UNIVERSITI SAINS MALAYSIA

First Semester Examination
2014/2015 Academic Session

December 2014/January 2015

MAT 514 - Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[Arahan: Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Discuss briefly on the concept of the boundary layer. Then, state the two-dimensional boundary-layer approximations.

   (b) Define:

   (i) incompressible flow,
   (ii) shear stress,
   (iii) convective mass-transfer process.

[16 marks]

1. (a) Bincangkan secara ringkas mengenai konsep lapisan sempadan. Seterusnya, nyatakan penghampiran-penghampiran lapisan sempadan dua dimensi.

(b) Takrifkan:

(i) aliran tak termampat,
(ii) tegasan ricih,
(iii) proses pemindahan jisim berolak.

[16 markah]

2. Suppose that the momentum and energy equations of constant surface heat rate for horizontal circular tube of radius r_s with fully developed velocity and temperature profiles are given by the following equations, respectively,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mu\frac{\partial u}{\partial r}\right)-\frac{dP}{dx}=0, \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\alpha\frac{\partial T}{\partial r}\right)-u\frac{dT_m}{dx}=0, \qquad (2)$$

where x and r are the axial and radial coordinates of circular tube, respectively, u is the fully develop velocity profile, the pressure P and the mass-averaged fluid temperature T_m are independent of r, and the dynamic viscosity μ and the thermal diffusivity α of the fluid are constants.

   (a) Derive a fully developed temperature profile T by using the momentum equation (1) and the energy equation (2) subject to the following boundary conditions

$$\frac{\partial u}{\partial r}=\frac{\partial T}{\partial r}=rx \quad \text{at} \quad r=0,$$
$$u=0, \quad T=T_s \quad \text{at} \quad r=r_s.$$

[T_s is the fluid temperature at the surface of circular tube.]

(b) What are the velocity and temperature profiles at the centerline of the circular tube?

[26 marks]

2. *Andaikan bahawa persamaan-persamaan momentum dan tenaga dengan kadar haba permukaan malar bagi tiub bulat mendatar berjejari r_s yang profil-pofil halaju dan suhu terbangun penuh diberikan oleh persamaan-persamaan berikut*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mu\frac{\partial u}{\partial r}\right)-\frac{dP}{dx}=0, \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\alpha\frac{\partial T}{\partial r}\right)-u\frac{dT_m}{dx}=0, \qquad (2)$$

dengan x dan r masing-masing adalah koordinat paksian dan jejarian tiub bulat, u adalah profil halaju terbangun penuh, tekanan P dan suhu jisim-purata bendalir T_m tidak bersandar pada r, dan kelikatan dinamik μ dan resapan terma α adalah malar.

(a) *Terbitkan profil suhu terbangun penuh T dengan menggunakan persamaan momentum (1) dan persamaan tenaga (2) tertakluk kepada syarat-syarat sempadan berikut*

$$\frac{\partial u}{\partial r}=\frac{\partial T}{\partial r}=rx \quad pada \quad r=0,$$
$$u=0, \quad T=T_s \quad pada \quad r=r_s.$$

[T_s ialah suhu bendalir pada permukaan tiub bulat.]

(b) *Apakah profil-profil halaju dan suhu pada garis tengah tiub bulat?*

[26 markah]

3. Consider a three-dimensional fluid flow in xyz plane. Let M represent mass flux (mass flow rate per unit of normal area), where M_x, M_y and M_z are the x, y and z components, respectively, ρ is the density of the fluid and t is the time. The mass flux flow in and flow out in x-direction of unit depth are given by $M_x\delta y\delta z$ and $\left(M_x+\frac{\partial M_x}{\partial x}\delta x\right)\delta y\delta z$, respectively.

(a) Draw a complete figure of a control volume of unit depth for three-dimensional fluid flow in xyz plane.

(b) Express the principle of conservation of mass.

(c) Based on 3(a) and 3(b), show that the continuity equation for three-dimensional flow is

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0. \qquad (3)$$

Then, state the assumption that can be made so that the time term in equation (3) can be neglected.

[20 marks]

3. *Pertimbangkan suatu aliran bendalir tiga dimensi dalam satah xyz. Andaikan M mewakili fluks jisim (kadar aliran jisim per unit luas serenjang), dengan* M_x, M_y *dan* M_z *masing-masing ialah komponen* x, y *dan* z, ρ *ialah ketumpatan bendalir dan* t *ialah masa. Fluks jisim mengalir masuk dan mengalir keluar pada arah* x *dengan satu unit kedalaman masing-masing diberikan oleh* $M_x \delta y \delta z$ *dan* $\left(M_x + \frac{\partial M_x}{\partial x}\delta x\right)\delta y \delta z$.

(a) *Lukis satu rajah lengkap isipadu kawalan dengan satu unit kedalaman bagi aliran bendalir tiga dimensi dalam satah xyz.*

(b) *Nyatakan prinsip keabadian jisim.*

(c) *Berdasarkan 3(a) dan 3(b), tunjukkan bahawa persamaan keselanjaran bagi aliran bendalir tiga dimensi adalah*

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0. \qquad (3)$$

Seterusnya, nyatakan andaian-andaian yang boleh dibuat supaya sebutan masa dalam persamaan (3) boleh diabaikan.

[20 markah]

4. Consider a steady boundary layer flow and heat transfer past a sphere of radius a in a forced convection flow of a viscous and incompressible fluid of free stream velocity U_∞ and ambient temperature T_∞. It is assumed that the sphere is kept at the uniform temperature T_w and the boundary layer approximations are valid. Under these assumptions, the dimensional governing equations are

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0, \tag{4}$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \rho \bar{u}_e \frac{d\bar{u}_e}{d\bar{x}} + \mu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{5}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{6}$$

subject to the boundary conditions

$$\begin{aligned} &\bar{u} = \bar{v} = 0, \quad \bar{T} = T_w \quad \text{at} \quad \bar{y} = 0, \\ &\bar{u} \to \bar{u}_e(\bar{x}), \quad \bar{T} \to T_\infty \quad \text{as} \quad \bar{y} \to \infty, \end{aligned} \tag{7}$$

where $\bar{x}$ and $\bar{y}$ are the Cartesian coordinates measured along the sphere and normal to it, respectively, $\bar{u}$ and $\bar{v}$ are velocity components along $\bar{x}$ and $\bar{y}$ axes, respectively, $\bar{T}$ is the temperature of the fluid, ρ is the constant density of the fluid, μ is the dynamic viscosity of the fluid, $\alpha = \mu/(\rho \text{Pr})$ is the thermal diffusivity of the fluid and Pr is the Prandtl number. Here $\bar{r}$ is the radial distance from the symmetrical axis to the surface of the sphere and $\bar{u}_e(\bar{x})$ is the local free stream velocity which are given by $a \sin(\bar{x}/a)$ and $(3/2)U_\infty \sin(\bar{x}/a)$, respectively. The non-dimensional variables are

$$\begin{aligned} &x = \bar{x}/a, \quad y = \text{Re}^{1/2}(\bar{y}/a), \quad u = \bar{u}/U_\infty, \quad v = \text{Re}^{1/2}(\bar{v}/U_\infty), \\ &r = \bar{r}/a, \quad \theta = (\bar{T} - T_\infty)/(T_w - T_\infty), \quad u_e(x) = \bar{u}_e(\bar{x})/U_\infty, \end{aligned}$$

where $\text{Re} = U_\infty a\rho/\mu$ is the Reynolds number.

Note: ψ is a stream function which is defined as $u = \dfrac{1}{r}\dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial x}$.

(a) By using non similarity variables of the following form

$$\psi = xrf(x,y), \quad \theta = g(x,y),$$

show that the boundary layer equations (4) – (6) can be reduced to the following system of differential equations

$$\begin{aligned} \frac{\partial^3 f}{\partial y^3} + \left(1 + \frac{x}{\sin x}\cos x\right) f \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{9}{4}\frac{\sin x \cos x}{x} \\ = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \end{aligned} \tag{8}$$

$$\frac{1}{\Pr}\frac{\partial^2 g}{\partial y^2} + \left(1 + \frac{x}{\sin x}\cos x\right) f\frac{\partial g}{\partial y} = x\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \qquad (9)$$

with the boundary conditions (7) becoming

$$f = \frac{\partial f}{\partial y} = 0, \quad g = 1 \quad \text{at} \quad y = 0, \qquad (10)$$
$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \quad g \to 0 \quad \text{as} \quad y \to \infty.$$

(b) Table 1 shows the numerical results of shear stress $\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$ and surface heat transfer rate $\left(-\frac{\partial g}{\partial y}\right)_{y=0}$ for the system of equations (8) and (9) subject to boundary conditions (10) when $x = 0$ with various values of Pr. Interpret the obtained results in this table.

Table 1: Numerical results of shear stress and surface heat transfer rate with various values of Pr

Pr	$\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$	$\left(-\frac{\partial g}{\partial y}\right)_{y=0}$
0.7	2.4104	0.8150
1.0	2.4104	0.9336
6.8	2.4104	1.8747

[38 marks]

4. *Pertimbangkan suatu aliran lapisan sempadan dan pemindahan haba yang mantap terhadap sfera berjejari a dalam aliran olakan paksa bagi bendalir likat dan tak termampat dengan halaju strim bebas U_∞ dan suhu persekitaran T_∞. Andaikan bahawa sfera itu mempunyai suhu yang seragam T_w dan penghampiran lapisan sempadan adalah sah. Berdasarkan andaian tersebut, persamaan-persamaan menakluk berdimensi adalah*

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0, \qquad (4)$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \rho\bar{u}_e\frac{d\bar{u}_e}{d\bar{x}} + \mu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \qquad (5)$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \qquad (6)$$

tertakluk kepada syarat-syarat sempadan

$$\bar{u} = \bar{v} = 0, \quad \bar{T} = T_w \quad \textit{pada} \quad \bar{y} = 0, \tag{7}$$
$$\bar{u} \to \bar{u}_e(\bar{x}), \quad \bar{T} \to T_\infty \quad \textit{apabila} \quad \bar{y} \to \infty,$$

dengan $\bar{x}$ *dan* $\bar{y}$ *masing-masing adalah koordinat-koordinat Cartesan yang diukur di sepanjang sfera dan berserenjang dengannya,* $\bar{u}$ *dan* $\bar{v}$ *masing-masing ialah komponen-komponen halaju pada paksi* $\bar{x}$ *dan* $\bar{y}$, $\bar{T}$ *ialah suhu bendalir,* ρ *ialah ketumpatan bendalir malar,* μ *ialah kelikatan dinamik bendalir,* $\alpha = \mu/(\rho Pr)$ *adalah resapan terma bendalir dan* Pr *ialah nombor Prandtl. Di sini* $\bar{r}$ *ialah jarak jejari dari paksi simetri dengan permukaan sfera dan* $\bar{u}_e(\bar{x})$ *ialah halaju strim bebas setempat yang masing-masing diberikan oleh* $a\sin(\bar{x}/a)$ *dan* $(3/2)U_\infty \sin(\bar{x}/a)$. *Pemboleh-pemboleh ubah tak berdimensi adalah*

$$x = \bar{x}/a, \quad y = Re^{1/2}(\bar{y}/a), \quad u = \bar{u}/U_\infty, \quad v = Re^{1/2}(\bar{v}/U_\infty),$$
$$r = \bar{r}/a, \quad \theta = (\bar{T} - T_\infty)/(T_w - T_\infty), \quad u_e(x) = \bar{u}_e(\bar{x})/U_\infty,$$

dengan $Re = U_\infty a\rho/\mu$ *ialah nombor Reynolds.*

Nota: ψ *adalah fungsi strim yang ditakrifkan sebagai* $u = \dfrac{1}{r}\dfrac{\partial\psi}{\partial y}$ *dan* $v = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial x}$.

(a) *Dengan menggunakan pemboleh-pemboleh ubah tak serupa berbentuk berikut*

$$\psi = xrf(x,y), \quad \theta = g(x,y),$$

tunjukkan bahawa persamaan-persamaan lapisan sempadan (4) – (6) boleh diturunkan kepada sistem persamaan pembezaan berikut

$$\frac{\partial^3 f}{\partial y^3} + \left(1 + \frac{x}{\sin x}\cos x\right) f\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{9}{4}\frac{\sin x \cos x}{x} = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \tag{8}$$

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial y^2} + \left(1 + \frac{x}{\sin x}\cos x\right) f\frac{\partial g}{\partial y} = x\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \tag{9}$$

dengan syarat-syarat sempadan (7) menjadi

$$f = \frac{\partial f}{\partial y} = 0, \quad g = 1 \quad \textit{pada} \quad y = 0, \tag{10}$$
$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \quad g \to 0 \quad \textit{apabila} \quad y \to \infty.$$

(b) *Jadual 1 menunjukkan keputusan berangka untuk tegasan ricih* $\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{y=0}$ *dan kadar pemindahan haba permukaan* $\left(-\dfrac{\partial g}{\partial y}\right)_{y=0}$ *bagi sistem persamaan*

(8) dan (9) tertakluk kepada syarat-syarat sempadan (10) bila $x = 0$ dengan beberapa nilai Pr. Tafsirkan keputusan yang diperoleh dalam jadual tersebut.

Jadual 1: Keputusan berangka untuk tegasan ricih dan kadar pemindahan haba permukaan dengan beberapa nilai Pr.

Pr	$\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$	$\left(-\frac{\partial g}{\partial y}\right)_{y=0}$
0.7	2.4104	0.8150
1.0	2.4104	0.9336
6.8	2.4104	1.8747

[38 markah]

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