APPROXIMATE ANALYTICAL METHODS FOR SOLVING FREDHOLM INTEGRAL EQUATIONS

MOHAMMAD SALAMEH TALEB ALMOUSA

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by

MOHAMMAD SALAMEH TALEB ALMOUSA

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LIST OF ABBREVIATIONS

1D-FIEs	The one-dimensional Fredholm integral equations
1st FIEs	The first kind Fredholm integral equations
2nd FIEs	The second kind Fredholm integral equations
2D-FIEs	The two-dimensional Fredholm integral equations
1st 2D-FIEs	The first kind two-dimensional Fredholm integral equations
2nd 2D-FIEs	The second kind two-dimensional Fredholm integral equations
FHIEs	The Hammerstein-Fredholm integral equations
2nd FHIEs	The second kind Hammerstein Fredholm integral equations
HPM	The homotopy perturbation method
ADM	The Adomian decomposition method
OHAM	The optimal homotopy asymptotic method
HAM	The homotopy analysis method

LIST OF SYMBOLS

$K(s,t)$	The kernel function
$\mathrm{E}_{n\ abs}$	The absolute errors
g	The truth value
$\hat{g}$	The approximation value
L	Highest order derivative which assumed to be invertible
R	The remainder of the linear operator
N	Nonlinear differentiable operator
L^{-1}	The inverse operator
A_i	The polynomials
λ	The parameter
A	General operator
$\frac{\partial}{\partial n}$	Differentiation the normal vector drawn outwards from Ω
g_0	The initial approximation of equations
B	Boundary operator
p	Embedding parameter
$H(p)$	Non-zero auxiliary function
c_j	Constants
$f'(x)$	The differentiation equation
D	The differential operator
P	Known functions
F_i	Linearly independent set
W_i	The weight function
R	The residual equation
$\tilde{g}(x)$	The approximate of $g(x)$
U	Function defined as $U : \Omega \times [0,1] \to \Re ; x \in \Omega, \mathrm{p} \in [0,1]$
C_f	Regularisation parameter

r	Regularisation parameter
g_{exact}	The exact solution
g_{OHAM}	The OHAM solution
g_{HPM}	The HPM solution
g_{ADM}	The ADM solution
g^{m}	The *m*th-order approximations of equations
g_1	The first order problem
g_2	The second order problem
g_3	The third order problem
$v\left(x, p\right)$	Convex homotopy
$g^{3}{}_{OHAM}$	The third OHAM order solution
$g^{20}{}_{HPM}$	The twentieth order HPM solution
$g^{20}{}_{ADM}$	The twentieth order ADM solution
$g^{9}{}_{1HPM}$	The ninth order HPM solution of g_1
$g^{9}{}_{2HPM}$	The ninth order HPM solution of g_2
$g^{32}{}_{1ADM}$	The 32 order ADM solution of g_1
g^{m}	The *m*th-order approximations of equations
$g^{32}{}_{2ADM}$	The 32 order ADM solution of g_2
$g^{32}{}_{1TFM}$	The 32 order TFM solution of g_1
$g^{32}{}_{2TFM}$	The 32 order TFM solution of g_2
$g^{5}{}_{1RBFN-ShA}$	The fifth order RBFN-SHA solution of g_1
$g^{5}{}_{2RBFN-ShA}$	The fifth order RBFN-SHA solution of g_2

KAEDAH ANALISIS HAMPIRAN UNTUK MENYELESAIKAN PERSAMAAN KAMIRAN FREDHOLM

ABSTRAK

Persamaan kamiran memainkan peranan penting dalam banyak bidang sains seperti matematik, biologi, kimia, fizik, mekanik dan kejuruteraan. Oleh yang demikian,pelbagai teknik berbeza telah digunakan untuk menyelesaikan persamaan jenis ini. Kajian ini, memfokus kepada analisis secara matematik dan berangka bagi beberapa kes persamaan kamiran Fredholm yang linear dan bukan linear. Kes-kes ini termasuklah persamaan kamiran Fredholm satu dimensi jenis pertama dan kedua, persamaan kamiran Fredholm dua dimensi jenis pertama dan kedua dan sistem persamaan kamiran Fredholm satu dimensi dan dua dimensi. Dalam tesis ini, kaedah analisis hampiran dicadangkan untuk mengkaji beberapa kes persamaan kamiran Fredholm yang linear dan bukan linear. Kaedah analisis hampiran ini termasuk: kaedah homotopi asimptotik optimum (OHAM)), kaedah usikan homotopi (HPM) and kaedah Dekomposisi Adomain (ADM). Melalui pendekatan pertama, keberkesanan OHAM untuk menyelesaikan beberapa kes dalam persamaan kamiran Fredholm dikaji. Penyelesaian secara analisis dan ralat mutlak yang diperoleh melalui kaedah OHAM akan dimasukkan ke dalam jadual dan dianalisiskan. Perbandingan dibuat dengan kaedah lain yang terdapat dalam literatur. Didapati bahawa penggunaan kaedah OHAM adalah lebih cepat, lebih mudah dilaksanakan dan lebih tepat jika dibandingkan dengan penggunaan kaedah lain. OHAM juga tidak memerlukan tekaan awal dan penggunaan memori komputer yang besar. Melalui pendekatan kedua dan ketiga, kaedah HPM dan ADM dirumuskan untuk menyelesaikan persamaan kamiran Fredholm-Hammerstein dan persamaan kamiran Fredholm dua dimensi. Keputusan yang diperoleh dibandingkan

dengan keputusan daripada kaedah OHAM dan kaedah lain dalam literatur. Secara jelas, teknik HPM dan ADM ialah teknik yang tepat dan berkesan, HPM adalah sepadan dengan ADM dengan homotopi $H = 0$ dan HPM dan ADM ialah kes OHAM yang khas untuk menyelesaikan jenis persamaan ini.

APPROXIMATE ANALYTICAL METHODS FOR SOLVING FREDHOLM INTEGRAL EQUATIONS

ABSTRACT

Integral equations play an important role in many branches of sciences such as mathematics, biology, chemistry, physics, mechanics and engineering. Therefore, many different techniques are used to solve these types of equations. This study focuses on the mathematical and numerical analysis of some cases of linear and nonlinear Fredholm integral equations. These cases are one-dimensional Fredholm integral equations of the first kind and second kind, two-dimensional Fredholm integral equations of the first kind and second kind and systems of one and two-dimensional Fredholm integral equations. In this thesis, approximate analytical methods are proposed to investigate some cases of linear and nonlinear Fredholm integral equations. Such approximate analytical methods include: optimal homotopy asymptotic method (OHAM), homotopy perturbation method (HPM) and Adomian decomposition method (ADM). In the first approach, the effectiveness of OHAM is investigated for solving some cases of Fredholm integral equations. The analytical solutions and absolute errors obtained by using this method are tabulated and analyzed and comparison is carried out by using other methods in literature. It was found that the OHAM is faster, easier to implement and more accurate compared to other methods and there is no need of initial guess and large computer memory. In the second and third approaches, HPM and ADM are formulated for solving Fredholm-Hammerstein integral equations and two-dimensional Fredholm integral equations. The results obtained by these methods are compared with OHAM and other methods in literature. It is clear that HPM and ADM are accurate and efficient

techniques, HPM is equivalent to ADM with the homotopy $H = 0$ and these methods are special cases of the OHAM in solving these types of equations.

CHAPTER 1

INTRODUCTION

This thesis introduces new solution methods to one-dimensional Fredholm integral equations of the first kind and second kind, two-dimensional Fredholm integral equations of the first kind and second kind and systems of one and two-dimensional Fredholm integral equations. This chapter reviews the background, some cases of Fredholm integral equations and special kinds of kernels. Beside this, we provide objective of research, the methodology and structure of this thesis.

1.1 Background

In 1888, the integral equations were first used by Paul du Bois-Reymond; See (Kress, 1999). These types of equations play an important role in many branches of sciences such as mathematics, biology, chemistry, physics, mechanics and engineering. In fact, many linear and nonlinear problems in sciences can be expressed in the form of integral equations. Examples include radiative heat transfer problems (Bednov, 1986), elasticity (Matsumoto, Tanaka and Hondoh, 1993), time series analysis (Zhukovskii, 2004), plasticity (Mashchenko and Churikov, 1980), potential theory and Dirichlet problems (Jiang and Rokhlin, 2004), problems of radiative equilibrium (Hopf, 1934), wave motion (Bandrowski, Karczewska and Rozmej, 2010), fluid and solid mechanics (Bonnet, 1999), control (Park Kim, Park and Choi, 2005), diffusion problems (Bobula, Twardowska and Piskorek, 1987), biomechanics (Herrebrugh, 1968), economics (Boikov and Tynda, 2003), game theory (Carl and Heikkilä, 2011), electrostatics (Xie and Scott, 2011), contact problems (Smetanin, 1991), reactor theory (Kaper and Kellogg, 1977), acoustics (Yang, 1999), electrical engineering (Shore and Yaghjian, 2005), medicine and queuing theory (Baker and Derakhshan, 1993). Many equations in sciences are obtained from experiments in the form of integral equations. Therefore, the treatments and exact solutions which are obtained by the different methods play an important role in these fields.

In recent years, much work has been carried out by researchers in sciences and engineering on applying and analyzing novel numerical and approximate analytical methods for obtaining solutions of integral equations. Among these are the homotopy analysis method (Awawdeh et al., 2009; Adawi et al., 2009; Vahdati et al., 2010), variational iteration method (Xu, 2007; Saadati et al., 2009), monic Chebyshev approximations (El-Kady and Moussa, 2013), Legendre-spectral method (Adibi and Rismani, 2010), rationalized Haar functions (Babolian, Bazm and Lima, 2011), traditional collocation method radial basis functions (Avazzadeh et al., 2011) and B-spline scaling functions (Maleknejad and Aghazadeh, 2009). Other examples include the Spectral Galerkin method (Nadjafi, Samadi and Tohidi, 2011), a neural network approach (Effati and Buzhabadi, 2012), CAS wavelet (Barzkar et al., 2012), operational Tau method (Abadi and Shahmorad, 2002), quadrature rule (Mirzaee, 2012), discrete Adomian decomposition (Bakodah and Darwish, 2012), collocation and iterated collocation (Brunner and Kauthen, 1989), triangular functions method (Maleknejad and Mirzaee, 2010), quasi interpolation method (Muller and Varnhorn, 2011) and radial basis functions (Avazzadeha et al., 2011). Also, automatic augmented Galerkin algorithms (Abbasbandy and Babolian, 1995), a modified ADM (Vahidi and Damercheli, 2012), Sinc-collocation method (Rashidinia and Zarebnia, 2007), neural network (Jafarian and Nia, 2013), a Chebyshev collocation method (Akyuz-Dascıoglu, 2004), Block–Pulse functions (Maleknejad et al., 2005), radial basis function networks (Golbabai et al., 2008), resolvent method (Wang et al., 2008) and Taylor expansion method (Huang et al., 2009).

This thesis focuses on one and two-dimensional Fredholm integral equations and systems of Fredholm integral equations which are essential in science and engineering.

1.2 Fredholm Integral Equations (FIEs)

The main founders of the integral equations are Fredholm (1903), Hammerstein (1930), Hilbert (1912), Volterra (1896), Schmidt (1907) and Lalescu (1908); see (Ben-Menahem, 2009). There are several types of integral equations such as Fredholm integral equations, Volterra integral equations, Hammerstein integral equations, mixed integral equation and two-dimensional integral equations. This study focuses on Fredholm type of equations. The following some cases of Fredholm integral equations are discussed.

1.2.1 One-Dimensional Fredholm Integral Equations (1D-FIEs)

The general type of one-dimensional Fredholm integral equation can be written as (Wazwaz, 2011a)

$$h(s)g(s)=f(s)+\lambda\int_a^b K(s,t)L(t,g(t))dt, \quad s\in[a,b] \tag{1.1}$$

where a and b are fixed, L is a known function called the appropriate integral operator, $h(s)$ and $f(s)$ are known functions, $K(s,t)$ is called the kernel function, $g(s)$ is unknown function and λ is a nonzero constant. Equation (1.1) is called a linear one-dimensional Fredholm integral equaa tion if all the unknown functions terms are linear. Otherwise, it is called nonlinear one-dimensional Fredholm integral equation. These equations include either Urysohn or Hammerstein integral equations.

The first kind Fredholm integral equation (1^{st} FIE) is obtained by setting $h(s)=0$ in the above equation (1.1) as (Wazwaz, 2011a)

$$f(s)+\lambda\int_a^b K(s,t)L(t,g(t))dt=0. \tag{1.2}$$

The second kind Fredholm-Hammerstein integral equation (2^{nd} FHIE) is obtained by setting $h(s)=1$ in equation (1.1) as (Rashidinia, Khosravian Arabb, and Parsa, 2011)

$$g(s)=f(s)+\lambda\int_a^b K(s,t)L(t,g(t))dt. \tag{1.3}$$

The homogeneous Fredholm-Hammerstein integral equation is obtained by setting $f(s)=0$ in equation (1.3) as (Wazwaz, 2011a)

$$g(s)=\lambda\int_a^b K(s,t)L(t,g(t))dt. \tag{1.4}$$

This is a special case of equation (1.3).

The Urysohn integral equation is (Saberi-Nadjafi and Heidari, 2010)

$$g(s)=f(s)+\lambda\int_a^b K(s,t,g(t))dt. \tag{1.5}$$

1.2.2 Two-Dimensional Fredholm Integral Equations (2D-FIEs)

The general type of two-dimensional Fredholm integral equation is as follows (Wazwaz, 2011a)

$$h(x,t)g(x,t)=f(x,t)+\lambda\int_a^b\int_c^d k(x,t,s,y)L(g(s,y))dsdy, \qquad (1.6)$$

where a,b,c and d are constants, $g(x,t)$ is unknown function, $h(x,t), f(x,t)$ and L are known functions, $k(x,t,s,y)$ is the kernel function and λ is a nonzero constant.

If $h(x,t)$ is identically zero, equation (1.6) is called first kind two-dimensional Fredholm integral equation given in the form (Wazwaz, 2011)

$$f(x,t)+\lambda\int_a^b\int_c^d k(x,t,s,y)L(g(s,y))dsdy=0. \qquad (1.7)$$

If $h(x,t)$ is identically one, equation (1.6) is second kind two-dimensional Fredholm integral equation given in the form (Wazwaz, 2011a)

$$g(x,t)=f(x,t)+\lambda\int_a^b\int_c^d k(x,t,s,y)L(g(s,y))dsdy. \qquad (1.8)$$

If b or d is a variable, equation (1.6) is called mixed integral equation.

Here, one can say the two-dimensional integral equation is linear, if all the terms of unknown functions are linear, otherwise called nonlinear.

1.3 Systems of the Second Kind Fredholm Integral Equations

This section presents some cases of the systems of the second kind Fredholm integral equations.

1.3.1 Systems of the Second Kind One-Dimensional Fredholm Integral Equations

Consider the general system of the second kind one-dimensional Fredholm integral equation given in the form (Babolian et al., 2004)

$$G(x)=F(x)+\int_a^b K\left(x,y,G(t)\right)dy, \tag{1.9}$$

where

$$G(x)=\left[g_i(x)\right], i=1,2,\ldots,n.$$

$$F(x)=\left[f_i(x)\right], i=1,2,\ldots,n.$$

$$K\left(x,y,G(t)\right)=\left[k_{i,j}\left(x,y,G(t)\right)\right], i,j=1,2,\ldots,n.$$

In system (1.9), the functions $F(x)$ and $k(x,y,G(t))$ are given and $G(t)$ is to be determined. We shall assume that this system has unique solution, then we have the i th linear and nonlinear systems as (Babolian et al., 2004)

$$g_i(x)=f_i(x)+\sum_{l=1}^{n}\int_a^b K_{i,l}(x,y)\,g_l(y)\,dy, \tag{1.10}$$

$$g_i(x)=f_i(x)+\sum_{l=1}^{n}\int_a^b K_{i,l}\left(x,y,g_l(y)\right)dy, \tag{1.11}$$

respectively. The following system of nonlinear second kind one-dimensional Fredholm integral equation is special case of system (1.11)

$$g_i(x) = f_i(x) + \sum_{l=1}^{n} \int_a^b K_{i,l}(x,y)\left(g_l(y)\right)^m dy,\ m = 2,3,\ldots \tag{1.12}$$

1.3.2 Systems of the Second Kind Two-Dimensional Fredholm Integral Equations

The system of the second kind two-dimensional Fredholm integral equation can be defined as (Saeed and Mahmud, 2009)

$$G(x,t) = F(x,t) + \int_a^b \int_c^d K\left(x,t,s,y,G(s,y)\right) dsdy, \tag{1.13}$$

where

$$G(x,t) = \left[g_i(x,t)\right], i = 1,2,\ldots,n.$$

$$F(x,t) = \left[f_i(x,t)\right], i = 1,2,\ldots,n.$$

$$K\left(x,\mathrm{t},\mathrm{s},y,G(s,y)\right) = \left[k_{i,j}\left(x,\mathrm{t},\mathrm{s},y,G(s,y)\right)\right], i,j = 1,2,\ldots,n.$$

From the system (1.13), one can obtain the linear and nonlinear systems as (Saeed and Mahmud, 2009)

$$g_i(x,t) = f_i(x,t) + \sum_{l=1}^{n} \int_a^b \int_c^d K_{i,l}(x,t,s,y)\, g_l(s,y)\, dsdy. \tag{1.14}$$

$$g_i(x,t) = f_i(x,t) + \sum_{l=1}^{n} \int_a^b \int_c^d K_{i,l}\left(x,t,s,y,g_l(s,y)\right) dsdy. \tag{1.15}$$

respectively. Based on (1.14), a special case of the second kind two-dimensional Fredholm integral equation system can be defined as

$$g_i(x,t) = f_i(x,t) + \sum_{l=1}^{n} \int_a^b \int_c^d K_{i,l}(x,t,s,y)\left(g_l(s,y)\right)^m dsdy, \quad m = 2,3,\ldots \tag{1.16}$$

1.4 Special Kinds of Kernels (Kanwal, 1971)

i. Separable kernel

The kernel $K(s,t)$ is said separable if it is of finite rank, i.e.,

$$K(s,t)=\sum_{i=1}^{n}u_i(s)v_i(t), \tag{1.17}$$

where $u_i(s)$ and $v_i(t)$ are linearly independent.

ii. Symmetric kernel

The kernel $K(s,t)$ is called symmetric if

$$K(s,t)=K(t,s). \tag{1.18}$$

iii. Skew symmetric kernel

The skew symmetric kernel $K(s,t)$ is of the form

$$K(s,t)=-K(t,s). \tag{1.19}$$

iv. Hilbert-Schmidt kernel

The kernel $K(s,t)$ is to be Hilbert-Schmidt kernel if for each

a. set of values of s,t in $a\le s\le b$ and $a\le t\le b$

$$\int_a^b\int_a^b |K(s,t)|^2\,dsdt<\infty, \tag{1.20}$$

b. value of s in $a<s<b$

$$\int_a^b |K(s,t)|^2\,dt<\infty, \tag{1.21}$$

c. value of t in $a<t<b$

$$\int_a^b \left|K(s,t)\right|^2 ds < \infty. \tag{1.22}$$

1.5 Objective of Research

The objectives of this research are

1. To develop and apply the use of approximate analytical method called the optimal homotopy asymptotic method (OHAM) for solving both the linear and nonlinear one and two-dimensional Fredholm integral equations.
2. To investigate the properties and ability of this method for these types of equations.
3. To develop and apply the optimal homotopy asymptotic method for solving systems of linear and nonlinear one and two-dimensional Fredholm integral equations.
4. To show that the homotopy perturbation method (HPM) and Adomian decomposition method (ADM) are equivalent for solving both second kind Fredholm-Hammerstein integral equations and two-dimensional Fredholm integral equations and a comparative study between these methods and OHAM.

1.6 Scope and Methodology

To begin with, the basics of methods for solving one and two-dimensional Fredholm integral equations and systems of one and two-dimensional integral equations will be presented. The literature on methods for solving the Fredholm integral equations will be studied. Attention will be concentrated on the OHAM, HPM and ADM.

Selected OHAM will be developed and applied to find the numerical solutions for both linear and nonlinear one and two-dimensional Fredholm integral equations. Further, it will be developed to solve the systems of linear and nonlinear one and two-dimensional Fredholm integral equations. Beside this, the selected HPM and ADM will be applied to solve both second kind Fredholm-Hammerstein integral equations and two-dimensional Fredholm integral equations. The equality of the two methods will be shown for solving these types of equations.

The analytical solutions and absolute errors obtained by using these methods will be tabulated and analyzed and comparison will be carried with the analytical solutions and absolute errors obtained by using other methods in literature. Based on these results, the effectiveness and accuracy of the methods will be determined for solving these types of equations.

Maple 14 software with long format and double accuracy will be used to carry out the computations.

1.7 Organization of Thesis

This thesis describes the application of the OHAM to linear and nonlinear problems of Fredholm integral equations. It consists of eight chapters. Chapter 1 will cover the background of analytical methods, some types of Fredholm integral equations, certain cases of systems of Fredholm integral equations, some special kinds of kernels, objective of research, scope and methodology and followed by organization of thesis. The basic idea of the methods will be discussed in Chapter 2. Chapter 3 will cover the literature review, beginning with history and development of methods with application for solving linear and nonlinear integral equations problems in various fields of sciences.

Chapter 4 will explain the application of the OHAM technique to first kind Fredholm integral equations and second kind Fredholm-Hammerstein integral equations. The proposed method is used to solve some numerical examples of these types of equations to show the effectiveness and validity of the method. A comparison between this method and other methods in literature is conducted.

Application of OHAM for the solution of two-dimensional integral equations is presented in Chapter 5. Two kinds of these equations are studied: linear and nonlinear first kind and second kind. This method is investigated to solve some different numerical examples and the analytical solutions obtained by this method is tabulated and analyzed. A comparison result by the OHAM with other methods literature is given.

Chapter 6 will cover the application of HPM and ADM for solving both of the second kind Fredholm-Hammerstein integral equations and two-dimensional Fredholm integral equations. The equivalence between the two methods to solve these types of equations is shown. A comparative study between these methods and OHAM is conducted.

The OHAM will be introduced for obtaining the solution of systems of one and two-dimensional Fredholm integral equations in Chapter 7. Some numerical examples of linear and nonlinear of these types of systems are tested to show that the proposed method can be applied to these types of systems. The results obtained by this method are compared with other methods which used in literature.

Chapter 8 will cover a summary of the results obtained by application the methods.

CHAPTER 2

REVIEW ON THE BASIC PRINCIPLES OF APPROXIMATE ANALYTICAL METHODS

2.1 Introduction

This chapter presents the basic approximate analytical methods. These methods are the optimal homotopy asymptotic method (OHAM), homotopy perturbation method (HPM) and Adomian decomposition method (ADM). In 1999, the homotopy perturbation method (HPM) was first introduced by He based on combination of topology and perturbation method. In fact, many authors have been developing and applying the HPM in linear and nonlinear problems, see He (1999; 2004; 2006; 2010), Abbasbandy (2006; 2007), Chun (2010), Merdan (2007), Javidi and Golbabai (2007), Yusufoglu (2009), Jazbi and Moini (2008), Biazar and Ghazvini (2009), Mohyud-Din and Noor (2009), Hemeda (2009), Yıldırım and Öziş (2007), Aminsadrabad (2012) and Zedan and El Adrous (2012).

The Adomian decomposition method (ADM) was suggested and developed by Adomian in 1980. This method has been used by authors in differential equations, algebraic equations and integral equations. Examples include Adomian and Rach (1985), Adomian (1994), Wazwaz (1999), Biazar, Babolian and Islam (2004), Abbasbandy (2006), Tatari, Dehghan and Razzaghi (2007), Pei, Yong and Zhi-Bin (2008), Wu, Shi and Wu (2011), Abassy (2010), Evirgen and Özdemir (2010), Abbaoui and Cherruault (1994), Kutafina (2011), Fadaei (2011), Cheniguel and Ayadi (2011) and Heidarzadeh, Joubari and Asghari (2012).

In recent years, Marinca and Herişanu (2008) suggested and developed a new technique called the optimal homotopy asymptotic method (OHAM). This method has been successfully applied by many researchers in sciences and engineering for solving linear and nonlinear problems. Examples include (Marinca and Herişanu, 2008), (Shah et al., 2010), (Iqbal et al., 2010), (Iqbal and Javed, 2011), (Temimi, Ansari and Siddiqui, 2011), (Kaliji et al., 2010), (Ghoreishi, Ismail and Alomari, 2012), (Esmaeilpour and Ganji, 2010), (Islam, Shah and Ali, 2010), (Jafari and Gharbavy, 2012), (Ali, Khan and Shah, 2012) and (Idrees et al., 2012). The following Table 2.1 displays the history of development for coupling of homotopy with perturbation.

Table 2.1: History of the development of coupling of homotopy with perturbation (Idrees, 2011).

Reference	**Type of Differential Equation**	**Family of Homotopy**
Liao 1992	$N\left[u(x)\right]=0$	$(1-p)L\left[U(x,p)-u_0(x)\right]+N\left[U(x,p)\right]=0,$
He 1999	$L(u)+N(u)=f(x)$	$(1-p)\left[L(u)-L(u_0)\right]+P[L(u)+N(u)-f(x)]=0,$
Liao 1999	$N\left[u(x)\right]=0$	$\left(1-B(p)\right)L\left[U(x,p)-u_0(x)\right]=c_0ApN\left[U(x,p)\right],$
Marinca and Herişanu 2008	$L\left(u(x)\right)+f(x)$ $+N\left(g(x)\right)=0,$ $B\left(g,\frac{dg}{ds}\right)=0$	$(1-p)[L\left(u(x,p)\right)+f(x)]=H(p)\left[L\left(u(x,p)\right)+\right.$ $\left.f(x)+N\left(g(x,p)\right)\right],$
Liao 2009	$N\left[u(x)\right]=0$	$\left(1-B(p)\right)L\left[U(x,p)-u_0(x)\right]=(c_0p+c_1p^2+c_2p^3)$ $\times N\left[U(x,p)\right].$

In Table 2.1, the function U is defined as $U(x,p):\Omega\times[0,1]\rightarrow\Re\,;\,x\in\Omega,\mathrm{p}\in[0,1]$, L is linear, A and H are auxiliary functions, N is nonlinear, u_0 is an initial guess, c_i are constants, $f(x)$ is known function and B is a boundary operator.

2.2 Definition of the Homotopy (Aubry, 1995)

Let X and Y be two topological spaces. If f and g are continuous map of the space $X\rightarrow Y$, it is called that f is called homotopic to g, if there exists a continuous map $H:X\times[0,1]\rightarrow Y$, such that $\forall x\in X$

$$H(x,0)=f(x),$$

$$H(x,1)=g(x).$$

Then the map is called homotopy between f and g.

2.3 Introduction of Least Squares Method of Residuals

This method was first published by Legendre in 1805. The objective of this method is to find the minimum of the sum of the squares in the integral equations problem. In this section, will review this method based on the principles set out by Grandin (1991). Firstly, consider the differential equation as

$$D(g(x))=P(x), \tag{2.1}$$

where D is a differential operator with g and P are known functions.

Assume that the function g is approximated by $\tilde{g}$ as

$$g\cong\tilde{g}(x)=\sum_{i=1}^{n}c_i\,\mathrm{F}_i(x), \tag{2.2}$$

where c_i are coefficients and F_i a linearly independent set.

By substituting equation (2.2) into equation (2.1), the result of the operations is not $P(x)$. Hence, the residual defined will exist as

$$R(x) = D(\tilde{g}(x)) - P(x) \neq 0, \tag{2.3}$$

Next, define function to make the residual to zero as follows

$$S = \int_X R(x) W_i dx, i = 1, 2, \ldots, n, \tag{2.4}$$

where W_i are called the weight function.

Using least squares method, the sum of the squares of the residuals can be minimized by

$$S = \int_X R(x) R(x) dx = \int_X R^2(x) dx, \tag{2.5}$$

and then minimizing it, yields

$$\frac{\partial S}{\partial c_i} = 0$$

$$= 2\int_X R(x) \frac{\partial R}{\partial c_i} dx. \tag{2.6}$$

2.3 Introduction of Galerkin Method

This method may be identical to the least squares method. It was originally introduced by Galerkin (1915). Let us look at the differential equation as follows

$$S(g(x,t)) = J(x,t), \tag{2.7}$$

where S is a differential operator and $g(x,t)$, $J(x,t)$ are known functions. By expand function $g(x,t)$ to N as a series of

$$g_N(x,t) = \sum_{i=1}^{N} c_i(t) \mathrm{F}_i(x), \tag{2.8}$$

and substituting equation (2.8) into equation (2.7), the residual defined will exist as

$$R_N(x,t) = S(g_N(x,t)) - J(x,\mathrm{t}) \neq 0. \tag{2.9}$$

The goal of Weighted Residuals is to choose the coefficients c_i such that the residual R_N becomes small (in fact 0) over a chosen domain. In integral form this can be achieved with the condition

$$S = \int_X R_N(x,t) W_i dx, i = 1, 2, \ldots, N. \tag{2.10}$$

By deriving the approximating function, function W_i can be obtained by

$$W_i = \frac{\partial g_N}{\partial c_i} \tag{2.11}$$

2.5 Definition of Taylor Series

Taylor series was first introduced by Taylor in 1712 and published in 1715. Application of Taylor series is in the field of calculus and ordinary differential equations. To explain this series, we let the function $f(x)$ as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \tag{2.12}$$

Differentiating equation (2.12) gives

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \tag{2.13}$$

Replacing $x = a$ in the equation (2.13), it holds that

$$f'(a) = c_1 \tag{2.14}$$

Differentiating equation (2.12) twice gives

$$f''(x) = 2c_2 + 6c_3(x-a) + \dots \tag{2.15}$$

and then at $x = a$

$$f''(a) = 2c_2 \tag{2.16}$$

and continuing in this way. The Taylor series generated by $f(x)$ at $x = a$ is defined as follows

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n . \tag{2.17}$$

The following Table 2.2 displays some functions by Taylor's series.

Table 2.2: Some functions by Taylor's series (Wazwaz, 2011a).

Function	**Taylor's series**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$
$\ln(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n}$
$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$

2.6 Description of Optimal Homotopy Asymptotic Method (OHAM)

This section describes OHAM which was proposed by Marinca and Herişanu (2008). Consider the differential equation as

$$L\left(g(s)\right)+f\ (s)+N\ \left(g(s)\right)=0,\quad B\left(g,\frac{dg}{ds}\right)=0, \tag{2.18}$$

where L is known function called the linear operator, $f\ (s)$ is known function, N is called the nonlinear operator, $g(s)$ is unknown function and B is called boundary operator.

Using the OHAM, consider a family of equations for an embedding parameter $p\in[0,1]$ as below

$$(1-p)\left[L\left(g(s,p)\right)+f\ (s)\right]=H\ (p)\left[L\left(g(s,p)\right)+f\ (s)+N\ \left(g(s,p)\right)\right],\quad B\left(g,\frac{dg}{ds}\right)=0, \tag{2.19}$$

where $H\ (p)$ denotes a non-zero auxiliary function for $p\neq 0$ and $H(0)=0.$ Obviously, when $p=0,$ it holds that

$$g(s,0)=g_0(s), \tag{2.20}$$

and when $p=1,$ it holds that

$$g(s,1)=g(s). \tag{2.21}$$

Assume that the auxiliary function $H\ (p)$ can be expressed as

$$H\ (p)=\sum_{j=1}^{m}c_j\,p^j, \tag{2.22}$$

where $c_j,\ j=1,2,...$ are constants.

Setting $p=0$ in equation (2.19), it holds that

$$L\left(g_0(s)\right)+f\ (s)=0, \quad B\left(g_0,\frac{dg_0}{ds}\right)=0. \tag{2.23}$$

By Taylor's series, the OHAM solution can be calculated as below

$$g\left(s,p,c_j\right)=g_0(s)+\sum_{k=1}^{m}g_k\left(s,c_j\right)p^m,\ j=1,2,\ldots \tag{2.24}$$

When $p=1$, the equation (2.24) becomes

$$g\left(s,p,c_j\right)=g_0(s)+\sum_{k=1}^{m}g_k\left(s,c_j\right),\ j=1,2,\ldots \tag{2.25}$$

Substituting equation (2.24) into equation (2.19) and equating the coefficients of like powers of p , yields

$$L\left(g_1(s)\right)=c_1N\left(g_0(s)\right), \quad B\left(g_1,\frac{dg_1}{ds}\right)=0, \tag{2.26}$$

$$L\left(g_m(s)-g_{m-1}(s)\right)=c_mN\left(g_0(s)\right)+\sum_{j=1}^{m-1}c_j\left[L(g_{m-j}(s)+N_{m-j}(g_0(s)+g_1(s)+\ldots+g_{m-1}(s)\right],$$

$$B\left(g_m,\frac{dg_m}{ds}\right)=0, m=2,3,\ldots \tag{2.27}$$

where $N_m\left(g_0(s),g_1(s),\ldots,g_m(s)\right)$ are the coefficient of p^m in the expansion of $N\left(g(s,p)\right)$ about p

$$N\left(g\left(s,p,c_j\right)\right)=N_0\left(g_0(s)\right)+\sum_{m=1}^{\infty}N_m\left(g_0(s),g_1(s),\ldots,g_m(s)\right)p^m. \tag{2.28}$$

The result of *m*th-order approximations are as follows

$$g^m\left(s,c_{i,j}\right)=g_0(s)+\sum_{k=1}^{m}g_k\left(s,c_j\right),\ j=1,2,\ldots m. \tag{2.29}$$

Replacing equation (2.29) into equation (2.18), the following residual equation can be obtain

$$R\left(s,c_j\right)=L\left(g^m\left(s,c_j\right)\right)+f\left(s\right)+N\left(g^m\left(s,c_j\right)\right). \tag{2.30}$$

If $R\left(s,c_j\right)=0$ then $g^m\left(s,c_j\right)$ will be an exact solution. For finding the constants $c_j,\ j=1,2,...$ using least squares method, at first consider

$$J\left(c_j\right)=\int_a^b R^2\left(S,c_j\right)ds, \tag{2.31}$$

or by using Galerkin's method as below

$$\frac{\partial J}{\partial c_j}=\int_a^b R\left(s,c_j\right)\frac{\partial R}{\partial c_j}ds. \tag{2.32}$$

Then the constants $c_j,\ j=1,2,...$ can be identified as below

$$\frac{\partial J}{\partial c_1}=\frac{\partial J}{\partial c_2}=\ldots=\frac{\partial J}{\partial c_m}=0. \tag{2.33}$$

Knowing $c_j,\ j=1,2,...$ the OHAM solution is determined.

2.7 Description of Homotopy Perturbation Method (HPM)

This section presents description of the HPM which was proposed by He (1997). First, let the operator equation as below

$$A\left(g\right)-f\left(s\right)=0,\quad B\left(g,\frac{\partial g}{\partial n}\right)=0, \tag{2.34}$$

where $s\in\Omega$, A is an operator, $f\left(s\right)$ is a known function, g is the sought function and $\frac{\partial}{\partial n}$ is differentiation the normal vector drawn outwards from Ω.

The operator A can be divided into L and N as

$$L\left(g\right)+N\left(g\right)-f\left(s\right)=0, \tag{2.35}$$

where L is a linear and N is a non-linear operator. By using the HPM technique, a homotopy can be define $v(s,p):\Omega\times[0,1]\to R$ for an embedding parameter $p\in[0,1]$ as

$$H(v,p)=(1-p)\left[L(v)-L(g_0)\right]+p\left[A(v)-f(s)\right], \tag{2.36}$$

where g_0 is the initial approximation of equation (2.34).

From above equation (2.36), if $p=0$ and $p=1$, it holds that

$$H(v,0)=L(v)-L(g_0)=0, \tag{2.37}$$

$$H(v,1)=L(v)+N(v)-f(s)=0, \tag{2.38}$$

the changing process of p from zero to unity is just that of $v(s,p)$ from $g_0(s)$ to $g(s)$.

Next step, consider the solution of equation (2.36) can be obtained in the form of power series

$$v(s,p)=\sum_{m=0}^{\infty}v_m(s)p^m. \tag{2.39}$$

When the series in equation (2.39) of $v(s,p)$ converges at $p=1$, then

$$g(s)=\lim_{p\to 1}v(s,p)=\sum_{m=0}^{\infty}v_m(s). \tag{2.40}$$

Using equation (2.39) into equation (2.36), one can obtain

$$H(v,p)=(1-p)\left[L\left(\sum_{m=0}^{\infty}v_m(s)p^m\right)-L(g_0)\right]+p\left[A\left(\sum_{m=0}^{\infty}v_m(s)p^m\right)-f(s)\right]. \tag{2.41}$$

For simplicity, one can choose $v_0(s)=g_0(s)=f(s)$, and replace $v_0(s)$ into equation (2.39) and then equate the coefficients of powers of p.

2.8 Description of Adomian Decomposition Method (ADM)

This section will discuss the idea of the ADM which was proposed by Adomian (1980). Consider the nonlinear differential equation as follows

$$L(g)+R(g)+N(g)-f(s)=0, \tag{2.42}$$

where L is the highest order derivative which assumed to be invertible, R is the remainder of the linear operator and N is a nonlinear differentiable operator. From equation (2.42), we obtain

$$L(g)=f(s)-R(g)-N(g). \tag{2.43}$$

By applying the inverse operator L^{-1} to equation (2.43) with the initial condition $g(0)=g_0$, it holds that

$$L^{-1}L(g)=L^{-1}\left[f(s)\right]-L^{-1}\left[R(g)\right]-L^{-1}\left[N(g)\right], \tag{2.44}$$

and gives

$$g=g_0-L^{-1}\left[R(g)\right]-L^{-1}\left[N(g)\right], \tag{2.45}$$

where L^{-1} would represent an integration and with any given initial or boundary condition.

The ADM defines the solution $g(s)$ as below

$$g(s)=\sum_{i=0}^{\infty} g_i(s). \tag{2.46}$$

Next, $N(g)$ will be decomposed by

$$N(g)=\sum_{i=0}^{\infty} A_i, \tag{2.47}$$

where A_i are the polynomials of $g_0, g_1, \ldots, g_i$ given by

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{i}\lambda^j g_i(s)\right)\right]_{\lambda=0}, i = 0,1,2,\dots \qquad (2.48)$$

where λ is a parameter introduced for convenience.

Using equations (2.46) and (2.47) in to equation (2.45), we will have

$$\sum_{i=0}^{\infty} g_i(s) = g_0 - L^{-1}\left[R\left(\sum_{i=0}^{\infty} g_i(s)\right)\right] - L^{-1}\left[\sum_{i=0}^{\infty} A_i\right]. \qquad (2.49)$$

Here, $g_i(s)$ will be determined as follows

$$g_0(s) = g_0,$$

$$g_{i+1}(s) = -L^{-1}\left[R(g)\right] - L^{-1}\left[N(g)\right], i = 0,1,\dots \qquad (2.50)$$

2.9 Definition of the Absolute Error

To know whether the analytical calculations are accurate or inaccurate, the amount of error between the true and approximation must be calculated. In this study, the absolute error, which is the difference between the truth value and approximation value, is used to show the efficiency of the present methods in our problem. Let us define the absolute error as follows (Abramowitz and Stegun, 1972)

$$\mathrm{E}_{abs} = \left|g - \hat{g}\right|, \qquad (2.51)$$

where E_{abs} denotes the absolute error, g is the true value and $\hat{g}$ its approximation.

On the other hand, the absolute errors in each value can be defined as follows

$$\mathrm{E}_{1\,abs} = \left|g_1 - \hat{g}_1\right|, \mathrm{E}_{2\,abs} = \left|g_2 - \hat{g}_2\right|, \dots, \mathrm{E}_{n\,abs} = \left|g_n - \hat{g}_n\right|, \qquad (2.52)$$

where $\mathrm{E}_{n\,abs}$ are absolute errors, n is measurement values, g_n are the truth values and $\hat{g}_n$ denote the approximations.