^2}, \quad 1 < m < n$$

(ii) $$\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{X_i-\mu}, \quad 1 \le i \le n$$

[20 marks]

(b) Let $\{X_n\}$ be a sequence of random variables defined on the probability space (Ω, S, P). If the sequence $\{X_n\}$ converges in probability to one, i.e.,

$$X_n \xrightarrow{P} 1,$$

show that $X_n^{-1} \xrightarrow{P} 1$.

[20 marks]

(c) Let $X_1, X_2, ..., X_n$ be random variables from a Bernoulli distribution with probability mass function (pmf)

$$f_\alpha(x) = \alpha^x (1-\alpha)^{1-x} \quad \text{for } x = 0 \text{ or } 1.$$

(i) Is X_n an unbiased estimator of $\tau(\alpha) = \alpha$?

(ii) Show that $\sum_{i=1}^{n} X_i$ is a sufficient statistic.

(iii) Use the Rao-Blackwell's theorem to find an unbiased estimator of α with variance not larger than that of X_n.

[60 marks]

3. *(a) Biarkan $X_1, X_2, ..., X_n$ mewakili pembolehubah rawak tak bersandar dan bertaburan $N(\mu,1)$ secara secaman. Takrifkan min sampel sebagai $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Cari taburan untuk statistik berikut:*

(i) $\dfrac{m\sum_{i=1}^{n}(X_i - \mu)^2}{n\sum_{i=1}^{m}(X_i - \mu)^2}$, $1 < m < n$

(ii) $\dfrac{\sqrt{n}(\overline{X}_n - \mu)}{X_i - \mu}$, $1 \le i \le n$

[20 markah]

(b) Biarkan $\{X_n\}$ mewakili suatu jujukan pembolehubah rawak yang ditakrifkan pada ruang kebarangkalian (Ω, S, P). Jika jujukan $\{X_n\}$ menumpu secara kebarangkalian kepada satu, yakni,

$$X_n \xrightarrow{P} 1,$$

tunjukkan bahawa $X_n^{-1} \xrightarrow{P} 1$.

[20 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ mewakili pembolehubah rawak daripada taburan Bernoulli dengan fungsi jisim kebarangkalian (fjk)

$$f_\alpha(x) = \alpha^x (1-\alpha)^{1-x} \quad \textit{untuk } x = 0 \textit{ atau } 1.$$

(i) Adakah X_n suatu penganggar saksama bagi $\tau(\alpha) = \alpha$?

(ii) Tunjukkan bahawa $\sum_{i=1}^{n} X_i$ ialah suatu statistik cukup.

(iii) Gunakan teorem Rao-Blackwell untuk mencari suatu penganggar saksama bagi α yang mempunyai varians yang tidak melebihi varians bagi X_n.

[60 markah]

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4. (a) A random sample of size n has the probability density function

$$f(x;\alpha_1,\alpha_2)=\begin{cases}\dfrac{1}{\alpha_2}e^{-(x-\alpha_1)/\alpha_2}, & x\geq\alpha_1,\ -\infty<\alpha_1<\infty,\ \alpha_2>0\\ 0, & \text{elsewhere}\end{cases}.$$

Find the maximum likelihood estimator of α_1 and α_2.

[30 marks]

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal, $N(\mu,\sigma^2)$ distribution, where both μ and σ^2 are unknowns.

(i) Find the Cramer-Rao lower bound for the variance of unbiased estimators of σ^2.

(ii) Find the efficiency of the sample variance, $S^2=\dfrac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$.

(iii) Is S^2 an asymptotically efficient estimator of σ^2?

[30 marks]

(c) (i) State the definition of pivotal quantity.

(ii) A continuous random variable X follows a $U(0,\theta)$ distribution. Show that $\dfrac{Y_n}{\theta}$ is a pivotal quantity, where Y_n is the largest order statistic from a sample of size n. Next, use this pivotal quantity to construct a 90% confidence interval for θ.

[40 marks]

4. *(a) Suatu sampel rawak saiz n mempunyai fungsi ketumpatan kebarangkalian*

$$f(x;\alpha_1,\alpha_2)=\begin{cases}\dfrac{1}{\alpha_2}e^{-(x-\alpha_1)/\alpha_2}, & x\geq\alpha_1,\ -\infty<\alpha_1<\infty,\ \alpha_2>0\\ 0, & \textit{di tempat lain}\end{cases}.$$

Cari penganggar kebolehjadian maksimum bagi α_1 dan α_2.

[30 markah]

(b) Biarkan $X_1, X_2, \ldots, X_n$ mewakili suatu sampel rawak daripada taburan normal, $N(\mu,\sigma^2)$, yang mana kedua-dua μ dan σ^2 adalah anu.

(i) Cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama bagi σ^2.

(ii) Cari kecekapan varians sampel, $S^2=\dfrac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$.

(iii) Adakah S^2 suatu penganggar cekap secara berasimptot bagi σ^2?

[30 markah]

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(c) (i) *Nyatakan takrif kuantiti pangsian.*

(ii) *Suatu pembolehubah rawak selanjar X mempunyai taburan* $U(0,\theta)$. *Tunjukkan bahawa* $\frac{Y_n}{\theta}$ *ialah suatu kuantiti pangsian, yang mana* Y_n *ialah statistik tertib terbesar daripada sampel saiz n. Seterusnya, gunakan kuantiti pangsian ini untuk membina selang keyakinan 90% bagi* θ.

[40 markah]

5. (a) State the Neyman-Pearson lemma and its role.

[20 marks]

(b) Let X be a discrete random variable with probability mass function

$$P_0\{X=x\}=\begin{cases}\dfrac{\theta}{2} & \text{if } x=\pm 2\\ \dfrac{1-2\theta}{2} & \text{if } x=\pm 1\\ \theta & \text{if } x=0\end{cases}$$

under the null hypothesis $H_0: p=0$, and

$$P_1\{X=x\}=\begin{cases}pc & \text{if } x=-2\\ \dfrac{1-c}{1-\theta}\left(\dfrac{1}{2}-\theta\right) & \text{if } x=\pm 1\\ \theta\left(\dfrac{1-c}{1-\theta}\right) & \text{if } x=0\\ (1-p)c & \text{if } x=2\end{cases}$$

under the alternative hypothesis $H_1: p\in(0,1)$, where θ and c are constants with $0<\theta<\frac{1}{2}$ and $\frac{\theta}{2-\theta}<c<\theta$. Find the Generalized Likelihood Ratio test of size θ to reject H_0.

[40 marks]

(c) Let X be a random variable with probability density function

$$f(x)=\theta x^{\theta-1},\ 0\le x\le 1,\ \theta>0,$$

where $\Theta=(0,\infty)$ and $\theta_0\in(0,\infty)$. We wish to test $H_0:\theta\le\theta_0$ versus $H_1:\theta>\theta_0$ for a random sample of size 2. The critical region of the test is $C=\{(x_1,x_2): x_1x_2\ge k;\ x_1,x_2\ge 0\}$ where k (< 1) is a fixed positive constant.

(i) Find the power function of the test.

(ii) What is the size of the test?

(iii) Based on (ii), find the size of the test for k = 0.5 if we test the hypothesis, $H_0:\theta\le 2$ versus $H_1:\theta>2$.

[40 marks]

5. (a) *Nyatakan lema Neyman-Pearson dan peranannya.*

[20 markah]

(b) *Biarkan X mewakili pembolehubah rawak diskrit dengan fungsi jisim kebarangkalian*

$$P_0\{X=x\}=\begin{cases}\dfrac{\theta}{2} & jika\ x=\pm 2\\ \dfrac{1-2\theta}{2} & jika\ x=\pm 1\\ \theta & jika\ x=0\end{cases}$$

di bawah hipotesis nol $H_0 : p = 0$, *dan*

$$P_1\{X=x\}=\begin{cases}pc & jika\ x=-2\\ \dfrac{1-c}{1-\theta}\left(\dfrac{1}{2}-\theta\right) & jika\ x=\pm 1\\ \theta\left(\dfrac{1-c}{1-\theta}\right) & jika\ x=0\\ (1-p)c & jika\ x=2\end{cases}$$

di bawah hipotesis alternatif $H_1 : p \in (0,1)$, *yang mana* θ *dan* c *adalah pemalar dengan* $0<\theta<\frac{1}{2}$ *dan* $\frac{\theta}{2-\theta}<c<\theta$. *Cari ujian Nisbah Kebolehjadian Teritlak saiz* θ *untuk menolak* H_0.

[40 markah]

(c) *Biarkan X mewakili suatu pembolehubah rawak dengan fungsi ketumpatan kebarangkalian*

$$f(x)=\theta x^{\theta-1},\ 0\le x\le 1,\ \theta>0,$$

yang mana $\Theta = (0,\infty)$ *dan* $\theta_0 \in (0,\infty)$. *Kita ingin menguji* $H_0 : \theta \le \theta_0$ *lawan* $H_1 : \theta > \theta_0$ *bagi suatu sampel rawak saiz 2. Rantau genting bagi ujian ini ialah C* $=\{(x_1,x_2): x_1x_2 \ge k;\ x_1,x_2 \ge 0\}$ *yang mana k (< 1) ialah suatu pemalar positif yang tetap.*

(i) *Cari fungsi kuasa bagi ujian ini.*

(ii) *Apakah saiz ujian ini?*

(iii) *Berdasarkan (ii), cari saiz ujian ini untuk k = 0.5 jika kita menguji hipotesis,* $H_0 : \theta \le 2$ *lawan* $H_1 : \theta > 2$.

[40 markah]

LAMPIRAN

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x) = \frac{1}{N} I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q + pe^t\right)^n$
Geometri	$f(x) = pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}, \quad qe^t < 1$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}, \quad t \neq 0$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\{-(x-\mu)^2 / 2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t + (\sigma t)^2 / 2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}, \quad t < \lambda$
Gama	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha, \quad t < \lambda$
Khi Kuasa Dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}, \quad t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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